INFINITE-DIMENSIONAL LINEAR ALGEBRA, ETC.

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Thanks to Dennis for explaining this stuff to me.

Orientation. Let's start by remembering where we are and where we're going. Let X be a smooth, proper, connected curve over $\mathbf{C}$ and G a connected reductive group over $\mathbf{C}$. In the past few talks we heard about the Hitchin map

$$T^* \operatorname{Bun}_G(X) \xrightarrow{p} \operatorname{Hitch}(X).$$

We noted a cool property of p: its induced map on functions $h^{cl} : A^{cl} \to \Gamma(T^* \operatorname{Bun}_G, \mathcal{O})$ is Poisson when A^{cl} has the trivial Poisson structure and $\Gamma(T^* \operatorname{Bun}_G, \mathcal{O})$ has the natural one. This raises the possibility of quantizing h^{cl}, that is, finding filtered algebras A and $\mathcal{D}$ with A commutative and a map $h : A \to \mathcal{D}$ whose associated graded recovers h^{cl}. This is what we'll be doing in the next little bit.

To get started, let's recall what we used to prove that cool property. First, we picked a point $x \in X$, and then we noted that the composition (the local Hitchin map)

$$p_x : T^* \operatorname{Bun}_G(X) \xrightarrow{p} \operatorname{Hitch}(X) \hookrightarrow \operatorname{Hitch}_x(X),$$

which uniquely determines p, admits a purely formal description in terms of the action of the group $G(\widehat{\mathcal{K}_x})$ on $\operatorname{Bun}_{G,x}(X)$. Using this description, we showed that actually, if $h_x^{cl} : A_x^{cl} \to \Gamma(T^* \operatorname{Bun}_G, \mathcal{O})$ denotes the induced map on functions of p_x, then this local h_x^{cl} is Poisson with trivial bracket on the source.

So it makes sense to start by quantizing h_x^{cl} – even more so because, just as in the classical case, the local h_x will uniquely determine the global h. And in fact, we learned in one of Sam's talks that this kind of quantization can be done purely formally any time we have a central extension $\widetilde{G(\widehat{\mathcal{K}_x})}$ of $G(\widehat{\mathcal{K}_x})$ by $\mathbb{G}_m$, a line bundle $\mathcal{L}$ on $\operatorname{Bun}_{G,x}(X)$, and an extended action of $\widetilde{G(\widehat{\mathcal{K}_x})}$ on $\mathcal{L}$ satisfying the nice "quantization conditions".

Unfortunately, we won't be able to get away with taking the trivial extension of $G(\widehat{\mathcal{K}_x})$, and $\mathcal{L}$ won't be trivial either; so actually, the goal for the rest of this talk is to introduce this central extension, this line bundle, and the action of the one on the other. Later we will return to the details of the local quantization, in particular to the verification of the quantization conditions.

Infinite-Dimensional Linear Algebra. We'll start with the construction of the central extension of $G(\widehat{\mathcal{K}_x}) \simeq G((t))$. This will come from the fact that if V is an infinite-dimensional vector space — in the right context — then $GL(V)$ automatically gets a canonical central extension by $\mathbb{G}_m$. Then, for instance, taking $V = \mathfrak{g}((t))$ and pulling back by the adjoint action $G((t)) \to GL(V)$ we will get the desired extension.

Date: wha.

To explain this picture, we need to introduce the right category of infinite-dimensional vector spaces for V to live in, and be able to construct $GL(V)$ at least as a group sheaf on $Aff_{\mathbf{C}}^{fppf}$.

The right category will be that of *locally linearly compact vector spaces*, also known as *Tate vector spaces*. Here is the definition:

Definition 1. *A* locally linearly compact vector space *over a field k is a vector space V over k together with a nonempty family $\mathcal{G}r$ of vector subspaces $L \subseteq V$, called* lattices*, satisfying the following conditions:*

1. *$\mathcal{G}r$ filters down to 0 and up to V;*
2. *Any L_1 and L_2 in $\mathcal{G}r$ are commensurable;*
3. *$\mathcal{G}r$ is closed under sandwiches;*
4. *V is complete: the natural map $V \to \varprojlim_{L \in \mathcal{G}r} V/L$ is an isomorphism.*

The third property isn't essential: we can always pass to the sandwich closure.

Example 1. If V is any vector space, we can take $\mathcal{G}r$ to be the family of its finite-dimensional subspaces. This kind of example is called *discrete*. Equivalently, V is discrete if and only if $0 \in \mathcal{G}r$.

Example 2. If $\{V_i\}_{i \in I}$ is any pro-system of finite-dimensional vector spaces, we can let V be its inverse limit and call a lattice anything containing the kernel of a projection $V \to V_i$. This kind of example is called *linearly compact*. Equivalently, V is linearly compact if and only if $V \in \mathcal{G}r$.

Example 3. There is only one possible structure of a locally linearly compact space on a finite dimensional V, and it is both discrete and linearly compact. Conversely, a discrete and linearly compact space is finite dimensional.

Example 4. We can take $V = k((t))$ and call a lattice any subspace sandwiched between $t^n k[[t]]$ and $t^{-n} k[[t]]$ for some $n \in \mathbf{N}$. This $k((t))$ is neither discrete nor linearly compact.

