Conformal blocks for a chiral algebra as quasi-coherent sheaf on Bun_G.

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1 Conformal blocks for a chiral algebra.

Recall that in Andrei's talk [4], we studied what it means to take conformal blocks for a $\mathcal{D}_X$-algebra. Namely the functor from k-alg $\to \mathcal{D}_X$-alg sending a k-algebra C to the constant $\mathcal{D}_X$-algebra $\mathcal{O}_X \otimes \mathcal{C}$, has a left adjoint functor:

$$H_\nabla(X,\cdot) : \mathcal{D}_X - \mathrm{alg} \to k - \mathrm{alg} \quad \mathrm{Hom}(H_\nabla(X,\mathcal{B}),C) \simeq \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{B},\mathcal{O}_X \otimes_k C)$$

and this is the functor of *conformal blocks.* Also recall that we had a more concrete description of $H_\nabla(X,\mathcal{B})$ in terms of $H^0_{dR}(X-x,\mathcal{B})$. In fact the short exact sequence

$$0 \to \mathcal{B} \to j_*j^*(\mathcal{B}) \to i_*i^*(\mathcal{B}) \to 0$$

induces a long exact sequence

$$0 \to H^0_{dR}(X,\mathcal{B}) \to H^0_{dR}(X-x,\mathcal{B}) \to \mathcal{B}_x \to H^1_{dR}(X,\mathcal{B}) \to 0$$

and in Andrei's talk we have seen that $H_\nabla(X,\mathcal{B}) \simeq \mathcal{B}_x/\mathcal{B}_x \cdot (\mathrm{Im}(H^0_{dR}(X-x,\mathcal{B}) \to \mathcal{B}_x))$. We want now to introduce the concept of conformal blocks in the setting of Chiral algebras, and show that in the case of a commutative chiral algebra, you obtain what we already know.

Definition 1.1. For a unital chiral algebra $\mathcal{A}$, the vector space

$$H_\nabla(X,\mathcal{A}) := H^2_{dR}(X \times X, \mathcal{A}^{(2)})$$

is called the space of *conformal blocks.* Where $\mathcal{A}^{(2)}$ denotes the kernel of the map

$$j_*j*(\mathcal{A} \boxtimes \mathcal{A}) \to \Delta_*(\mathcal{A})$$

We now want to give a description of $H_\nabla(X,\mathcal{A})$ similar to the one we had for commutative $\mathcal{D}_X$-algebras. For this purpose we will need the following lemma.

Lemma 1.1. *The space of conformal blocks $H_\nabla(X,\mathcal{A})$ is isomorphic to the following:*

$$H_\nabla(X,\mathcal{A}) = H^2_{dR}(X\times X,\mathcal{A}^{(2)}) \simeq Coker(H^1_{dR}(X\times X-\Delta(X), j^*(\mathcal{A}\boxtimes\mathcal{A})) \to H^1_{dR}(X,\mathcal{A})).$$

Proof. Consider the short exact sequence $0 \to \mathcal{A}^{(2)} \to j_*j^*(\mathcal{A}\boxtimes\mathcal{A}) \to \Delta_*(\mathcal{A}) \to 0$. This gives rise to the following long exact sequence:

$$\cdots \to H^1_{dR}(X\times X-\Delta(X), j^*(\mathcal{A}\boxtimes\mathcal{A})) \to H^1_{dR}(X,\mathcal{A}) \to$$

$$\to H^2_{dR}(X\times X,\mathcal{A}^{(2)}) \to H^2_{dR}(X\times X-\Delta(X), j^*(\mathcal{A}\boxtimes\mathcal{A})) \to 0.$$

