SEMINAR NOTES: AFFINE GRASSMANNIAN AND THE LOOP GROUP (OCT. 13, 2009)

DENNIS GAITSGORY

Notes written by D. Gaitsgory and N. Rozenblyum

1. THE BEAUVILLE-LASZLO THEOREM

1.1. **Vector bundles on the formal disc.** Let A be a commutative k-algebra.

Definition 1.1.1.

(1) *An A-family of vector bundles on the formal disc $\mathcal{D}$ is a projective finitely generated module over the ring $A[\![t]\!]$.*

(2) *An A-family of vector bundles on the formal punctured disc $\overset{\circ}{\mathcal{D}}$ is a projective finitely generated module over the ring $A(\!(t)\!)$.*

A-families of vector bundles on $\mathcal{D}$ (resp., $\overset{\circ}{\mathcal{D}}$) form a category, where morphisms are just morphisms between modules. We denote these categories by $\mathrm{Vect}_A(\mathcal{D})$ and $\mathrm{Vect}_A(\overset{\circ}{\mathcal{D}})$, respectively. These categories are not abelian, but they have structures of exact categories.

Lemma 1.1.2. *The category $\mathrm{Vect}_A(\mathcal{D})$ is equivalent to the category of systems*

$$(M_n, \alpha_n, \ n \in \mathbb{N}),$$

where each M_n is a projective finitely generated module over $A[t]/t^n$, and α_n is an isomorphism $M_{n+1}/t^{n+1}M_{n+1} \simeq M_n$.

Proof. The only thing to check is that for a system (M_n, α_n) as above, the projective limit $\varprojlim_n M_n$ is finitely generated and projective over $A[\![t]\!]$.

For this, with no restriction of generality, we can assume that M_0 is free over A, in which case, by induction we obtain that each M_n is free over $A[t]/t^n$, and hence M is free over $A[\![t]\!]$. $\square$

1.2. **The gluing theorem.** Here's one of the formulations of the Beauville-Laszlo theorem. Let X be a smooth (but not necessarily comlete) curve over k. Let x be a point on X; we'll think of $k[\![t]\!]$ (resp., $k(\!(t)\!)$) as the completed local ring (resp., field of X at x). Let $\overset{\circ}{X}$ denote the punctured curve $X - x$.

For a commutative k-algebra A, consider the category of triples $(M_{\overset{\circ}{X}}, M_{\mathcal{D}}, \beta)$, where $M_{\overset{\circ}{X}} \in \mathrm{Vect}_A(\overset{\circ}{X})$ (i.e., a vector bundle on $\mathrm{Spec}(A) \times \overset{\circ}{X}$), $M_{\mathcal{D}} \in \mathrm{Vect}_A(\mathcal{D})$, and β is an isomorphism in $\mathrm{Vect}_A(\overset{\circ}{\mathcal{D}})$

$$M_{\overset{\circ}{X}}\big|_{\overset{\circ}{\mathcal{D}}} \simeq M_{\mathcal{D}}\big|_{\overset{\circ}{\mathcal{D}}}.$$

In the above formula, $M_{\overset{\circ}{X}}|_{\overset{\circ}{\mathcal{D}}}$ (resp., $M_{\mathcal{D}}|_{\overset{\circ}{\mathcal{D}}}$) is the pull-back of $M_{\overset{\circ}{X}}$ (resp., $M_{\mathcal{D}}$) with respect to the map of schemes $\mathrm{Spec}(A((t))) \to \mathrm{Spec}(A) \times \overset{\circ}{X}$ (resp., $\mathrm{Spec}(A((t))) \to \mathrm{Spec}(A[\![t]\!])$).

Morphisms in the above category are defined naturally, as morphisms between vector bundles, compatible with the data of β. We call this category "the category of gluing data".

1.2.1. Note that if M_X is an object of $\mathrm{Vect}_A(X)$ (i.e., a vector bundle on $\mathrm{Spec}(A) \times X$), we obtain an object in the category of gluing data by setting $M_{\overset{\circ}{X}} := M_X|_{\overset{\circ}{X}}$, $M_{\mathcal{D}} := M_X|_{\mathcal{D}}$, and β defined naturally. In the above formula $M_X|_{\mathcal{D}}$ is the pull-back of M_X with respect to the morphism $\mathrm{Spec}(A[\![t]\!]) \to \mathrm{Spec}(A) \times X$.

Here is one of the possible formulations of the Beauville-Laszlo theorem:

Theorem 1.2.2. *The restriction functor defines an equivalence from the category of A-families of vector bundles on X to the category of gluing data.*

This theorem is not so easy for a general commutative k-algebra A. However, if A is Noetherian, it follows immediately from faithfully flat descent:

Proof. It is sufficient to note that for A Noetherian, the map $\mathrm{Spec}(A[\![t]\!]) \to \mathrm{Spec}(A) \times X$ is flat, so the disjoint union of $\mathrm{Spec}(A[\![t]\!])$ and $\mathrm{Spec}(A) \times \overset{\circ}{X}$ forms a faithfully flat cover of $\mathrm{Spec}(A) \times X$, and

$$(\mathrm{Spec}(A[\![t]\!])) \underset{\mathrm{Spec}(A)\times X}{\times} \left(\mathrm{Spec}(A) \times \overset{\circ}{X} \right) \simeq \mathrm{Spec}(A[\![t]\!]).$$

$\square$

Remark. It will turn out that for our purposes the Noetherian version of the BL theorem would be sufficient.