Now it's time to make llcvs into a category. But actually, we should do more: we need to be able to talk about, not just morphisms of llcvs, but families of morphisms of llcvs parametrized by an arbitrary base scheme; we will recover the ordinary category structure by taking k-valued points. Here is the definition:

Definition 2. *Let V and V' be llcvs over a field k. Define an fppf sheaf $Hom(V, V')$ on Aff_k by letting its value on $A \in Aff_k$ be the set*

$$\varprojlim_{L' \in \mathcal{G}r'} \varinjlim_{L \in \mathcal{G}r} Hom_A((V/L) \otimes A, (V'/L') \otimes A).$$

Equivalently, $Hom(V, V')$ is the set of continuous *A-module maps $f : V \widehat{\otimes} A \to V' \widehat{\otimes} A$, where*

$$V \widehat{\otimes} A := \varprojlim_{L \in \mathcal{G}r} ((V/L) \otimes A)$$

gets its inverse limit topology.

These Hom sheaves admit composition laws making llcvs into a category enriched in fppf sheaves on Aff_k.

We can also define an fppf sheaf $\mathcal{G}r$; essentially, we pick up new lattices over A by taking the sandwich closure of the old lattices $L \widehat{\otimes} A$. Here is the definition:

Definition 3. *Let V be a llcvs over k. Define an fppf sheaf $\mathcal{G}r$ on Aff_k by letting its value on $A \in Aff_k$ be defined by*

$$\mathcal{G}r(A) := \varinjlim_{L_0 \subseteq L_1} Summ((L_1/L_0) \otimes A),$$

where for an A-module M we let $Summ(M)$ denote the set of summands of M.

Equivalently $\mathcal{G}r(A)$ is the set of co-flat submodules N of $V \widehat{\otimes} A$ for which there exists a lattice L such that $L \widehat{\otimes} A \subseteq N$ with $N/(L\widehat{\otimes}A)$ finitely presented.

Henceforth $\mathcal{G}r$ will refer to the sheaf and $\mathcal{G}r(k)$ will be the original set of lattices in V. With this definition we have that if $L \subseteq L'$ are lattices in $\mathcal{G}r(A)$ then L'/L is finitely-presented flat (the in-families analog of finite-dimensional), and that any two $L_1, L_2 \in \mathcal{G}r(A)$ are commensurate in a similar sense.

The reason for introducing this Grassmannian sheaf $\mathcal{G}r$ instead of just working with the lattices $L\widehat{\otimes}A$ is that $\mathcal{G}r$ is functorial: it carries a natural action of the group sheaf $GL(V) := Isom(V, V)$. This fact will be crucial for the proof of the following fundamental proposition:

Proposition 1. *Let V be a llcvs. Then there is a canonical homomorphism of group stacks*

$$det : GL(V) \to B\mathbb{G}_m.$$

Let me explain the statement a little bit. $\mathbb{G}_m$ is an abelian group sheaf; this means that the multiplication map $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ is actually a group homomorphism, and so we can apply B to it to get $B\mathbb{G}_m \times B\mathbb{G}_m \simeq B(\mathbb{G}_m \times \mathbb{G}_m) \to B(\mathbb{G}_m)$. This then gives a "commutative group stack" structure on $B\mathbb{G}_m$, that is, it makes each $B\mathbb{G}_m(A)$ a symmetric monoidal groupoid and each pullback a symmetric monoidal functor. In familiar terms, this symmetric monoidal structure is just given by tensor product of line bundles, or, even more explicitly, by multiplying transition functions.

Then what we are requiring of *det* is that it be a monoidal functor on each A-valued point, compatible with pullbacks (where $GL(V)$ is a discrete monoidal stack). Note that this is more *data* than just an ordinary map of stacks $GL(V) \to B\mathbb{G}_m$, since there are isomorphisms to be specified proving that our functor is monoidal (namely, the $det(gg') \simeq det(g) \otimes det(g')$). In fact, while a map of stacks $G \to B\mathbb{G}_m$ just corresponds to a $\mathbb{G}_m$-bundle on G, a map of *group* stacks $G \to B\mathbb{G}_m$ corresponds to giving a group law on that $\mathbb{G}_m$-bundle, compatibly with the one on G and the action of $\mathbb{G}_m$. More precisely,

Exercise 1. *Let G be a group sheaf and E an abelian group sheaf. Show that taking the kernel gives rise to an equivalence between homomorphisms $G \to BE$ and central extensions $E \to G' \to G$ of G by E.*

So, in fact, Proposition 1 will give us a canonical central extension $GL(V)'$ of $GL(V)$ by $\mathbb{G}_m$, as we were shooting for.

Now let's actually prove Proposition 1. Here is the idea: just as the ordinary determinant map $GL(V) \to \mathbb{G}_m$ (for V finite-dimensional) is perhaps best seen as a consequence of the existence of a canonical $\mathbb{G}_m$-torsor attached to any V (namely the top exterior power), our $det : GL(V) \to B\mathbb{G}_m$ will be explained by the fact that any llcvs has a canonical $B\mathbb{G}_m$-torsor attached to it. Then the required *det* will come from functoriality: since $B\mathbb{G}_m$ is abelian, the automorphisms of any $B\mathbb{G}_m$-torsor are canonically identified with $B\mathbb{G}_m$ itself (acting via the structural action of the torsor).