We claim that $H^2_{dR}(X\times X-\Delta(X), j^*(\mathcal{A}\boxtimes\mathcal{A}))$ is zero. In fact, in general, for any $\mathcal{D}_X$ module $\mathcal{F}$ we have $H^2_{dR}(X\times X-\Delta(X),\mathcal{F})$ by considering the projection $p : X\times X-\Delta(X) \to X$. Hence we obtain

$$H_\nabla(X,\mathcal{A}) = H^2_{dR}(X\times X,\mathcal{A}^{(2)}) \simeq \mathrm{Coker}(H^1_{dR}(X\times X-\Delta(X), j^*(\mathcal{A}\boxtimes\mathcal{A})) \to H^1_{dR}(X,\mathcal{A})).$$

□

Let now x be a point of the curve, and recall that $\mathcal{A}_x$ is naturally a $H^0_{dR}(X-x,\mathcal{A})$-module. We are ready to show the following proposition.

Proposition 1.2. *The space $H_\nabla(X,\mathcal{A})$ is isomorphic to the space of coinvariants*

$$Coker(H^0_{dR}(X-x,\mathcal{A})\otimes A_x \to A_x)$$

Proof. Consider the following maps: $x \overset{i_x}{\hookrightarrow} X \overset{j_x}{\hookleftarrow} X-x$. These give rise to the long exact sequence

$$0 \to H^0_{dR}(X,\mathcal{A}) \to H^0_{dR}(X-x,\mathcal{A}) \to \mathcal{A}_x \to H^1_{dR}(X,\mathcal{A}) \to 0 \tag{1}$$

We can also consider the maps

$$(X-x)\times x \overset{id\times i_x}{\hookrightarrow} X\times X-\Delta(X) \overset{k}{\hookleftarrow} (X\times(X-x))-\Delta(X)$$

which would give as the sequence

$$\cdots \to H^0_{dR}(X-x,\mathcal{A})\otimes\mathcal{A}_x \to H^1_{dR}(X\times X-\Delta(X), j^*(\mathcal{A}\boxtimes\mathcal{A})) \to 0$$

where the last 0 is because $(X \times (X - x)) - \Delta(X)$ is affine. Hence we obtain a commutative diagram

$$\begin{array}{ccccc}
H^0_{dR}(X-x,\mathcal{A}) \otimes \mathcal{A}_x & \longrightarrow & \mathcal{A}_x & & \\
\downarrow & & \downarrow & & \\
H^1_{dR}(X \times X - \Delta(X), j^*(\mathcal{A} \boxtimes \mathcal{A})) & \longrightarrow & H^1_{dR}(X,\mathcal{A}) & \longrightarrow & H_\nabla(X,\mathcal{A}) \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

Surjectivity is clear since the left verical arrow is surjective. To prove the injectivity it is enough to show that if an element $a \in \mathcal{A}_x$ is mapped to zero, then it comes from an element in $H^0_{dR}(X-x,\mathcal{A}) \otimes \mathcal{A}_x$. However by the construction of the left vertical arrow given in (1) a must be the image of an element $a^{'}$ under the map $H^0_{dR}(X-x,\mathcal{A}) \to \mathcal{A}_x$. Now recall that our chiral algebra was unital, hence there is a canonical element $unit_x \in \mathcal{A}_x$. Now if we consider the element $a^{'} \otimes unit_x$ we see that it maps to a as desired.

□

There is actually a more general definition of conformal blocks that involves chiral modules supported at some point $x \in X$. If $\mathcal{M}$ is such a module and $M = H^0(i^!(M))$, then we define the space of conformal blocks with coefficients in the module M to be:

$$H_\nabla(X,\mathcal{A},M) = M/\mathrm{Im}(H^0_{dR}(X-x,\mathcal{A}) \otimes M \to M)$$

as you can see, this is exactly $H_\nabla(X,\mathcal{A})$ when we take $M = \mathcal{A}_x$. There is another way to compute conformal blocks with coefficients in a module M in the case our chiral algebra happens to be of the form $\mathcal{A} = \mathfrak{U}(L)$, for a *Lie**-algebra L. This construction will be very useful in the future.

