An Atlas For $\mathrm{Bun}_r(X)$

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1 $\mathrm{Bun}_r(X)$ Is Not Of Finite Type

The goal of this lecture is to find a smooth atlas locally of finite type for the stack $\mathrm{Bun}_r(X)$ of rank-r vector bundles on a smooth projective curve X. Let's see first that it is impossible to find an atlas of finite type:

Theorem 1.1. *thm:no.global.atlas There is no surjection from a scheme of finite type to* $\mathrm{Bun}_r(X)$.

Actually, there are two reasons for this. The first is that the determinant of a vector bundle varies continuously in families, and since there are infinitely many components of $\mathrm{Pic}(X)$, it follows that $\mathrm{Bun}_2(X)$ has infinitely many components. On top of that, I'll show now that each connected component is not of finite type.

Denote the structure sheaf of X by O. Fix an ample line bundle $O(1)$ on X. For a vector bundle E, we denote $E(n) = E \otimes O(1)^{\otimes n}$. If S is a scheme, let $X_S = X \times S$, and let E_S be the pullback of E by the projection $X_S \to X$. Consider the rank-2 vector bundles $O(n) \oplus O(-n)$ as points of $\mathrm{Bun}_2(X)$. Theorem 1.1 follows from the following two theorems:

Theorem 1.2. *thm:connected.SL For every n, there is a connected variety Y (actually, an affine space), a map $Y \to \mathrm{Bun}_2(X)$, and two points $y_0, y_1 \in Y$, such that y_0 is mapped to $O \oplus O$, and y_1 is mapped to $O(n) \oplus O(-n)$.*

After constructing an atlas, this means that all the $O(n) \oplus O(-n)$'s are in the same connected component.

Theorem 1.3. *thm:infinite.type There is no map $Y \to \mathrm{Bun}_2(X)$ from a scheme of finite type Y and points $y_n \in Y$, $n = 0, 1, 2, \ldots$, such that y_n is mapped to $O(n) \oplus O(-n)$.*

I am going to use the following theorem of Serre:

Theorem 1.4. *Let F be a sheaf on a curve X, and let $O(1)$ be an ample line bundle on X.*

1. *(absolute version) If n is big enough then $E(n)$ is generated by global sections, and $H^1(X, E(n)) = 0$.*

2. *(relative version) For every scheme S of finite type, if n is big enough, then $E_S(n)$ is generated by global sections, and $R^1 p_* E(n) = 0$, where $p : X_S \to S$ is the projection.*

Proof of Theorem 1.2. I want to show first that $F = O_X \oplus O_X$ is an extension of $O_X(-n)$ by $O_X(n)$. $F(n)$ is globally generated by Serre's theorem. Therefore, there is a never-zero section $s \in \Gamma(F(n))$. This gives a short exact sequence

$$0 \to O_X \to F(n) \to F(n)/O_X = L \to 0.$$

Note that L must be a line bundle. By untwisting, we get

$$0 \to O_X(-n) \to F \to L(-n) \to 0.$$

By looking at the determinant, the last term must be $O_X(n)$.

Finally, the space of extensions of $O_X(n)$ by $O_X(-n)$ is the affine space $\mathrm{Ext}(O_X(n), O_X(-n))$. □

Proof of Theorem 1.3. A map $Y \to \mathrm{Bun}_2 X$ is a rank 2 vector bundle E on X_Y. By Serre, there is n such that $E(n)$ is globally generated, and hence E_y is globally generated for all y. But $E_{y_{n+1}}(n) = O(-1) \oplus O(2n+1)$ is not globally generated. □

2 $\mathrm{Bun}_r(X)$ Is Locally Of Finite Type

On the other hand, it turns out that $\mathrm{Bun}_r(X)$ is an increasing union of Artin stacks of finite type. More precisely,

Theorem 2.1. thm:loc.fin.atlas *There are open sub-functors $\mathcal{U}_n \hookrightarrow \mathrm{Bun}_r(X)$, varieties Y_n of finite type, and smooth surjective maps $Y_n \to \mathcal{U}_n$, such that $\mathcal{U}_n$ are a cover of* $\mathrm{Bun}_r(X)$.

As said in the beginning, the degree of the vector bundle is constant in each connected component. Let's fix this degree, and consider only vector bundles of a given degree. For every vector bundle E on X, if n is large enough then $E(n)$ is generated by global sections and $H^1(X, E(n)) = 0$. Let $\mathcal{U}_n$ be the moduli stack of bundles that satisfy these two conditions. More precisely, for every scheme T, let $\mathcal{U}_n(T)$ be the full sub-groupoid of $\mathrm{Bun}_r(X)$ whose objects are vector bundles E on X_T that are generated by global sections and such that $R^1 p_*(E(n)) = 0$ (p is the projection $X \times T \to T$). By the theorem of Serre (in the version that applies to families), $\mathcal{U}_n$, as n runs over $\mathbb{N}$, cover $\mathrm{Bun}_r(X)$.