The reason we get a $B\mathbb{G}_m$-torsor out of V is basically the following: we can "subtract" two lattices $L \subseteq L'$ in $\mathcal{G}r(A)$ and get a line bundle on A, namely $\Lambda^{top}(L'/L) =: d(L'/L)$, and this operation comes with a canonical identification $d(L''/L') \otimes d(L'/L) \simeq d(L''/L)$ when $L \subseteq L' \subseteq L''$.[1] Formally, though, we proceed as follows:

Definition 4. *Let V be a llcvs. Define an fppf sheaf (of groupoids) on Aff_k, the* determinant torsor $D(V)$*, by having its sections on A be maps of sheaves $d : \mathcal{G}r|_A \to B\mathbb{G}_m|_A$ together with, for all B over A and $L \subseteq L' \in \mathcal{G}r(B)$, isomorphisms $d(L) \otimes d(L'/L) \simeq d(L')$, satisfying a compatibility for $L \subseteq L' \subseteq L''$.*

In this definition it is in fact equivalent to consider just the global sections $d : \mathcal{G}r(A) \to B\mathbb{G}_m(A)$ and not the whole map of sheaves; this is because, for $A \to B$, the sandwich-closure of the image of $\mathcal{G}r(A)$ in $\mathcal{G}r(B)$ is all of $\mathcal{G}r(B)$, and so we can uniquely formally extend the definition of d to $\mathcal{G}r(B)$ using the compatibilities we already have. We will work with this alternate description of $D(V)(A)$ because it lets us carry less baggage around, but one should remember the original definition to see the presheaf structure.

Now, $D(V)$ was called the determinant torsor, so we should say why it's a torsor. Well, there is a pointwise action of $B\mathbb{G}_m$ on $D(V)$, and actually I claim that it makes $D(V)$ into a trivial torsor, a trivialization being determined by choosing a lattice. Indeed, given a lattice L, we have a map $D(V) \to B\mathbb{G}_m$ given by evaluation at L; it is clearly $B\mathbb{G}_m$-equivariant, and one can check that it is both fully faithful and essentially surjective (i.e. that such a d is uniquely determined by what it does on L) using the fact that any two lattices are commensurate.

So $D(V)$ is a $B\mathbb{G}_m$-torsor, and by functoriality we do get the desired $det : GL(V) \to Aut(D(V)) \simeq B\mathbb{G}_m$; this finishes the proof of Proposition 1.

Note that in the above proof we showed that choosing a lattice in V gives a trivialization of $D(V)$. There is also another way to get a trivialization, by choosing what's called a *co-lattice* in V. Basically, a co-lattice is just something that's nearly a complementary summand to all lattices:

Definition 5. *Let V be a llcvs and $A \in Aff_k$. A* co-lattice *in V over A is a flat submodule $\Gamma \subseteq V\widehat{\otimes}A$ such that for some (equivalently, all sufficiently small) $L \in \mathcal{G}r(A)$ we have that $\Gamma \cap L = 0$ and $V\widehat{\otimes}A/(\Gamma + L)$ is finitely-presented flat.*

Note that co-lattices form a presheaf on Aff_k, via $\Gamma \mapsto \Gamma \otimes_A B$.

Now, why do these guys also trivialize $D(V)$? Well, since $D(V)$ is a torsor, to trivialize is just to give a section; so the claim is that any co-lattice Γ gives rise to a $d_\Gamma : \mathcal{G}r(A) \to B\mathbb{G}_m(A)$ as in Definition 4. Well, for sufficiently small lattices L we will set

$$d_\Gamma(L) = d(V\widehat{\otimes}A/(\Gamma + L))^{-1}.$$

This has the right compatibilities for $L \subseteq L'$, and one can argue that it then extends uniquely to a d_Γ defined for all lattices[2]; instead, however, we will give an alternate

[1] these identifications moreover satisfying various natural compatibilities; we should really be working in $K_{\leq 1}(A)$...

[2] for symmetry's sake, I'll mention that for all sufficiently large lattices L we'll have $d_\Gamma(L) = d(\Gamma \cap L)$

formula which works for every lattice L, namely,

$$d_\Gamma(L) = d(\Gamma \to V\widehat{\otimes}A/L).$$

Here on the right we are taking d of a complex; what does that mean? Well, to make sense of it requires the complex to be *perfect*[3]. For a perfect complex $C^\cdot$, by $d(C^\cdot)$ one means, represent $C^\cdot$ as a bounded complex of finitely-presented flats, then take the alternating tensor power of the terms. This d of a complex is well-defined and is multiplicative in distinguished triangles, with suitable compatibilities (briefly, it factors through $K_{\leq 1}(Perf(A))$, where here by $Perf(A)$ we mean the right kind of enhancement of the triangulated category of perfect complexes...).