Proposition 1.3. *Let $\mathfrak{U}(L)$ be the enveloping algebra of a Lie*-algebra L and $\mathcal{M}$ a $\mathfrak{U}(L)$-module supported at x.Then the map*

$$M_x/(H^0_{dR}(X-x,L) \otimes M) \to M/(H^0_{dR}(X-x,\mathfrak{U}(L)) \otimes M) = H_\nabla(X,\mathfrak{U}(L),M)$$

is an isomorphism.

Proof. For the proof we first need the following lemma, whose proof can be found in [5].

Lemma 1.4. *$\mathfrak{U}(L)$ has a filtration $\mathfrak{U}(L) \simeq \bigcup_n \mathfrak{U}(L)_n$ such that if we consider $j_*j^*(L \boxtimes \mathfrak{U}(L)) \to \Delta_*(\mathfrak{U}(L))$ then*

a) $Im(L \boxtimes \mathfrak{U}(L)_n) = \Delta_*(\mathfrak{U}(L)_n)$

b) $Im(j_*j^*(L \boxtimes \mathfrak{U}(L)_n)) = \Delta_*(\mathfrak{U}(L)_{n+1})$

We will prove that the surjection

$$M_x/(H^0_{dR}(X-x, L) \otimes M) \to M/(H^0_{dR}(X-x, \mathfrak{U}(L)_n) \otimes M)$$

is an isomorphism for every $n \geq 1$. Suppose that it is an isomorphism when we consider $\mathfrak{U}(L)_n$. If we take a section $a \in \Gamma(X-x, \mathfrak{U}(L)_{n+1})$ (recall that $\Gamma(X/x, h(\mathcal{A})) = H^0_{dR}(X-x, \mathcal{A})$ for every chiral algebra $\mathcal{A}$, where $h(M) = M/M\Theta_X$) and an element $m \in M$, it is enough to show that $h(a).m$ is not just in $\mathrm{Im}(H^0_{dR}(X-x, \mathfrak{U}(L)_{n+1}) \otimes M)$ but that it belongs to $\mathrm{Im}(H^0_{dR}(X-x, \mathfrak{U}(L)_n) \otimes M)$. By point $b)$ of the previous lemma, we can find a section $b \boxtimes a' \cdot f(x,y)$ of $\Gamma(X-x \times X-x, j_*j^*(L \boxtimes \mathfrak{U}(L)_n))$ such that $(h \boxtimes id)(\left\{ b \boxtimes a' \cdot f(x,y) \right\}) = a$. Now we can use the Jacobi identity and we have

$$(h \boxtimes h \boxtimes h)(a'.b.(m \cdot f(x,y)) - b.a'.m \cdot f(y,x)) = a.m.$$

But now these terms belong to $\mathrm{Im}(H^0_{dR}(X-x, \mathfrak{U}(L)_n) \otimes M)$ as desired.

□

The case of a commutative chiral algebra $\mathfrak{Z}$.

As we have seen so far, at the level of vector spaces, the expression in 1.2 is indeed equal to the one Andrei defined in his talk. However for a chiral algebra $\mathcal{A}$ the space of conformal blocks will not in general have any structure of an algebra. Nonetheless we have seen that a commutative chiral algebra $\mathfrak{Z}$ is the same as a $\mathcal{D}_X$-scheme, and we will see shortly that in this particular case $H_\nabla(X, \mathfrak{Z})$ inherits a structure of commutative algebra. Before that, let's recall what it means for a chiral algebra to be commutative.