1.3. **G-bundles.** Vector bundles on $\mathcal{D}$ and $\overset{\circ}{\mathcal{D}}$ (and when saying "vector bundles" we mean A-families of such) obviously form a tensor category.

Definition 1.3.1.
(1) *An A-family of G-bundles on $\mathcal{D}$ is an exact tensor functor from the category $\mathrm{Rep}(G)$ to the category $\mathrm{Vect}_A(\mathcal{D})$.*

(2) *An A-family of G-bundles on $\overset{\circ}{\mathcal{D}}$ is an exact tensor functor from the category $\mathrm{Rep}(G)$ to the category $\mathrm{Vect}_A(\overset{\circ}{\mathcal{D}})$.*

A-families of G-bundles on $\mathcal{D}$ (resp., $\overset{\circ}{\mathcal{D}}$) naturally form a groupoid.

By Lemma 1.1.2, A-families of G-bundles on $\mathcal{D}$ can be also described as compatible families of G-bundles on $\mathrm{Spec}(A[t]/t^n)$.

1.3.2. Theorem 1.2.2 immediately implies the corresponding statement for G-bundles:

Corollary 1.3.3. *The restriction functor defines an equivalence from the groupoid of A-families of G-bundles on X to the groupoid of triples $(P_{\overset{\circ}{X}}, P_{\mathcal{D}}, \beta)$, where $P_{\overset{\circ}{X}}$ is an A-family of G-bundles on $\overset{\circ}{X}$, $P_{\mathcal{D}}$ is an A-family of G-bundles on $\mathcal{D}$, and β is an isomorphism between A-families of G-bundles on $\overset{\circ}{\mathcal{D}}$:*

$$P_{\overset{\circ}{X}}|_{\overset{\circ}{\mathcal{D}}} \simeq P_{\mathcal{D}}|_{\overset{\circ}{\mathcal{D}}}.$$

2. The affine Grassmannian

2.1. We let the "local" affine Grassmannian to be the following functor on the category of commutative k-algebras: $\mathrm{Gr}_G^{loc}(A)$ is the set pairs $(P_{\mathcal{D}}, \gamma)$, where $P_{\mathcal{D}}$ is an A-family of G-bundles on $\mathcal{D}$, and γ is an isomorphism $P_{\mathcal{D}}|_{\overset{\circ}{\mathcal{D}}} \simeq P_{\mathcal{D}}^0|_{\overset{\circ}{\mathcal{D}}}$, where $P_{\mathcal{D}}^0$ denotes the trivial A-family of G-bundles on $\mathcal{D}$.

We let the "global" affine Grassmannian to be the following functor on the category of commutative k-algebras: $\mathrm{Gr}_G^{glob}(A)$ is the set pairs (P_X, γ), where P_X is an A-family of G-bundles on X, and γ is an isomorphism $P_X|_{\overset{\circ}{X}} \simeq P_X^0|_{\overset{\circ}{X}}$ where P_X^0 denotes again the trivial G-bundle (this time on $\mathrm{Spec}(A) \times X$).

There is a natural map given by restriction

$$\mathrm{Gr}_G^{glob} \to \mathrm{Gr}_G^{loc}(A).$$

Corollary 1.3.3 implies:

Corollary 2.1.1. *The above map $\mathrm{Gr}_G^{glob} \to \mathrm{Gr}_G^{loc}(A)$ is an isomorphism.*

Hence, in what follows, we'll often omit the superscripts "loc" or "glob", and write just Gr_G.

Remark. As stated, Corollary 2.1.1 relies on the general (i.e., non-Noetherian) version of Theorem 1.2.2. However, as we shall see shortly, one case easily avoid that and prove the above corollary using only the Noetherian version.

2.2. Let us recall the following definitions:

Definition 2.2.1.

(1) *A strict ind-scheme is a functor on the category of commutative k-algebras to sets, which can be written as filtered direct limit $\varinjlim_i X_i$, where X_i are functors representable by schemes, and the maps $X_i \to X_j$ for $i \leq j$ are closed embeddings.*

(2) *A strict ind-scheme is said to be (a) of ind-finite type, (b) ind-affine, (c) ind-projective, (d) etc. if the family X_i above can be chosen so that each of the schemes X_i is (a) of finite type, (b) affine, (c) projective, (d) etc.*

2.2.2. We are going to prove the following:

Proposition 2.2.3.

(1) *The functor Gr_G is an ind-scheme of ind-finite type.*

(2) *If G is reductive, then Gr_G is ind-projective.*

Let us show how this proposition allows to reduce the statement of Corollary 2.1.1 to the Noetherian case. First, we remark that in proving Prop 2.2.3(1) we'll do it for both the local and the global versions. Now, Prop 2.2.3 implies that in either case

$$\mathrm{Gr}_G(A) \simeq \varinjlim_{A'} \mathrm{Gr}_G(A'),$$

where A' runs through the set of finitely-generated subalgebras of A. (Functors of this sort are called "locally of finite type", see Section 4.3 below.) This reduces the assertion to the Noetherian case.

2.3. Reduction to GL_n. We'll establish Prop 2.2.3 explicitly for GL_n in the next section. Granting that for a moment, the case of a general G reduces to the case of GL_n using the next proposition:

Proposition 2.3.1. *Let $G_1 \to G_2$ be an injective homomorphism of affine algebraic groups.*

(1) *The functor $\mathrm{Gr}_{G_1} \to \mathrm{Gr}_{G_2}$ is a locally closed embedding.*

(2) *If the quotient G_2/G_1 is affine, then the above map is a closed embedding.*

We'll give a proof for Gr_G^{loc}; the case of Gr_G^{glob} is similar.