We should then explain why $\Gamma \to V\widehat{\otimes}A/L$ is perfect. Well, certainly if L is sufficiently small it is so, being quasi-isomorphic to just $V\widehat{\otimes}A/(\Gamma + L)$ sitting in degree 1; then I claim that $\Gamma \to V\widehat{\otimes}A/L$ is perfect for all lattices L if and only if it is so for one lattice L. Indeed, this follows from the distinguished triangle

$$(\Gamma \to V\widehat{\otimes}A/L) \to (\Gamma \to V\widehat{\otimes}A/L') \to (V\widehat{\otimes}A/L \to V\widehat{\otimes}A/L') \to$$

for $L \subseteq L'$ and the fact that any two lattices are commensurate. (Note that the last term in this triangle is quasi-isomorphic to just L'/L).

The last thing we need to explain is why we get the right compatibilities with this extended definition of d_Γ. But this also follows from the above distinguished triangle, and the multiplicativity of d in distinguished triangles.

Oh, and another important technical point is that this d_Γ commutes with pull-backs; this is what necessitated Γ being flat.

Thus we have explained why co-lattices also trivialize $D(V)$. Now, what's the upshot of all of this? Well, the fact that either a lattice or a co-lattice can trivialize $D(V)$ translates into the following:

Claim 1. *Let V be a llcvs and $A \in Aff_k$. If V_0 is either a lattice or a co-lattice in V over A, then the homomorphism $det : GL(V) \to B\mathbb{G}_m$ admits, over A, a canonical trivialization when restricted to $GL(V; V_0)$, the subgroup sheaf of automorphisms g satisfying $gV_0 = V_0$.*

Now let's apply this stuff to get our action.

The action. From now on, for simplicity of notation I won't write as if I'm working over arbitrary A, even though subtextually I am. Recall $x \in X$ our pointed curve, and let $n \in \mathbb{N}$. The important llcvs for us will be $V := \widehat{\mathcal{K}_x}^{\oplus n}$, where we call each $L_m := \{f \in V \mid \text{the pole orders of } f \text{ are no worse than } m\}$ a lattice, and then take the sandwich closure (c.f. Example 4). For instance, $L := L_0 = \widehat{\mathcal{O}_x}^{\oplus n}$ is a canonical lattice. But moreover, any $\mathcal{E} \in Bun_{n,x}(X)$ (viewed as a vector bundle of rank n with formal trivialization at x) gives rise to a co-lattice in V, namely $\Gamma := \Gamma(X \setminus x, \mathcal{E})$, mapping in via

$$\Gamma(X \setminus x, \mathcal{E}) \hookrightarrow \Gamma(\widehat{\mathcal{K}_x}, \mathcal{E}) \simeq V,$$

the last isomorphism coming from the formal trivialization at x.

[3] i.e. to satisfy any of the following equivalent conditions: 1. To be quasi-isomorphic to a bounded complex of finitely-presented flat modules; 2. To be in the smallest triangulated subcategory of $D(A)$ containing A and closed under shifts and retracts; 3. To be dualizable in the tensor structure on $D(A)$; 4. To be a compact object of $D(A)$.

Why is this a co-lattice? Well, we need to find a lattice L for which $\Gamma \cap L = 0$ and $V/(L+\Gamma)$ is finite-dimensional, but let me leave that aside for the moment, and calculate what $d_\Gamma(L)$ would be if Γ were a co-lattice. For this I ought to identify the complex $\Gamma \to V/L$; and indeed I claim that it is simply $R\Gamma(\mathcal{E})$, the total cohomology complex of $\mathcal{E}$. To check this, let $j : X \setminus x \to X$ be the open inclusion and $i : \widehat{x} \to X$ the inclusion of the formal neighborhood of x in X, and consider the short exact sequence of quasi-coherent sheaves on X

$$0 \to \mathcal{E} \to j_*j^*\mathcal{E} \to i_*(V/L) \to 0$$

(regular functions go to functions with pole at x go to "what pole was that, exactly?"), the map $j_*j^*\mathcal{E} \to i_*(V/L)$ being explained just like the inclusion $\Gamma \subseteq V$ above. Since j is both flat and affine and i is affine, we can also read this as a distinguished triangle (with all the operations being derived); then taking $R\Gamma$ and using the fact that $X \setminus x$ and $\widehat{x}$ are affine we find a distinguished triangle

$$R\Gamma(\mathcal{E}) \to \Gamma(X \setminus x; \mathcal{E}) \to V/L \to,$$

the last two complexes just sitting in degree zero; this proves the claim.

Now, what happens here if use an element $g \in GL_n(\widehat{\mathcal{K}_x})$ to change the gluing data of $\mathcal{E}$ in the puncured disk at x? All that changes in the above is that the co-lattice Γ gets replaced by $g^{-1}\Gamma$; therefore, we find quasi-isomorphisms of complexes

$$(\Gamma_{\mathcal{E}} \to V/gL) \simeq (g^{-1}\Gamma_{\mathcal{E}} \to V/L) \simeq (\Gamma_{g\mathcal{E}} \to V/L) \simeq R\Gamma(g\mathcal{E}).$$

Here we can draw two consequences: first of all, Γ is indeed a co-lattice (and hence all of these complexes are perfect): choosing g to be multiplication by a sufficiently high power m of a uniformizer at x and looking at cohomology in the above complexes, we find $\Gamma \cap gL \simeq H^0(X; g\mathcal{E}) = H^0(X; \mathcal{E}(-m)) = 0$ by Serre vanishing, and then similarly $V/(\Gamma + gL) \simeq H^1(X; \mathcal{E}(-m))$ will be finitely-presented flat by cohomology and base-change, as required. But secondly, taking determinants on both sides of this quasi-isomorphism we find $d_\Gamma(gL) \simeq dR\Gamma(g\mathcal{E})$; on the other hand, though, $d_\Gamma(gL) \simeq det(g) \otimes d_\Gamma(L) \simeq det(g) \otimes dR\Gamma(\mathcal{E})$ by definition of det, and hence we find (and here is the fundamental isomorphism)

$$dR\Gamma(g\mathcal{E}) \simeq det(g) \otimes dR\Gamma(\mathcal{E}),$$

with compatibilities when g and g' multiply and under pullback.