Lemma 2.2. *$\mathcal{U}_n$ is an open sub-functor of* $\mathrm{Bun}_r(X)$.

Proof. The claim is that, for every scheme S and every vector bundle E on X_S, the set of points s in S such that E_s (which is the restriction of E to the fiber $X \times \{s\}$) is globally generated and has zero first cohomology, is open. The subset of S for which E_s is globally generated is clearly open. The subset of S for which $H^1(E_s) = 0$ is the complement of the support of $R^1 p_* E$, and hence open. □

Suppose that T is affine and connected, and let $E \in \mathrm{Ob}\,\mathcal{U}_n(T)$. For every $t \in T$, the Euler characteristic of the restriction $E_t(n)$ of $E(n)$ to $X \times \{t\}$ is $\chi(E_t(n)) = \dim H^0(E_t(n)) - \dim H^1(E_t(n)) = \dim H^0(E_t(n))$. By flatness, $\chi(E_t(n))$ is constant. This means that the sheaf $p_* \underline{\mathrm{Hom}}(O_T, E(n))$ is a vector bundle. To compute its rank, note that, for every $t \in T$

$$\deg(E_t \otimes O(n)) = r \deg(O(n)) + \deg(E_t),$$

and so the dimension of $H^0(E_t(n))$, which is the rank of $p_* \underline{\mathrm{Hom}}(O_T, E(n))$, is

$$d = \deg(E_t(n)) + r(g-1) = r \deg(O(n)) + \deg(E) + r(g-1).$$

For every T, let $\mathcal{Z}_n(T)$ be the groupoid whose objects are pairs (E, ϕ), where $E \in \mathrm{Ob}\,\mathcal{U}_n(T)$ and $\phi : O(n)^d \to E$ is an epimorphism, and the morphisms between (E, ϕ) and (E', ϕ') are maps $f : E \to E'$ such that $f \circ \phi = \phi'$. Note that $\mathcal{Z}_n$ is actually an equivalence relation, rather than a groupoid (i.e. the stabilizers of objects are trivial).

Lemma 2.3. *$\mathcal{Z}_n$ is representable by an open subscheme of the Quot-scheme.*

Proof. Let E be a coherent sheaf over X. Recall that the Quot functor sends a scheme S to

$$\mathrm{Quot}_{E/X}(S) = \{(G, \phi) | G \text{ is a sheaf over } X_S \text{ which is flat over } S, \text{ and } \phi : E_S \to G \text{ is an epimorphism}\}.$$

Let F be the following functor:

$$F(S) = \{(G, \phi) | G \text{ is a sheaf over } X_S \text{ which is flat over } X_S, \text{ and } \phi : E_S \to G \text{ is an epimorphism}\}.$$

$\mathcal{Z}_n$ is an open sub-functor of F, so it is enough to prove that F is representable by an open sub-scheme of Quot.

Suppose we have a map $S \to \mathrm{Quot}_{E/X}$, i.e. a coherent sheaf G over X_S and a surjection $\phi : E_S \to G$. We need to show that the set of points $s \in S$ such that G_s is flat over X_s is open. Let $p : X_S \to S$ be the projection. Let $A \subset X_S$ be the locus where G is non-flat. A is closed, and so $S \setminus p(A)$, which is the set we are after, is open (note that X, and so p, is proper). □

If $(E, \phi) \in \mathcal{Z}_(T)$, then ϕ gives a map between O_T^d and $p_*(E(n))$. Let $\mathcal{Y}_n(T) \subset \mathcal{Z}_n(T)$ be the collection of such (E, ϕ) such that this last map is an isomorphism. It is clear that $\mathcal{Y}_n$ is open in $\mathcal{Z}_n$, and hence in the Quot scheme. In particular, it is representable.

Finally, we want to show that the natural map $\mathcal{Y}_n \to \mathcal{U}_n$ is a smooth surjection. In fact,

Lemma 2.4. *The map $\mathcal{Y}_n \to \mathcal{U}_n$ is a GL_d-torsor.*

Proof. Let $S \to \mathcal{U}_n$ be an S-valued point. This is a vector bundle E over X_S. Lifting this point to $\mathcal{Y}_n$ is, by definition, choosing an identification between O_S^d and $p_*(E)$. This set of identifications is a GL_d torsor, as these are vector bundles of degree d. □

3 Level Structure

Pick $x \in X$. For every n, let $(nx) \hookrightarrow X$ be the n'th infinitesimal neighborhood of x in X, and let $\mathrm{Bun}_r^{(nx)} X$ be the following functor:

$$\mathrm{Bun}_r^{(nx)} X(S) = \{(E, \phi) | E \text{ is a rank } r \text{ vector bundle on } X_S \text{ and } \phi : E_{(nx)_T} \xrightarrow{\cong} O^r_{(nx)_T}\}$$