Definition 1.2. A commutative chiral algebra $\mathfrak{Z}$ is a chiral algebra such that the map

$$\mathfrak{Z} \boxtimes \mathfrak{Z} \to j_*j^*(\mathfrak{Z} \boxtimes \mathfrak{Z}) \to \Delta_*(\mathfrak{Z})$$

vanishes. From the natural short exact sequence below, we have that this is equivalent to the fact that the bracket factors as follow:

$$\begin{array}{ccccc} \mathfrak{z} \boxtimes \mathfrak{z} & \longrightarrow & j_*j^*(\mathfrak{z} \boxtimes \mathfrak{z}) & \longrightarrow & \Delta_*(\mathfrak{z} \overset{!}{\otimes} \mathfrak{z}) \\ & \searrow & \downarrow & \swarrow & \\ & & \Delta_*(\mathfrak{z}) & & \end{array}$$

hence we have a map $\mathfrak{z} \overset{!}{\otimes} \mathfrak{z} \to \mathfrak{z}$.

Proposition 1.5. *Fora chiral commutative algebra $\mathfrak{z}$ the space $H_\nabla(X, \mathfrak{z})$ has a stucture of an algebra.*

Proof. By 1.2 we have $H_\nabla(X, \mathfrak{z}) = \mathfrak{z}_x / \mathrm{Im}(H^0_{dR}(X - x, \mathfrak{z}) \otimes \mathfrak{z}_x \to \mathfrak{z}_x)$. However the map $H^0_{dR}(X - x, \mathfrak{z}) \otimes \mathfrak{z}_x \to \mathfrak{z}_x$ was obtained from the map $j_*j^*(\mathfrak{z} \boxtimes \mathfrak{z}) \to \Delta_*(\mathfrak{z})$ by taking De Rham cohomology, and because of the commutative diagram above, we have that the map factors through $\mathfrak{z}_x \otimes \mathfrak{z}_x$

$$\begin{array}{ccc} H^0_{dR}(X - x, \mathfrak{z}) \otimes \mathfrak{z}_x & \longrightarrow & \mathfrak{z}_x \\ \downarrow{\scriptstyle can \otimes id} & \nearrow & \\ \mathfrak{z}_x \overset{!}{\otimes} \mathfrak{z}_x & & \end{array}$$

where *can* is the canonical map we have seen in 1. The diagram shows that $\mathrm{Im}(H^0_{dR}(X - x, \mathfrak{z}) \otimes \mathfrak{z}_x \to \mathfrak{z}_x) = \mathrm{Im}(H^0_{dR}(X - x, \mathfrak{z}) \to \mathfrak{z}_x) \cdot \mathfrak{z}_x$, or in other words that the image of that map is the ideal in $\mathfrak{z}_z$ generated by $\mathrm{Im}(H^0_{dR}(X - x, \mathfrak{z}) \to \mathfrak{z}_x$. From this it follows that $H_\nabla(X, \mathfrak{z})$ is an algebra. □

The center of a chiral algebra.

Definition 1.3. The center of chiral algebra $\mathcal{A}$ is the maximal $\mathcal{D}_X$-submodule $\mathfrak{z} \subset \mathcal{A}$ such that the map

$$\mathfrak{z} \boxtimes \mathcal{A} \to j_*j^*(\mathfrak{z} \boxtimes \mathcal{A}) \to \Delta_*(\mathcal{A})$$

vanishes. Equivalently, it is the maximal subalgebra for which we have the following:

$$\begin{array}{ccccc} \mathfrak{z} \boxtimes \mathcal{A} & \longrightarrow & j_*j^*(\mathfrak{z} \boxtimes \mathcal{A}) & \longrightarrow & \Delta_*(\mathfrak{z} \overset{!}{\otimes} \mathcal{A}) \\ & \searrow & \downarrow & \swarrow & \\ & & \Delta_*(\mathcal{A}) & & \end{array}$$

which implies the existence of a map $\mathfrak{Z} \overset{!}{\otimes} \mathcal{A} \to \mathcal{A}$. In other words $\mathfrak{Z}$ is the $\mathcal{D}_X$-submodule spanned by all the section z, such that $\{z \boxtimes a\} = 0$ for all sections $a \in \mathcal{A}$.