Now, what does this mean in terms of our central extension $GL_n(\widehat{\mathcal{K}_x})'$, which, recall, formally came from the $det : GL_n(\widehat{\mathcal{K}_x}) \to B\mathbb{G}_m$ which appears in the above formula? Well, it precisely means that $GL_n(\widehat{\mathcal{K}_x})'$ *acts* on the *determinant line bundle* $dR\Gamma$ on $Bun_{n,x}$ defined by

$$(\mathcal{E}, triv_x) \mapsto d(R\Gamma(\mathcal{E})).$$

Note that the trivialization does not enter into the definition here, so actually $dR\Gamma$ is pulled back from just Bun_n; we'll also give it the same name there.

Here we can remark the necessity of working with perfect complexes instead of just sheaves: the individual cohomology sheaves $H^0(\mathcal{E})$ and $H^1(\mathcal{E})$ are not necessarily finitely-presented flat, since their ranks can jump in families. It is only the object $R\Gamma(\mathcal{E})$ that behaves like a real family over $Spec(A)$ (formally, it is compatible with the formation of pullbacks), and which we can legitimately take d of.

So we have some kind of action of a central extension on a line bundle, namely $GL_n(\widehat{\mathcal{K}_x})'$ acts on $dR\Gamma$ over the space $Bun_{n,x}$. Now, though, let's bring our reductive group G into play. We have the adjoint map

$$G \to GL(\mathfrak{g}),$$

which induces maps $Bun_G \to Bun_{GL(\mathfrak{g})}$ and $Bun_{G,x} \to Bun_{GL(\mathfrak{g}),x}$, as well as $G(\widehat{\mathcal{K}_x}) \to GL(\mathfrak{g})(\widehat{\mathcal{K}_x})$, all of these being compatible in the evident sense; therefore we can pull back our action of $GL(\mathfrak{g})(\widehat{\mathcal{K}_x})$ on $Bun_{GL(\mathfrak{g}),x}$ along these maps, and we obtain the following:

Proposition 2. *Let $x \in X$ be our pointed curve and G our reductive group. Let $\mathcal{L}$ denote the line bundle on $Bun_{G,x}(X)$ defined by $(\mathcal{P}, triv_x) \mapsto d(R\Gamma(X; \mathcal{P} \times^G \mathfrak{g}))$, and let $\widetilde{G(\widehat{\mathcal{K}_x})} \to G(\widehat{\mathcal{K}_x})$ denote the pullback of the $\mathbb{G}_m$-extension $GL(\mathfrak{g} \otimes \widehat{\mathcal{K}_x})' \to GL(\mathfrak{g} \otimes \widehat{\mathcal{K}_x})$ (gotten from the determinant homomorphism of Proposition 1) along the adjoint action.*

Then there is a canonical extension of the action of $G(\widehat{\mathcal{K}_x})$ on $Bun_{G,x}$ to an action of $\widetilde{G(\widehat{\mathcal{K}_x})}$ on $\mathcal{L}$, for which the central $\mathbb{G}_m$ acts as it should.

Note that $\mathcal{L}$ is actually pulled back from a bundle ω on Bun_G; in fact, ω is nothing but the canonical bundle of Bun_G, that is, the determinant of the cotangent complex of Bun_G.

More explicitly. So we have our action of $\widetilde{G(\widehat{\mathcal{K}_x})}$ on the line bundle $\mathcal{L}$ on $Bun_{G,x}$. Now, we can call $\widetilde{\mathfrak{g} \otimes \widehat{\mathcal{K}_x}}$ the Lie algebra of $\widetilde{G(\widehat{\mathcal{K}_x})}$; it is a central extension of $\mathfrak{g} \otimes \widehat{\mathcal{K}_x}$ by the one-dimensional Lie algebra. What we're actually going to be concerned with is not so much the action of the group $\widetilde{G(\widehat{\mathcal{K}_x})}$ on $\mathcal{L}$, but the induced map from the universal enveloping algebra of $\widetilde{\mathfrak{g} \otimes \widehat{\mathcal{K}_x}}$ to $Diff(\mathcal{L}, \mathcal{L})$. Therefore it behooves us, at least morally, to get a more explicit description of this Lie algebra extension.

And in fact this can be done:

Proposition 3. *$\widetilde{\mathfrak{g} \otimes \widehat{\mathcal{K}_x}}$ is a Kac-Moody extension of $\mathfrak{g} \otimes \widehat{\mathcal{K}_x}$: it carries a canonical splitting (as vector spaces) for which the corresponding 2-cocycle is*

$$(X \otimes f, Y \otimes g) \mapsto \langle X, Y \rangle \cdot res(fdg),$$

where the pairing on the right is the Killing form.