Proof. Fix an A-point of Gr_{G_2}. I.e., a G-bundle $P_{\mathcal{D}}$ over $\mathrm{Spec}(A[\![t]\!])$ and its trivialization γ over $\mathrm{Spec}(A(\!(t)\!))$. Then the fiber product

$$\mathrm{Gr}_{G_1} \underset{\mathrm{Gr}_{G_2}}{\times} \mathrm{Spec}(A)$$

represents the following functor on the category of A-algebras:

For $f : \mathrm{Spec}(A') \to \mathrm{Spec}(A)$, let $P'_{\mathcal{D}}$ denote the pull-back of $P_{\mathcal{D}}$ to $\mathrm{Spec}(A'[\![t]\!])$, and consider the unit section of the associated bundle $(G_2/G_1) \overset{G_2}{\times} P'_{\mathcal{D}}|_{\overset{\circ}{\mathcal{D}}}$, corresponding to

$$\beta : P_{\mathcal{D}}|_{\overset{\circ}{\mathcal{D}}} \simeq P^0_{\mathcal{D}}|_{\overset{\circ}{\mathcal{D}}} \quad \Rightarrow \quad (G_2/G_1) \overset{G_2}{\times} P'_{\mathcal{D}}|_{\overset{\circ}{\mathcal{D}}} \simeq (G_2/G_1) \times \mathrm{Spec}(A(\!(t)\!)).$$

Exercise 2.3.2. *Show that the map f lifts to a map to the fiber product if and only if the above section*

$$\mathrm{Spec}(A(\!(t)\!)) \to (G_2/G_1) \overset{G_2}{\times} P'_{\mathcal{D}}|_{\overset{\circ}{\mathcal{D}}}$$

extends to a map

$$\mathrm{Spec}(A[\![t]\!]) \to (G_2/G_1) \overset{G_2}{\times} P'_{\mathcal{D}}.$$

We claim that the latter corresponds to a locally closed subscheme of $\mathrm{Spec}(A)$, and if G_2/G_1 is affine, then this subscheme is actually closed.

Indeed, more generally, let Y be a scheme over $\mathrm{Spec}(A[\![t]\!])$, and let $\overset{\circ}{s}$ be a section

$$\mathrm{Spec}(A(\!(t)\!)) \to Y \underset{\mathrm{Spec}(A[\![t]\!])}{\times} \mathrm{Spec}(A(\!(t)\!)).$$

Consider the functor on the category of A algebras that associates to $f : \mathrm{Spec}(A') \to \mathrm{Spec}(A)$ the one-element set or the empty set according to whether or not the map $\overset{\circ}{s}$ extends to a map $s : \mathrm{Spec}(A[\![t]\!]) \to Y$. We claim that this functor corresponds to a locally closed subscheme of $\mathrm{Spec}(A)$, which is closed if Y is affine.

Exercise 2.3.3. *Reduce the assertion to the affine case.*

Exercise 2.3.4. *Reduce the assertion in the affine case to the case $Y = \mathrm{Spec}(A[\![t]\!]) \times \mathbb{A}^1$.*

In the case $Y = \mathrm{Spec}(A[\![t]\!]) \times \mathbb{A}^1$ we are dealing with the question of extension of a map $\mathrm{Spec}(A(\!(t)\!)) \to \mathbb{A}^1$ to a map $\mathrm{Spec}(A[\![t]\!]) \to \mathbb{A}^1$. The former functor is given by definition by $A' \mapsto A'(\!(t)\!)$, and the latter by $A' \mapsto A'[\![t]\!]$. So, we are indeed dealing with a closed condition: the vanishing of polar terms of the Laurent expansion.

$\square$

2.4. Warning. Above we've said that the question of extension of a given map $\mathrm{Spec}(A((t))) \to Y$ to a map $\mathrm{Spec}(A[\![t]\!]) \to Y$ corresponds to a locally closed sub-scheme of $\mathrm{Spec}(A)$, which is in fact a closed subscheme if Y is affine.

As we shall see later on (Section 4), when Y is affine, the functor $A \mapsto \mathrm{Hom}(\mathrm{Spec}(A((t))), Y)$ is a strict ind-scheme denoted $Y((t))$, and the functor $A \mapsto \mathrm{Hom}(\mathrm{Spec}(A[\![t]\!]), Y)$ is a scheme denoted $Y[\![t]\!]$ (by the above, $Y[\![t]\!] \to Y((t))$ is a closed embedding).

For any Y, the functor $Y[\![t]\!]$ defined as above is a scheme. However, $Y((t))$ isn't in general a strict ind-scheme.

2.5. The case of a unipotent group. Let's give an explicit description how Gr_G looks like for $G = \mathbb{G}_a$. Namely, consider the (infinite-dimensional) vector space $k((t))/k[\![t]\!]$ as an ind-scheme. We claim that

$$\mathrm{Gr}_{\mathbb{G}_a} \simeq k((t))/k[\![t]\!].$$

(For a general unipotent group a similar argument will show that Gr_G is ind-affine.)