Remark 1. We see immediately that the Jacobi identity implies that $\mathfrak{Z}$ is actually a chiral subalgebra of $\mathcal{A}$.

Note that in particular $\mathfrak{Z}$ is commutative, hence (from 1.5) $H_\nabla(X, \mathfrak{Z})$ is an algebra. This algebra acts on the space $H_\nabla(X, \mathcal{A})$ according to the following proposition.

Proposition 1.6. *The algebra of conformal blocks $H_\nabla(X, \mathfrak{Z})$ of the center of chiral algebra $\mathcal{A}$, acts on the space $H_\nabla(X, \mathcal{A})$.*

Proof. As we have seen before, the diagram above implies the following:

$$\begin{array}{ccc} H^0_{dR}(X - x, \mathfrak{Z}) \otimes \mathcal{A}_x & \longrightarrow & \mathcal{A}_x \\ \downarrow{\scriptstyle can \otimes id} & \nearrow & \\ \mathfrak{Z}_x \overset{!}{\otimes} \mathcal{A}_x & & \end{array}$$

and therefore from the map $\mathfrak{Z}_x \otimes \mathcal{A}_x \to \mathcal{A}_x$ we obtain an action of $H_\nabla(X, \mathfrak{Z})$ on
$\mathcal{A}_x / \mathrm{Im}(H^0_{dR}(X - x, \mathfrak{Z}) \to \mathfrak{Z}_x) \cdot \mathcal{A}_x$. This action indeed descends to an action on
$\mathcal{A}_x / \mathrm{Im}(H^0_{dR}(X - x, \mathcal{A}) \otimes \mathcal{A}_x \to \mathcal{A}_x)$. In fact this follows from the following commutative diagram:

$$\begin{array}{ccc} H^0_{dR}(X - x, \mathcal{A}) \otimes \mathcal{A}_x & \longrightarrow & \mathcal{A}_x \\ \uparrow & & \uparrow \\ H^0_{dR}(X - x, \mathcal{A}) \otimes \mathcal{A}_x \otimes \mathfrak{Z}_x & \longrightarrow & \mathcal{A}_x \otimes \mathfrak{Z}_z \end{array}$$

which tells us that the map $H^0_{dR}(X - x, \mathcal{A}) \otimes \mathcal{A}_x \to \mathcal{A}_x$ is a map of $\mathfrak{Z}_x$ modules. The commutativity of the above diagram follows from the Jacobi identity applied to $j_* j^*(\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathfrak{Z}) \to \Delta_*^{x_1 = x_2 = x_3}(\mathcal{A})$ where j is the inclusion of the complement of the three diagonals in $X \times X \times X$. In fact we have $\mu_{1,(23)} = \mu_{(12)3} + \mu_{2(13)}$ where

$$\mu_{1(23)} : j_* j^*(\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathfrak{Z}) \to \Delta_*^{x_2 = x_3}(j_*^{x_1 \neq x_2} j^{*, x_1 \neq x_2}(\mathcal{A} \boxtimes \mathcal{A}))$$

$$\mu_{(12)3)} : j_* j^*(\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathfrak{Z}) \to \Delta_*^{x_1 = x_2}(j_*^{x_3 \neq x_1} j^{*, x_3 \neq x_1}(\mathcal{A} \boxtimes \mathfrak{Z}))$$

$$\mu_{2(13)} : j_*j^*(\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathfrak{Z}) \to \Delta_*^{x_1=x_3}(j_*^{x_2\neq x_1} j^{*,x_2\neq x_1}(\mathcal{A} \boxtimes \mathcal{A})).$$

Now we can consider the open set $U = \{x_2 \neq x_3, x_2 \neq x_1\} \xhookrightarrow{\tilde{j}} X \times X \times X$ and we obtain

$$\tilde{j}_*\tilde{j}^*(\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathfrak{Z}) \xrightarrow{k} j_*j^*(\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathfrak{Z}) \to \Delta_*^{x_1=x_3}(j_*^{x_2\neq x_1} j^{*,x_2\neq x_1}(\mathcal{A} \boxtimes \mathcal{A}))$$

and this composition is zero by the definition of $\mathfrak{Z}$. Hence we have that $\mu_{1,(23)} \circ k = \mu_{(12)3} \circ k$ and if we now restrict to $X \times x \times x$ we get the diagram above. □

The center $\mathfrak{Z}$ as the chiral algebra of endomorphisms of $\mathcal{A}$.