To make this calculation we first retreat back to the generality of an arbitrary llcvs V and try to understand the central extension of Lie algebras $\mathfrak{gl}(V)' \to \mathfrak{gl}(V)$ induced by $GL(V)' \to GL(V)$. The first thing to say is that, as expected, $\mathfrak{gl}(V)$ identifies with $End(V)(k)$, and is a Lie algebra in llcvs; this is the context we'll be working in from now on.

Now, if we let $\mathfrak{gl}(V)^\ell$ (resp. $\mathfrak{gl}(V)^\gamma$) denote the subset of $\mathfrak{gl}(V)$ consisting of operators whose image is contained in a lattice (resp. a co-lattice), we see that our central extension $\mathfrak{gl}(V)' \to \mathfrak{gl}(V)$ is canonically trivialized on both $\mathfrak{gl}(V)^\ell$ and $\mathfrak{gl}(V)^\gamma$: indeed, if for instance $X \in \mathfrak{gl}(V)^\ell$ has image contained in L, then $X \in Lie(GL(V; L))$, and one sees that the image of X in $Lie(GL(V)')$ under the canonical trivializing section $GL(V; L) \to GL(V)'$ is independent of L.

To get mileage from these two trivializations we should assume that V admits some decomposition $V = L \oplus \Gamma$ into a lattice plus a co-lattice[4]; then we will furthermore have $\mathfrak{gl}(V) = \mathfrak{gl}(V)^{\ell} + \mathfrak{gl}(V)^{\gamma}$, and so the k-central extension $\mathfrak{gl}(V)' \to \mathfrak{gl}(V)$ is uniquely determined by the two above canonical trivializations together with the "gluing datum" $\mathfrak{gl}(V)^{\ell} \cap \mathfrak{gl}(V)^{\gamma} \to k$ which is the difference between them on the intersection.

What is this difference? Well, $\mathfrak{gl}(V)^{\ell} \cap \mathfrak{gl}(V)^{\gamma}$ is just the operators with finite-dimensional image, and I claim that the map is just the ordinary trace. Indeed, this is just the derivative of the easy-to-check fact that if $g \in GL(V; L) \cap GL(V; \Gamma)$, then the difference between the two canonical trivializations of $det(g)$ is $det(g|_{\Gamma \cap L}) \cdot det(g|_{V/(\Gamma + L)})^{-1}$ (here, ordinary determinant of an operator on a finite-dimensional vector space).

Now we should use this information to calculate a 2-cocycle. Let $V = \widehat{\mathcal{K}_x}^{\oplus n}$ as above, and choose a uniformizer t to get a decomposition $V = L \oplus \Gamma$, where $L = \widehat{\mathcal{O}_x}^{\oplus n}$ and Γ consists of Laurent tails. Then we can decompose $\mathfrak{gl}(V) = \mathfrak{gl}(V)^L \oplus \mathfrak{gl}(V)^{\Gamma}$ where $\mathfrak{gl}(V)^W$ denotes the operators with image contained in W, and the canonical splittings of $\mathfrak{gl}(V)' \to \mathfrak{gl}(V)$ on $\mathfrak{gl}(V)^L$ and $\mathfrak{gl}(V)^{\Gamma}$ combine to split $\mathfrak{gl}(V)' \to \mathfrak{gl}(V)$ itself as a map of vector spaces; furthermore, the above considerations imply that the corresponding 2-cocycle is given by

$$(X, Y) \mapsto tr([A^L, B^L] - [A, B]^L),$$

where by A^L we mean the projection of A to $\mathfrak{gl}(V)^L$, etc.

It is not difficult to calculate the above expression when we pull back to $\mathfrak{gl}_n \otimes \widehat{\mathcal{K}_x}$: we find that there the 2-cocycle is given by

$$(X \otimes f, Y \otimes g) \mapsto tr(XY) \cdot res(fdg),$$

where, recall, $res(fdg)$ stands for the coefficient of t^{-1} in $f(t)g'(t)$. Under the adjoint action, this pulls back to the statement of Proposition 3, moduli the claim of canonicity; but the canonicity follows from the fact that the adjoint action factors through $\mathfrak{sl}_n$, where the splitting giving the cocycle is automatically unique because $\mathfrak{sl}_n$ is perfect, and hence has no characters.

Fun stuff with llcvs and curves. Above we shot straight for the action of $\widetilde{G(\mathcal{K}_x)}$ on $\mathcal{L}$; but if we linger a little bit along the way, we come across some fun stuff. N.B. As far as I know what follows has no real bearing on the rest of the seminar.