Indeed, a $\mathbb{G}_a$-torsor over a scheme S is a short exact sequence of q.c. sheaves:

$$0 \to \mathcal{O}_S \to E \to \mathcal{O}_S \to 0.$$

Hence, $\mathrm{Hom}(\mathrm{Spec}(A), \mathrm{Gr}_{\mathbb{G}_a})$ is the set of short exact sequences of $A[\![t]\!]$-modules

$$0 \to A[\![t]\!] \to M \to A[\![t]\!] \to 0,$$

with a given splitting when tensored with $A((t))$ over $A[\![t]\!]$. However, the latter is the same as the set of maps of $A[\![t]\!]$-modules $A[\![t]\!] \to A((t))/A[\![t]\!]$, i.e., the set $A((t))/A[\![t]\!]$, i.e., $A \underset{k}{\otimes} k((t))/k[\![t]\!]$, as desired.

3. Affine Grassmannian for GL_n

3.1. Note that for $G = GL_n$, the functors Gr_G^{loc} and Gr_G^{glob} can be interpreted, respectively as follows:

$\mathrm{Gr}_G^{loc}(A)$ is the set of $A[\![t]\!]$-submodules $M_{\mathcal{D}}$ of $M_{\mathcal{D}}^0 \underset{A[\![t]\!]}{\otimes} A((t))$, such that for some integer m, we have

$$t^m \cdot M_D^0 \subset M_{\mathcal{D}} \subset t^{-m} \cdot M_{\mathcal{D}}^0,$$

and such that $M_{\mathcal{D}}$ is projective and finitely generated over $A[\![t]\!]$. Here, as usual, $M_{\mathcal{D}}^0 = A[\![t]\!]^{\oplus n}$.

$\mathrm{Gr}_G^{glob}(A)$ is the set of quasi-coherent subsheaves M_X of $j_* j^*(M_X^0)$ on $\mathrm{Spec}(A) \times X$, such that for some integer k

$$M_X^0(-m \cdot x) \subset M_X \subset M_X^0(m \cdot x),$$

and such that M_X is a vector bundle on $\mathrm{Spec}(A) \times X$. Here $M_X^0 = \mathcal{O}_{\mathrm{Spec}(A) \times X}^{\oplus n}$, and j denotes the open embedding $\mathrm{Spec}(A) \times \overset{\circ}{X} \hookrightarrow \mathrm{Spec}(A) \times X$.

In both local and global cases, for a fixed integer m, let us denote by $\mathrm{Gr}_n^{\leq m}$ the corresponding closed subfunctor of Gr_{GL_n}.

3.2. Consider now a third functor, denoted Z^m, that associates to a k-algebra A, the set of $A[t]/t^{2m}$-quotient modules N of

$$t^m \cdot M_{\mathcal{D}}^0 / t^{-m} \cdot M_{\mathcal{D}}^0 \simeq M_X^0(m \cdot x)/M_X^0(-m \cdot x),$$

which are projective and finitely generated as A-modules.

It is easy to see that the functor Z^m is representable by a closed subscheme of a usual finite-dimensional Grassmannian.

Indeed, let W denote the vector space $t^m \cdot k[\![t]\!]^{\oplus n}/t^{-m} \cdot k[\![t]\!]^{\oplus n}$; it's endowed with an action of the nilpotent operator t.

Exercise 3.2.1. *Show that Z^m identifies with the closed subset of $Gr(W)$ corresponding to t-stable subspaces.*

We claim that the functors $\mathrm{Gr}_n^{loc,\leq m}$, $\mathrm{Gr}_n^{glob,\leq m}$ are both isomorphic to Z^m. This will prove Prop 2.2.3 for $G = GL_n$, and hence for any G.

3.3. We construct the map $\mathrm{Gr}_n^{loc,\leq m} \to Z^m$ as follows. We send the data of

$$t^m \cdot M_D^0 \subset M_{\mathcal{D}} \subset t^{-m} \cdot M_{\mathcal{D}}^0$$

to

$$t^m \cdot M_{\mathcal{D}}^0 / t^{-m} \cdot M_{\mathcal{D}}^0 \twoheadrightarrow t^{-m} \cdot M_{\mathcal{D}}^0 / M_{\mathcal{D}} =: N.$$

We construct the map $\mathrm{Gr}_n^{glob,\leq m} \to Z^m$ as follows. We send the data of

$$M_X^0(-m \cdot x) \subset M_X \subset M_X^0(m \cdot x)$$

to

$$M_X^0(m \cdot x)/M_X^0(-m \cdot x) \twoheadrightarrow M_X^0(m \cdot x)/M_X =: N.$$

We claim that in both cases the resulting $A[t]/t^{2m}$-module N is projective and finitely generated as an A-module. We'll prove it in the local case as the global one is similar.

Indeed, we have an embedding $N \hookrightarrow t^{-2m} \cdot M_{\mathcal{D}}/M_{\mathcal{D}}$, and we claim that this embedding splits. This would establish our claim as $t^{-2m} \cdot M_{\mathcal{D}}/M_{\mathcal{D}}$ is A-free of finite rank. [1]

To prove the required splitting, it's enough to show that the embedding $t^{-m} \cdot M_{\mathcal{D}}^0 \hookrightarrow t^{-2m} \cdot M_{\mathcal{D}}$ splits, *as a map of A-modules.* For the latter, it sufficient to show that the the composed embedding

$$t^{-m} \cdot M_{\mathcal{D}}^0 \hookrightarrow t^{-2m} \cdot M_{\mathcal{D}} \hookrightarrow t^{-3m} \cdot M_{\mathcal{D}}^0$$

splits, again as a map of A-modules. However, the latter is evident.