Proposition 3.1. *If $n > m$, then the map $\mathrm{Bun}_r^{(nx)} X \to \mathrm{Bun}_r^{(mx)} X$ is representable.*

Proof. Let $S \to \mathrm{Bun}_r^{(mx)} X$ be a map, i.e. a pair consisting of a vector bundle E on X_S together with an isomorphism $\phi : E_{(mx)_S} \to O^r_{(mx)_S}$. Let $\mathcal{F}$ be the fiber product. For every scheme T, $\mathcal{F}(T)$ is an equivalence relation, whose objects are triples of

1. A map $T \to S$.
2. A vector bundle F on X_T and an isomorphism $\psi : F_{(nx)_T} \to O^r_{(nx)_T}$.
3. An isomorphism $\xi : E_T \to F$.

such that $\phi_T = \psi_{(mx)_T} \circ \xi_{(mx)_T}$. Equivalently, the objects are pairs consisting of

1. A map $T \to S$.
2. An isomorphism $\psi : E_{(nx)_T} \to O^r_{(nx)_T}$.

such that the restriction of ψ to $E_{(mx)_T}$ is equal to ϕ_T. We want to show that this functor is representable. For any k, consider the group scheme $\mathrm{GL}(E_{(kx)_S})$—a group scheme over $(kx)_S$—and its restriction of scalars, G_k, to S. Denote the kernel of the map $G_n \to G_m$ by G^n_m. The functor $\mathcal{F}$ is a torsor over the group scheme G^n_m, and, therefore, representable. □

Let $p_n : \mathrm{Bun}_r^{(nx)} X \to \mathrm{Bun}_r X$ be the projection.

Proposition 3.2. *For every n there is N such that $p_N^{-1}\mathcal{U}_n \subset \mathrm{Bun}_r^{(Nx)} X$ is a scheme.*

Proof. Take $n = 0$ for example (the proof in general is the same). We want to show that, if N is large enough, the functor

$$F(S) = \{(E, \phi) | E \in \mathcal{U}_0(S), \phi : E_{(Nx)_S} \xrightarrow{\cong} O^r_{(Nx)_S} \text{ as } O_{(Nx)_S}\text{-modules}\}$$

is representable.

Lemma 3.3. *If N is large enough, then, for all $E \in \mathcal{U}_0$, global sections are determined by their restriction to (Nx). More precisely, for every S and every $E \in \mathcal{U}_0(S)$, the restriction map $p_*(E) \to p_*(E_{(Nx)_S})$ is injective as a bundle map.*

Proof. W.l.o.g we can assume that S is local. If N is large enough, then, by Riemann Roch, $\dim H^0(E_s(-Nx)) = 0$ for every $s \in S$, so any global section of E_s that vanishes to order N at x must be zero. □

Suppose that N is as in the lemma. Given $(E, \phi) \in F(S)$, we map it to (E, η), where η is the composition

$$\eta : p_*(E) \xrightarrow{\text{res}} p_*(E_{(Nx)_S}) \xrightarrow{p_*\phi} p_*(O^n_{(Nx)_S}) = O_S^{Nr}.$$

The range of the map $(E, \phi) \to (E, \eta)$, which we denote by f, is the following functor:

$$G(S) = \{(E, \eta) | E \in \mathcal{U}_0(S), \eta : p_*(E) \hookrightarrow O_S^{Nr} \text{ such that } O_S^{Nr}/\eta(p_*(E)) \text{ is flat}\}.$$

The representability of F follows from the following lemmas:

Lemma 3.4. *G is representable.*

Proof. G is isomorphic to the product of $\mathcal{Y}_0$ from the last lecture, and the Grassmannian of d-planes in $\mathbb{C}^{Nr}$, twisted by the $\mathrm{Isom}(O_S^d, O_S^d)$-torsor $\mathrm{Isom}(O_S^d, p_*(E))$. □

Lemma 3.5. *f is a composition of an open embedding and a closed embedding. In particular, it is representable.*

Proof. It is enough to test the claim on local rings S. In this case, the claim is that, for any vector bundle $E \in \mathcal{U}_0(S)$ and any inclusion $\eta : p_*(E) \to O_S^{Nr}$, there is at most one isomorphism $\phi : p_*(E_{Nx}) \to O^n_{(Nx)_S}$ inducing η, and the set over which there is such ϕ is locally closed. The uniqueness is clear, since $p_*(E)$ generates $p_*(E_{(Nx)_S})$ as an $O_{(Nx)_S}$-module. To prove the locally closed condition, note that η is in the image if and only if it factors through the map $p_*(E) \to p_*(E_{Nx})$. Since the image of this map generates $p_*(E_{Nx})$ over O_{Nx}, the requirements are

1. $\eta(p_*(E))$ generates $O^r_{(Nx)}$ as a module over O_{Nx}.

2. The kernel of η coincide with the kernel of $p_*(E) \to p_*(E_{(Nx)}$.

The first condition is open by Nakayama's lemma. The second condition is clearly closed. □

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