Recall that in Nick's talk, we have seen that $End(\mathcal{A})$ is a commutative chiral algebra that parametrizes the endomorphisms of $\mathcal{A}$. Hence we have a map $End(\mathcal{A}) \otimes \mathcal{A} \to \mathcal{A}$. Note that the fact the fact that $End(\mathcal{A})$ is commutative allows as to define a chiral action of this algebra on $\mathcal{A}$, by composition:

$$j_*j^*(End(\mathcal{A}) \boxtimes \mathcal{A}) \to \Delta_*(End(\mathcal{A}) \overset{!}{\otimes} \mathcal{A}) \to \Delta_*(\mathcal{A})$$

in other words we have that the action factors through a map $j_*j^*(End(\mathcal{A}) \boxtimes \mathcal{A}) \to \Delta_*(End(\mathcal{A}) \otimes \mathcal{A})$, and that $End(\mathcal{A})$ is the universal commutative chiral algebra with this property. However also the center $\mathfrak{Z} \subset \mathcal{A}$ had the same property, hence we obtain a map $\mathfrak{Z} \to End(\mathcal{A})$. Moreover we also have that $\mathfrak{Z}$ is preserved under $End(\mathcal{A})$, hence we obtain a map $End(\mathcal{A}) \overset{!}{\otimes} \mathfrak{Z} \to \mathfrak{Z}$. Consider now the composition with the unit map of $\mathfrak{Z}$:

$$End(\mathcal{A}) = End(\mathcal{A}) \overset{!}{\otimes} \Omega_X \to End(\mathcal{A}) \overset{!}{\otimes} \mathfrak{Z} \to \mathfrak{Z}.$$

This map provides an inverse to the previous one, showing that $\mathfrak{Z} \simeq End(\mathcal{A})$.

2 Group $\mathcal{D}_X$-scheme acting on a chiral algebra and conformal blocks as a quasi-coherent sheaf on Bun_G

In this section we want to make clear what it means for a group $\mathcal{D}_X$-scheme to act on a chiral algebra $\mathcal{A}$. First of all note that if $\tilde{G}$ is a group $\mathcal{D}_X$-scheme, then its coordinate ring $\mathcal{O}_{\tilde{G}}$ is a commutative chiral algebra endowed with a map

$$\delta : \mathcal{O}_{\tilde{G}} \to \mathcal{O}_{\tilde{G}} \otimes \mathcal{O}_{\tilde{G}}$$

of chiral algebras, i.e. a map such that the following diagram is commutative

$$\begin{array}{ccc} j_*j^*(\mathcal{O}_{\tilde{G}} \boxtimes \mathcal{O}_{\tilde{G}}) & \longrightarrow & \Delta_*(\mathcal{O}_{\tilde{G}}) \\ \downarrow & & \downarrow \\ j_*j^*(\mathcal{O}_{\tilde{G}} \otimes \mathcal{O}_{\tilde{G}} \boxtimes \mathcal{O}_{\tilde{G}} \otimes \mathcal{O}_{\tilde{G}}) & \longrightarrow & \Delta_*(\mathcal{O}_{\tilde{G}} \otimes \mathcal{O}_{\tilde{G}}) \end{array}$$