3.4. Let us construct the inverse maps

$$Z^m \to \mathrm{Gr}_n^{loc,\leq m} \text{ and } Z^m \to \mathrm{Gr}_n^{glob,\leq m} .$$

For N as above we set $M_{\mathcal{D}} \subset t^{-m} \cdot M_{\mathcal{D}}^0$ to be the preimage of N under

$$t^{-m} \cdot M_{\mathcal{D}}^0 \twoheadrightarrow t^{-m} \cdot M_{\mathcal{D}}^0 / t^m \cdot M_{\mathcal{D}}^0,$$

and similarly for M_X. We have to show that $M_{\mathcal{D}}$ and M_X are projective and finitely generated.

Note also that Z^m is a scheme of finite type, our initial map $\mathrm{Spec}(A) \to Z^m$ factors though $\mathrm{Spec}(A') \to Z^m$, where A' is a finitely generated subalgebra of A. Let us denote the resulting

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[1]This idea belongs to Nir Avni

$A'[t]/t^{2m}$-module by N', so that $N \simeq N' \underset{A'}{\otimes} A$, and since N' is A'-flat, $\mathrm{Tor}_i^{A'[\![t]\!]}(A[\![t]\!], N') = 0$ for $i > o$. Let $M'_{\mathcal{D}}$ be the corresponding $A'[\![t]\!]$-module. By the above vanishing of Tor_1, we have:

$$M_{\mathcal{D}} \simeq M'_{\mathcal{D}} \underset{A'[\![t]\!]}{\otimes} A[\![t]\!].$$

So, it's enough to show that $M'_{\mathcal{D}}$ is projective and finitely generated over $A'[\![t]\!]$. A similar reasoning applies in the global case. Thus, we have reduced to the case when A is finitely generated. In particular, we can assume that A is Noetherian, and we can treat only the case of Gr_n^{glob}.

By construction, M_X is coherent (there is no difference between finitely generated and finitely presented, since we're Noetherian). Hence, it's enough to show that it's flat over $\mathrm{Spec}(A) \times X$. For this, it's enough to show that for any maximal ideal of A, the coherent sheaf $M_X \underset{A}{\otimes} k$ is flat over X. By the reasoning employed above (which uses flatness of N), this reduces the assertion to the case when $A = k$. However, in this case M_X is flat because it's torsion-free, being a subsheaf of $M_X^0(-m \cdot x)$.

4. ARC SPACES AND LOOP SPACES

4.1. Arcs. Let Y be a scheme. We let arcs of Y be the functor $Y[\![t]\!]$ given by

$$\mathrm{Hom}(\mathrm{Spec}(A), Y[\![t]\!]) := \mathrm{Hom}(\mathrm{Spec}(A[\![t]\!]), Y).$$

We will show that $Y[\![t]\!]$ is a scheme. First, we can express it as an inverse limit of truncated arcs. Let $Y(k[t]/t^n)$ be the functor given by

$$\mathrm{Hom}(\mathrm{Spec}(A), Y(k[t]/t^n)) := \mathrm{Hom}(\mathrm{Spec}(A[t]/t^n), Y).$$

We then have $Y[\![t]\!] = \varprojlim Y(k[t]/t^n)$ since $A[\![t]\!] = \varprojlim A[t]/t^n$.

4.2.

Proposition 4.2.1. *Each $Y(k[t]/t^n)$ is a scheme, and $Y[\![t]\!]$ is a scheme. Moreover, if Y is (locally) of finite type, then so is each of $Y(k[t]/t^n)$ (but not $Y[\![t]\!]$). If Y is affine, then so is each of $Y(k[t]/t^n)$ and $Y[\![t]\!]$.*

Proof. We will prove this in several steps.

Step 1. First, suppose $Y = \mathbb{A}^1$. In this case, we claim that $Y(k[t]/t^n) \simeq \mathbb{A}^{n+1}$. Indeed, as a set

$$\mathrm{Hom}(\mathrm{Spec}(R), \mathbb{A}^1(k[t]/t^n)) = \mathrm{Hom}(\mathrm{Spec}(R[t]/t^n), \mathbb{A}^1) = (R[t]/t^n) \cong R^{\oplus(n+1)} \simeq \mathbb{A}^{(n+1)}(R).$$

Step 2. By construction, the functors $Y \mapsto Y(k[t]/t^n)$ and $Y \mapsto Y[\![t]\!]$ commute with limits. Since every affine scheme (resp., affine scheme of finite type) is a limit (resp., finite limit) of a diagram involving copies of $\mathbb{A}^1$, the statements concerning $Y(k[t]/t^n)$ and $Y[\![t]\!]$ for Y affine follow.

Remark. Explicitly, an affine scheme of finite type can be written as $\mathrm{pt} \underset{\mathbb{A}^n}{\times} \mathbb{A}^m$ for some m, n and a map $\mathbb{A}^m \to \mathbb{A}^n$. This also shows that if $Y_1 \to Y_2$ is a closed embedding of affine schemes, then the corresponding maps $Y_1(k[t]/t^n) \to Y_2(k[t]/t^n)$ and $Y_1[\![t]\!] \to Y_2[\![t]\!]$ are closed embeddings.