Firstly, above we singled out a point $x \in X$; it'd be much better not to make such a choice. Actually we should also let ourselves have finitely many points in play, because we can't always trivialize a bundle over $X \setminus x$. The canonical llcvs to consider when we allow ourselves to select finitely many points of X for consideration is the llcvs of adeles:

Definition 6. *Let X be our curve. For a finite subset $S \subseteq X$, define the vector space of* partial adeles

$$\widehat{\mathcal{A}_S} := \prod_{x \in S} \widehat{\mathcal{K}_x} \times \prod_{x \notin S} \widehat{\mathcal{O}_x},$$

[4]This is no assumption under the axiom of choice; one simply splits off a lattice. In practice, though, we'll get such a splitting by choosing a uniformizer at x, so we don't need this abstract existence argument.

and the llcvs of adeles

$$\widehat{\mathcal{A}} := \varinjlim_{S} \widehat{\mathcal{A}_S},$$

where we make each $\prod_{x\in S} L_n \times \prod_{x\notin S} \widehat{\mathcal{O}_x}$ *(as* S *and* n *vary) a lattice. For an* $f \in \widehat{\mathcal{A}}$*, we denote by* $f_x \in \widehat{\mathcal{K}_x}$ *its* x^{th} *component.*

Now, as always we have our determinant map $det : GL(\widehat{\mathcal{A}}^{\oplus n}) \to B\mathbb{G}_m$, and we can pull it back to $GL_n(\widehat{\mathcal{A}}) \to B\mathbb{G}_m$. The first claim is then that for $f \in GL_n(\widehat{\mathcal{A}})$, we have a canonical identification

$$det(f) \simeq \otimes_{x\in X} det(f_x).$$

Now, this will be true and trivial as long as we can say what we mean in the right way. The infinite tensor product makes sense because $f_x \in GL_n(\widehat{\mathcal{O}_x})$ for almost all x, and so $det(f_x)$ is canonically trivialized. But there's still an issue, which is how to make sure that the ordering of the tensor product canonically doesn't matter.

And indeed this is a real issue: as we've set it up, we can only get an isomorphism which is canonical up to sign. The problem is basically that if you have $V = V_1 \oplus V_2$ (finite-dimensional here), then the identification $d(V) \simeq d(V_1) \otimes d(V_2)$ depends on which order we consider V_1 and V_2 in. To fix this, we need to remember at least the $\mathbb{Z}/2$-graded dimension of our line bundles and to use the $\mathbb{Z}/2$-graded sign rule in our symmetric monoidal structure. But actually we might as well remember the whole $\mathbb{Z}$-graded dimension of our line bundles, since the formalism will still work out with that extra information.

So here's the deal: instead of the symmetric monoidal stack $B\mathbb{G}_m$ of ordinary lines, from now on we'll consider the symmetric monoidal stack $B\mathbb{G}_m^{gr}$ of *graded lines*: its value over $A \in Aff_k$ is the groupoid of pairs $(\mathcal{L}, n)$ where $\mathcal{L}$ is a line bundle on $Spec(A)$ and n is a locally constant function on $Spec(A)$ with values in $\mathbb{Z}$, and the symmetric monoidal structure incorporates the sign rule. Given a finitely-presented flat module M (or a perfect compex...), we redefine $d(M)$ as an element of $B\mathbb{G}_m^{gr}$ by $d(M) = (\Lambda^{top} M, rk(M))$; we still have all the compatibilities, and so we get a canonical group homomorphism

$$det : GL(V) \to B\mathbb{G}_m^{gr}$$

for any llcvs V. Note that there are canonical maps $B\mathbb{G}_m^{gr} \to \mathbb{Z}$ and $B\mathbb{G}_m^{gr} \to \mathbb{G}_m$; the former is symmetric monoidal, but the latter is merely monoidal.

Then we do have the canonical isomorphism $det(f) \simeq \otimes_{x\in X} det(f_x)$ in $B\mathbb{G}_m^{gr}$. The next claim is that $k(X)^{\oplus n}$ is a co-lattice in $\widehat{\mathcal{A}}^{\oplus n}$; this is checked as before, with a little cohomology argument. This implies that $det : GL_n(\widehat{\mathcal{A}}) \to B\mathbb{G}_m^{gr}$ is canonically trivialized when restricted to $GL_n(k(X))$. Similarly det is canonically trivialized on $GL_n(\prod_{x\in X} \widehat{\mathcal{O}_x})$; hence it descends to give a graded line

$$Bun_n \simeq GL_n(k(X)) \backslash GL_n(\widehat{\mathcal{A}}) / GL_n(\prod_{x\in X} \widehat{\mathcal{O}_x}) \to B\mathbb{G}_m^{gr}.$$

What is this graded line? Well, a first guess would be that it's just the (graded) determinant of the cohomology. But this can't be, because $det(id)$ is trivial and so our graded line is trivial on $\mathcal{O}^{\oplus n}$. But it's the next best thing: one can easily check as before that it just sends

$$\mathcal{E} \mapsto d(R\Gamma(\mathcal{E})) \otimes d(R\Gamma(\mathcal{O}^{\oplus n}))^{-1}.$$

This formula tells us a way of computing the right-hand side purely locally: given a vector bundle $\mathcal{E}$ of rank n, we first trivialize it over $k(X)$ and formally trivialize it at every $x \in X$; then the difference between $\mathcal{E}$ and $\mathcal{O}^{\oplus n}$ is given by a "transition function" $g \in GL_n(\widehat{\mathcal{A}})$, and we just take the determinant of g, a process which can be done locally by the above product formula.

Even for $n = 1$, this tells us a lot: it has three very classical consequences: the Riemann-Roch formula, Weil's reciprocity law, and the sum of residues formula. Let's see why.