Definition 2.1. An action of a group $\mathcal{D}_X$- scheme $\tilde{G}$ on a chiral algebra $\mathcal{A}$ is a coaction on $\mathcal{O}_{\tilde{G}}$ on $\mathcal{A}$. In other words it is a map $\mathcal{A} \xrightarrow{\pi} \mathcal{A} \otimes \mathcal{O}_{\tilde{G}}$ of chiral algebras, such that $(\pi \otimes id) \circ \pi = (id \otimes \delta) \circ \pi$

The condition about π being a map of chiral algebras translates into the following diagramm:

$$\begin{array}{ccc} j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) & \longrightarrow & \Delta_*(\mathcal{A}) \\ \downarrow & & \downarrow \\ j_*j^*(\mathcal{A} \otimes \mathcal{O}_{\tilde{G}} \boxtimes \mathcal{A} \otimes \mathcal{O}_{\tilde{G}}) & \longrightarrow & \Delta_*(\mathcal{A} \otimes \mathcal{O}_{\tilde{G}}) \end{array}$$

where the bottom arrow is the bracket of the Chiral algebra $\mathcal{A} \otimes \mathcal{O}_{\tilde{G}}$ defined by using the chiral bracket of $\mathcal{A}$ and $\mathcal{O}_{\tilde{G}}$, i.e. for $f(x,y)a \boxtimes h_1$ and $g(x,y)b \boxtimes h_2$ sections of $\mathcal{A} \otimes \mathcal{O}_{\tilde{G}}$, $\{f(x,y)a \boxtimes h_1\, g(x,y)b \boxtimes h_2\} = \{f(x,y)a \boxtimes b\}_{\mathcal{A}} \otimes \{g(x,y)h_1 \boxtimes h_3\}_{\mathcal{O}_{\tilde{G}}}$

We will denote with $\mathcal{A}^{\tilde{G}}$ the set

$$\mathcal{A}^{\mathcal{O}_{\tilde{G}}} = \{a \in \mathcal{A} \,|\pi(a) = a \otimes 1\}\,. \tag{2}$$

Twisting of a chiral algebra by a G-torsor.
Now suppose that our group $\tilde{G}$ is of the form $\mathcal{J}G$ and let P be a G-torsor. Then we can make sense to the concept of twisting a chiral algebra $\mathcal{A}$ by P. Before going into that, note that from our G-torsor P, we can form a $\mathcal{D}_X$-scheme $\mathcal{J}G$-torsor by taking $\mathcal{J}P$. This actually gives us an equivalence of category

$$\left\{\ G\text{-torsors on } X\ \right\} \ \tilde{\rightarrow}\ \left\{\ \mathcal{D}_X\text{-schemes } \mathcal{J}G\text{-torsors}\ \right\}\,.$$

where the inverse is given by push forward functor along the canonical map $\mathcal{J}G \to G$. Now if we are given a G-torsor P, then the coordinate ring of $\mathcal{J}P$ will be a commutative chiral algebra $\mathcal{O}_{\mathcal{J}P}$, endowed with an action of $\mathcal{J}G$ as a chiral algebra. Moreover for any chiral algebra $\mathcal{A}$ we can consider the tensor product $\mathcal{A} \otimes \mathcal{O}_{\mathcal{J}P}$ as a chiral algebra.

Definition 2.2. Given a chiral algebra $\mathcal{A}$ acted on by $\mathcal{J}G$ and a G-torsor P on X, the twisted chiral algebra $\mathcal{A}_P$ is defined to be the space of invariants for the action of $\mathcal{J}G$ on the chiral algebra $\mathcal{A} \otimes \mathcal{O}_{\mathcal{J}P}$. i.e. using the notation as in (2)

$$\mathcal{A}_P = (\mathcal{A} \otimes \mathcal{O}_{\mathcal{J}P})^{\mathcal{J}G}.$$