Step 3. Note that if $U \subset Y$ is an open subscheme, the diagram

$$\begin{array}{ccc} U(k[t]/t^n) & \longrightarrow & Y(k[t]/t^n) \\ \downarrow & & \downarrow \\ U & \longrightarrow & Y \end{array}$$

is Cartesian. In particular, we obtain that for an open affine cover $\underset{i}{\cup} U_i = Y$, the subfunctors $U_i[\![t]\!]$ provide an open affine cover of $Y[\![t]\!]$, proving the proposition.

Note also that we have shown that the projections

$$Y[\![t]\!] \to Y(k[t]/t^n) \text{ and } Y(k[t]/t^{n+1}) \to Y(k[t]/t^n)$$

are affine.

$\square$

If Y is a smooth variety over k, then we can identify $Y(k[t]/t^2)$ with the tangent bundle TY of Y.

Exercise 4.2.2. *Suppose that Y is a smooth variety over k. Show that the maps $Y(k[t]/t^{n+1}) \to Y(k[t]/t^n)$ are smooth. In fact, $Y(k[t]/t^{n+1})$ is a torsor over $Y(k[t]/t^n)$ with respect to the vector group-scheme*

$$Y(k[t]/t^n) \underset{Y}{\times} TY.$$

4.2.3. Recall that $\mathrm{Bun}_G^{\infty x}(X)$ is the functor given by

$$\mathrm{Hom}(\mathrm{Spec}(A), \mathrm{Bun}_G(X)) = \left\{ (P_X, \alpha) : \begin{array}{l} P_X \text{ a } G\text{-bundle on } \mathrm{Spec}(A) \times X \text{ together with} \\ \text{an isomorphism } \alpha : P_X|_{\mathrm{Spec}(A[\![t]\!])} \sim P^0|_{\mathrm{Spec}(A[\![t]\!])} \end{array} \right\}$$

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By construction, the group-scheme $G[\![t]\!]$ acts on $\mathrm{Bun}_G^{\infty x}(X)$ by changing the data of α. Indeed, the group of automorphisms of $P^0|_{\mathrm{Spec}(A[\![t]\!])}$ is nothing but $\mathrm{Hom}(\mathrm{Spec}(A), G[\![t]\!])$. Moreover, we have:

$$\mathrm{Bun}_G(X) \simeq \mathrm{Bun}_G^{\infty x}(X)/G[\![t]\!],$$

as sheaves in the fpqc topology.

As we know (and will be shown in Nir's notes), $\mathrm{Bun}_G^{\infty x}(X)$ is in fact a scheme.

4.3. Functors locally of finite type. Let Y be a stack, or more generally any functor from rings to groupoids.

Definition 4.3.1. *Y is locally of finite type if for any ring A*

$$Y(A) = \underset{A' \subset A}{\varinjlim} Y(A')$$

where $A' \subset A$ are subrings of finite type.

Remark. In other words, functors which are locally of finite type are those which are left Kan extension of functors defined on the category of rings of finite type.

Exercise 4.3.2. *Let Y be an algebraic stack. Show that it is locally of finite type if and only if it can be smoothly covered by a scheme locally of finite type.*

Example 4.3.3. (1) Bun_G *is locally of finite type.*
(2) Gr_G *is locally of finite type.*
(3) $Y[\![t]\!]$ *is not locally of finite type. For instance, for $Y = \mathbb{A}^1$, $Y[\![t]\!] \cong \mathbb{A}^\infty$.*
(4) $\mathrm{Bun}_G^{\infty x}(X)$ *is not locally of finite type. Indeed, the pullback*

$$\mathrm{Bun}_G^{\infty x} \underset{\mathrm{Bun}_G}{\times} \mathrm{pt}$$

identifies with $G[\![t]\!]$, where $\mathrm{pt} \to \mathrm{Bun}_G$ is the point corresponding to the trivial G-bundle P_X^0.

4.4. Loops. Let Y be a scheme. Define loops of Y, $Y((t))$ to be the functor

$$\mathrm{Hom}(\mathrm{Spec}\, A, Y((t))) := \mathrm{Hom}(\mathrm{Spec}\, A((t)), Y).$$

In general, $Y((t))$ will not be a scheme. However, we have

Lemma 4.4.1. *If Y is an affine scheme, then $Y((t))$ is a strict ind-scheme, which is ind-affine.*

Proof. The functor $Y \to Y((t))$ commutes with limits, so it will suffice to prove this for $Y = \mathbb{A}^1$. In this case, we have

$$\mathrm{Hom}(\mathrm{Spec}(R), \mathbb{A}^1((t))) = \mathrm{Hom}(\mathrm{Spec}(R((t))), \mathbb{A}^1) = R((t)) \cong \varinjlim_n t^{-n} R[\![t]\!] \cong \varinjlim \mathbb{A}^1[\![t]\!](R),$$

where the maps $\mathbb{A}^1[\![t]\!] \to \mathbb{A}^1[\![t]\!]$ in the latter system are given by multiplication by powers of t, and these maps are clearly closed embeddings. □

We also have that the morphism $Y[\![t]\!] \to Y((t))$ is a closed embedding, which again follows from the $Y = \mathbb{A}^1$ case.

Remark. Note that $Y((t))$ is not an ind-scheme of ind-finite type.

4.4.2. By the above, for an affine scheme Y, $Y((t)) = \varinjlim Z_i$, where Z_i are schemes which are themselves of the form $Z_i = \varprojlim Z_i^j$, with Z_i^j affine schemes of finite type.