The Riemann-Roch formula comes about when we forget about the line bundle part of $B\mathbb{G}_m^{gr}$ and just remember the dimension: then we have $det(g_x) = ord_x(g_x)$, and so

$$det(g) = \sum_{x \in X} ord_x g_x;$$

then the fact that this is trivial on $GL_1(k(X))$ is just the fact that the zeros minus the poles of a rational function is trivial; and the above "line bundle" is just $\mathcal{L} \mapsto \chi(\mathcal{L}) - \chi(\mathcal{O})$, and the above calculation gives the Riemann-Roch formula

$$\chi(\mathcal{L}) - \chi(\mathcal{O}) = deg(\mathcal{L}),$$

since the degree of a line bundle is just the zeros minus poles of any rational trivialization.

To deduce Weil Reciprocity, we need to remember the graded lines. Here's how it goes: the determinant map $det : GL_1(\widehat{\mathcal{K}_x}) \to B\mathbb{G}_m^{gr}$ is, recall, a monoidal functor on k-valued points; however, both the source and target are in fact *symmetric* monoidal, so one can ask what the obstruction is to *det* being symmetric monoidal.

Here is the setup: let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal groupoids, and $F : \mathcal{C} \to \mathcal{D}$ a monoidal functor between them. Recall that a symmetric monoidal groupoid has exactly three invariants: there is π_0, the abelian group of isomorphism classes of objects; π_1, the abelian group of automorphisms of the unit object (or the automorphisms of the identity functor, or the automorphisms of any object; there are canonical identifications), and finally a "Postnikov invariant" $k : \pi_0 \otimes \mathbb{Z}/2 \to \pi_1$ connecting them, which sends the isomorphism class of X to the automorphism of $X \otimes X$ given by the braiding on our category. For instance, for $GL_1(\widehat{\mathcal{K}_x})(k)$ we have $\pi_0 = \widehat{\mathcal{K}_x}^*$ and $\pi_1 = 0$, whereas for $B\mathbb{G}_m^{gr}(k)$ we have $\pi_0 = \mathbb{Z}$ and $\pi_1 = k^*$ and $k(n) = (-1)^n$.

Now, what is the obstruction to $F : \mathcal{C} \to \mathcal{D}$ being symmetric monoidal? Well, to be symmetric monoidal means that for any $X, X' \in \mathcal{C}$, we have $F(b_{X,X'}) = b_{FX,FX'}$, where b stands for the braiding; so the obstruction is just a pairing

$$\langle \cdot, \cdot \rangle : \pi_0(\mathcal{C}) \times \pi_0(\mathcal{C}) \to \pi_1(\mathcal{D})$$

given by $(X, X') \mapsto F(b_{X,X'}) \circ (b_{FX,FX'})^{-1}$.

But what properties does $\langle \cdot, \cdot \rangle$ have? Well, certainly

$$\langle X, X \rangle = k(FX) - Fk(X);$$

but I furthermore claim that $\langle \cdot, \cdot \rangle$ is bilinear and anti-symmetric. To see these facts, it is convenient to make the following construction: given any $X_1, \ldots, X_n \in \mathcal{C}$ and any $\sigma \in S_n$, we can let $\langle X_1, \ldots, X_n; \sigma \rangle \in \pi_1(\mathcal{D})$ measure the difference between applying σ to $X_1 \otimes \ldots \otimes X_n$ and then applying F, or going the other way around. Then we can see that this extended form is multiplicative in σ and satisfies another multiplicative compatibility (relating different n's) if $\sigma = \sigma_I \coprod \sigma_J$

along some partition $\{1, \dots, n\} = I \coprod J$; then we have $\langle X, X' \rangle = \langle X, X'; flip \rangle$, and the bilinearity and anti-symmetry follow easily from the relations of the extended form.

Applied to *det*, this gives an antisymmetric bilinear form

$$\langle \cdot, \cdot \rangle : \widehat{\mathcal{K}_x}^* \otimes \widehat{\mathcal{K}_x}^* \to k^*;$$

it is called the *tame symbol*, and can be explicitly given by

$$\langle f, g \rangle = (-1)^{ord(f) ord(g)} \frac{f^{ord(g)}}{g^{ord(f)}}(x).$$

This is actually a simple calculation given the formal properties we've etablished: one can choose a uniformizer t and remember that we knew a priori that our pairing is trivial when both f and g stabilize $\widehat{\mathcal{O}_x}$ to essentially reduce to the case $f \in \widehat{\mathcal{O}_x}^*$, $g = t^{-1}$, which can be done by hand.

Then the fact that the adelic *det* is trivial when restricted to the function field gives us Weil's formula: for $f, g \in k(X)$, we have

$$\prod_{x \in X} \langle f_x, g_x \rangle_x = 1.$$

As for the residue theorem, one can make a similar deduction using $\mathfrak{gl}_1$ instead of GL_1.

References.

(1) Beilinson-Drinfeld, *Quantization of Hitchin's Fibration etc.*

(2) Beilinson-Drinfeld, *Chiral Algebras.*

(3) Drinfeld, *Infinite-Dimensional Vector Bundles in Algebraic Geometry.*

(4) Lefschetz, *Algebraic Topology.*

(5) Tate, *Residues of Differentials on Curves.*