Remark 2. There is a more concrete way of constructing the twisted algebra $\mathcal{A}_P$ that will be used later rather than the one just mentioned. Recall that in Dennis's talk [3] we have seen that a G-torson on a curve X is locally trivial in the Zarisky topology. Let $X = \bigcup_{i=1}^n U_i$ be a covering of X where the G-torsor P is locally trivial and let $\psi_{i,j}$ be the gluing datum of P on $U_i \cap U_j$. Then we can define $\mathcal{A}_P$ locally in the following way: $\mathcal{A}_P|_{U_i} = \mathcal{A}|_{U_i}$, and on $U_i \cap U_j$ we identify

$$\mathcal{A}_P|_{U_i \cap U_j} = (\mathcal{A}|_{U_i})|_{U_i \cap U_j} = \mathcal{A}_P|_{U_i \cap U_j} \overset{\psi_{i,j}}{\simeq} (\mathcal{A}|_{U_j})|_{U_i \cap U_j} = \mathcal{A}_P|_{U_i \cap U_j}$$

using the gluing datum $\psi_{i,j} : U_i \cap U_j \to \mathcal{J}G$ of $\mathcal{J}P$ composed with the map given by the action of $\mathcal{J}G$ from $\mathcal{J}G$ to the automorphism of $\mathcal{A}$. Note that in fact this orizontal section of $\mathcal{J}G$ (by definition of Jets) corresponds to the gluing datum $\psi_{i,j} : U_i \cap U_j \to G$ of P. The last thing that we have to check now, is that the bracket is still well defined. Since $\Delta(X)$ is covered by the open sets of the form $U_i \times U_i$, we actually have to check that the following diagram commutes:

$$\begin{array}{ccc}
j_*j^*((\mathcal{A}|_{U_i})|_{U_i \cap U_j} \boxtimes \mathcal{A}|_{u_i})|_{U_i \cap U_j}) & \longrightarrow & \Delta_*((\mathcal{A}|_{U_i})|_{U_i \cap U_j}) \\
\downarrow{\scriptstyle j_*j^*(\psi_{i,j} \boxtimes \psi_{i,j})} & & \downarrow{\scriptstyle \Delta_*(\psi_{i,j})} \\
j_*j^*((\mathcal{A}|_{U_{ij}})|_{U_i \cap U_j} \boxtimes \mathcal{A}|_{U_j})|_{U_i \cap U_j}) & \longrightarrow & \Delta_*((\mathcal{A}|_{U_j})|_{U_i \cap U_j})
\end{array}$$

The commutativity of this diagram follows directly from the definition of action of a group $\mathcal{D}_X$-scheme on a chiral algebra.

Conformal blocks as a quasi-coherent sheaf on Bun_G.
Now we are ready to see how we can regard the assignment $\{P \to H_\nabla(X, \mathcal{A}_P)\}$ as a quasi-coherent sheaf on Bun_G. As we have seen before, it make sense to twist a chiral algebra $\mathcal{A}$ acted upon by $\mathcal{J}G$ by the $\mathcal{J}G$-torsor $\mathcal{J}P$. This means that for every point of Bun_G, taking conformal blocks, we get a vector space $H_\nabla(X, \mathcal{A}_P)$. This can also be done in families. In fact if we have $S \xrightarrow{f} X$, and P a G-torsor on $X \times S$, then we can consider $p_1^*(\mathcal{A})$ (where $p_1 : X \times S \to X$) and twist this by $\mathcal{J}P$. Moreover recall that taking De-Rham cohomology of $\mathcal{A}$, is the same as taking the cohomology of the complex $p_*(\mathcal{A})$, where

$p : X \to \{pt\}$. Hence, if we consider the map $p : X \times S \to \{pt\} \times S$, we can form the group $H^2(p_*((p_1^*(\mathcal{A})^{(2)})))$, which would be equal to the vector space $H_\nabla(X, \mathcal{A})$ if S were a point. From this we see that the assignment

$$P \in \mathrm{Bun}_G(S) \to H^2(p_*((p_1^*(\mathcal{A})_P^{(2)})))$$

define a quasi-coherent sheaf on Bun_G.

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