Ideally, we'd like to be able to find a collection of schemes Z_i^j for $i \in I$ and $j \in J$, where I and J are some filtered sets, such that there also exist closed embeddings $Z_i^j \to Z_{i'}^j$ for $i \leq i'$ and such that for $j' \geq j$ the following diagrams

$$
\begin{array}{ccc}
Z_i^{j'} & \longrightarrow & Z_{i'}^{j'} \\
\downarrow & & \downarrow \\
Z_i^j & \longrightarrow & Z_{i'}^{j'}
\end{array}
$$

commute and are Cartesian. Unfortunately, we don't know how to show that this is possible in general (perhaps, it's known to be impossible).

Another unpleasant phenomenon is that even when Y is smooth, we cannot show that one can find a collection of schemes Z_i as above, which can be written as $Z_i = \varprojlim Z_i^j$ with the maps $Z_i^{j'} \to Z_i^j$ smooth.

However, as we'll see, all of the above is possible when Y is an algebraic group.

4.5. Loops into non-affine schemes. Let Y be a non-affine scheme. Then $Y((t))$ need not be a strict ind-scheme.

Proposition 4.5.1. *It is not true that $\mathbb{P}^1((t))$ is not a strict ind-scheme and $\mathbb{P}^1[\![t]\!]$ is its closed subscheme.*

Proof. We'll argue in several steps.

Exercise 4.5.2. *Show that the valuative criterion of properness tells us that the map $\mathbb{P}^1[\![t]\!] \to \mathbb{P}^1((t))$ induces an isomorphism at the level of k'-points for any field $k' \supset k$.*

Exercise 4.5.3. *Let $Y' \hookrightarrow Y$ be a closed embedding of strict ind-schemes, which induces an isomorphism at the level of k'-points for any field $k' \supset k$. Show that $Y'(A) \to Y(A)$ is an isomorphism for any reduced k-algebra A.*

Exercise 4.5.4. *Five an example of a map* $\mathbb{A}^1 \to \mathbb{P}^1(\!(t)\!)$ *which does not come from a map* $\mathbb{A}^1 \to \mathbb{P}^1[\![t]\!]$.

$\square$

Remark. With slightly more work one can show that $\mathbb{P}^1(\!(t)\!)$ is not a strict ind-scheme (without mentioning $\mathbb{P}^1[\![t]\!]$.)

Here's one more exercise:

Exercise 4.5.5. *Deduce that for* $Y = \mathbb{A}^2 - 0$, *the functor* $Y(\!(t)\!)$ *isn't a strict ind-scheme, containing* $Y[\![t]\!]$ *as a closed subscheme.*

So, even for Y quasi-affine, we have a problem.

Exercise 4.5.6. *Show that for the open embedding* $(\mathbb{A}^1 - 0) \hookrightarrow \mathbb{A}^1$, *the map* $(\mathbb{A}^1 - 0)[\![t]\!] \to \mathbb{A}^1[\![t]\!]$ *is an open embedding, but* $(\mathbb{A}^1 - 0)(\!(t)\!) \to \mathbb{A}^1(\!(t)\!)$ *isn't.*

4.5.7. Since for Y non-affine, $Y(\!(t)\!)$ is unwieldy functor, it is not clear, e.g., how to define D-modules on it. For $Y = G/B$ this is a problem that has occupied people in geom. representation theory for years (and continues to do so).

5. Loop Groups

5.1. Let G be an affine algebraic group. We will construct a map

$$\pi : G(\!(t)\!) \to \mathrm{Gr}_G$$

which makes $G(\!(t)\!)$ a torsor over Gr_G with respect to $G[\![t]\!]$.

5.1.1. We have

$$\mathrm{Hom}(\mathrm{Spec}(A), G(\!(t)\!)) = \mathrm{Hom}(\mathrm{Spec}(A(\!(t)\!)), G) =$$

$$= \{\text{Automorphisms of the trivial } A\text{-family of } G\text{-bundles on } \overset{\circ}{\mathcal{D}}\}.$$

Now, for $g \in \mathrm{Hom}(\mathrm{Spec}(A), G(\!(t)\!)) = \mathrm{Hom}(\mathrm{Spec}(A(\!(t)\!)), G)$, we can construct $(P_{\mathcal{D}}, \gamma) \in \mathrm{Hom}(\mathrm{Spec}(A), \mathrm{Gr}_G)$. As a G bundle, $P_{\mathcal{D}} = P_{\mathcal{D}}^0$, the trivial bundle, and $\gamma : P_{\mathcal{D}}^0|_{\overset{\circ}{\mathcal{D}}} \to P_{\mathcal{D}}^0|_{\overset{\circ}{\mathcal{D}}}$ is given by g. This gives us the desired map $G(\!(t)\!) \to Gr_G$.

Note that the fibers of π are acted on simply-transitively by the group of automorphisms of the trivial G-bundle on $\mathrm{Spec}(A[\![t]\!])$. Thus, it's indeed a $G[\![t]\!]$-torsor.

5.1.2. Recall that $\mathrm{Gr}_G = \underset{i}{\varinjlim}\, Z_i$ where Z_i are schemes of finite type.

Let $\tilde{Z}_i = \pi^{-1}(Z_i) \subset G(\!(t)\!)$. We then have that $G(\!(t)\!) = \varinjlim \tilde{Z}_i$. Now, let

$$\tilde{Z}_i^j := \tilde{Z}_i \overset{G^j}{\times} G[\![t]\!]/G^j,$$

where $G^j = \ker\left(G[\![t]\!] \to G(k[t]/t^j)\right)$.

By construction, $\tilde{Z}_i^j$ is a torsor over Z_i with respect to the group $G(k[t]/t^j)$. In particular, $\tilde{Z}_i^j$ is a scheme of finite type. The maps $\tilde{Z}_i^{j+1} \to \tilde{Z}_i^j$ are smooth.

This gives us the desired description of $G(\!(t)\!)$ alluded to in Section 4.4.2.

5.1.3. Consider the functor $G((t))/G^j$ (fpqc quotient), which by the above is a strict ind-scheme of ind-finite type, isomorphic to $\varinjlim_i Z_i^j$. Moreover, this is a $G(k[t]/t^j)$-torsor over Gr_G.

Exercise 5.1.4. *Describe $G((t))/G^j$ as a solution to a moduli problem. In other words, say that $\mathrm{Hom}(\mathrm{Spec}(A), G((t))/G^j)$ is the set of A-families of G-bundles on $\mathcal{D}$ plus what?*

5.2. Regluing Map. Consider the map of functors

$$G((t)) \times \mathrm{Bun}_G^{\infty x}(X) \to \mathrm{Bun}_G^{\infty x}(X)$$

defined as follows. Given $g \in \mathrm{Hom}(\mathrm{Spec}(A), G((t))) = \mathrm{Hom}(\mathrm{Spec}(A((t))), G)$ and a G-bundle P_X on $\mathrm{Spec}(A) \times X$ with full level structure $\alpha : P_X|_{\mathrm{Spec}(A((t)))} \to P^0|_{\mathrm{Spec}(A((t)))}$, we construct a G-bundle P_X' on $\mathrm{Spec}(A) \times X$ with level structure as follows:

Let $P_{\overset{\circ}{X}}'$, which is supposed to be a G-bundle on $\mathrm{Spec}(A) \times (X - x)$ be $P_X|_{\mathrm{Spec}(A) \times (X-x)}$. Let $P_{\mathcal{D}}'$, which is supposed to be a G-bundle on $\mathrm{Spec}(A[\![t]\!])$, be $P^0|_{\mathrm{Spec}(A[\![t]\!])}$. We let the gluing data

$$\beta' : P_{\overset{\circ}{X}}'|_{\overset{\circ}{\mathcal{D}}} \simeq P_{\mathcal{D}}'|_{\overset{\circ}{\mathcal{D}}}$$

be the composition

$$P_{\overset{\circ}{X}}'|_{\mathrm{Spec}(A((t)))} \cong P_X|_{\mathrm{Spec}(A((t)))} \xrightarrow{\alpha} P^0|_{\mathrm{Spec}(A((t)))} \xrightarrow{g} P^0|_{\mathrm{Spec}(A((t)))}.$$

By the Beauville-Laszlo theorem (Theorem 1.3.3), this data gives us a G-bundle on $\mathrm{Spec}(A) \times X$ with level structure at x.

5.3. In order to construct the regluing map, we used the full Beauvill-Laszlo theorem. However, although seemingly we have functors not of locally finite involved, such as $G((t))$ and $\mathrm{Bun}_G^{\infty x}(X)$, we can get away with just using it in the Noetherian case, as we shall presently explain.

5.3.1. Consider the stack of G-bundles with finite level structure $\mathrm{Bun}_G^{nx}(X)$. We have that

$$\mathrm{Bun}_G^{\infty x}(X) = \varprojlim_n \mathrm{Bun}_G^{nx}(X),$$

so to define the map

$$G((t)) \times \mathrm{Bun}_G^{\infty x}(X) \to \mathrm{Bun}_G^{\infty x}(X),$$

we need to define a compatible system of maps

$$G((t)) \times \mathrm{Bun}_G^{\infty x}(X) \to \mathrm{Bun}_G^{nx}(X),$$

for each n. However, the resulting composite regluing map factors as

$$\begin{array}{ccc}
G((t)) \times \mathrm{Bun}_G^{\infty x}(X) & \longrightarrow & \mathrm{Bun}_G^{\infty x}(X) \\
\downarrow & & \downarrow \\
G^n \backslash G((t)) \times \mathrm{Bun}_G^{\infty x}(X) & \longrightarrow & \mathrm{Bun}_G^{nx}(X)
\end{array}$$

5.3.2. We have that $G^n \backslash G((t))$ is an ind-scheme of ind-finite type.

Note also also that the action of the group $G[\![t]\!] \subset G((t))$ on $\mathrm{Bun}_G^{\infty x}(X)$ is the canonical action from Section 4.2.3, which changes the level strcutre and leaves the G-bundle fixed. Therefore, we do not need to use the Beauville-Laszlo theorem to define the map

$$G[\![t]\!] \times \mathrm{Bun}_G^{\infty x}(X) \to \mathrm{Bun}_G^{\infty x}(X).$$

Thus to define the map

$$(G^n \backslash G((t))) \times \mathrm{Bun}_G^{\infty x}(X) \to \mathrm{Bun}_G^{nx}(X)$$

it suffices define the map

$$(G^n \backslash G((t))) \overset{G[\![t]\!]}{\times} \mathrm{Bun}_G^{\infty x}(X) \to \mathrm{Bun}_G^{nx}(X).$$

But now, $(G^n \backslash G((t))) \overset{G[\![t]\!]}{\times} \mathrm{Bun}_G^{\infty x}(X)$ is a functor locally of finite type, and it suffices to only use the Beauville-Laszlo theorem in the Noetherian case to construct this map.