OPERS

JONATHAN BARLEV

The geometric Langlands correspondence conjectures a correspondence

$$\mathfrak{Qco}(\text{loc-sys}_{{}^LG}(x)) \cong \text{D-mod}(\text{Bun}_G(X))$$

on the level of derived categories. As remarked previously in the seminar, to each local system over X we associate the skyscraper sheaf over it. In this talk we shall discuss a subspace[1] of loc-sys$_{{}^LG}(X)$ for whom the RHS counterpart (of the corresponding skyscraper sheaves) is known and consists of Hecke eigensheaves.

1. Definitions

Let G be a connected reductive group over $\mathbb{C}$, a G-oper will be defined shortly as a G- local system on X with extra structure. Fix a Borel subgroup $B \subseteq G$. This choice induces a descending filtration on $\mathfrak{g}$ of length twice the rank + 1 for which $\mathfrak{g}^{-r} = \mathfrak{g}$ and $\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{n}]$ (so $\mathfrak{g}^0 = \mathfrak{b}$, $\mathfrak{g}^1 = \mathfrak{n}$ etc.), which is preserved by $\mathfrak{b}$. We also denote $H = B/N$, which acts on $\text{gr}\mathfrak{g}$ via ad.

Let $\mathfrak{F}_B$ be B torsor on X, and let $(\mathfrak{F}_G, \nabla)$ be a local system on the induced G torsor. Let $\mathcal{E}_{\mathfrak{F}_G} = T\mathfrak{F}_G/G$, this is the vector bundle over X whose sections are G-invariant vector fields on ${\mathfrak{F}_G}$[2]; think of the connection as a map $\nabla : TX \to \mathcal{E}_{\mathfrak{F}_G}$ (which is a section to the projection $\mathcal{E}_{\mathfrak{F}_G} \to TX$). Composing $TX \xrightarrow{\nabla} \mathcal{E}_{\mathfrak{F}_G} \to \mathcal{E}_{\mathfrak{F}_G}/\mathcal{E}_{\mathfrak{F}_B} = (\mathfrak{g}/\mathfrak{b})_{\mathfrak{F}}$ we obtain a section $c(\nabla) \in \Gamma(X, (\mathfrak{g}/\mathfrak{b})_{\mathfrak{F}} \otimes \Omega_X)$, which measures to which extent $\mathfrak{F}_B$ is preserved by the connection; ∇ preserves $\mathfrak{F}_B$ if and only if $c(\nabla) = 0$.

We start by defining opers for groups of adjoint type, let $\mathfrak{g}$ be a semi-simple Lie Algebra, let G denote it's adjoint group.

Definition 1.1. A $\mathfrak{g}$-oper on X is a pair $(\mathfrak{F}_B, \nabla)$ where $\mathfrak{F}_B$ is a B torsor over X, and ∇ is a G-connection on the induced G-torsor, $\mathfrak{F}_G := \mathfrak{F}_B \times_B G$, such that:

(1) $c(\nabla)$ actually lands in $\Gamma(X, (\mathfrak{g}^{-1}/b)_{\mathfrak{F}})$.

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[1]At least morally, in our definitions it won't allways be a subspace.

[2]I.e. the Lie algebroid of infinitesimal symmetries of $\mathfrak{F}_G$. It a vector bundle over X equipped with a Lie Bracket satisfying an appropriate Leibnitz rule. It's fiber is non-canonically $\mathfrak{g} \times T_xX$.

(2) For each simple negative root α, the composition
$$c(\nabla)^\alpha : TX \xrightarrow{c(\nabla)} (\mathfrak{g}^{-1}/\mathfrak{b})_{\mathfrak{F}} \to (\mathfrak{g}^{-1}/\mathfrak{b})^\alpha$$
is an isomorphism.

Often we shall think of a G-oper as a G-local system $(\mathfrak{F}_G, \nabla)$ with the extra data of a specified reduction of the structure of the torsor to B [3] satisfying the connection conditions above.

Example 1.2. If X is a curve with a coordinate dx, and $G = PGL_n$ (with usual upper triangular Borel) we can give an oper structure to the trivial torsor $X \times B$ by using a connection of the form $\nabla = d + dx \otimes \begin{pmatrix} 0 & * & * \\ 1 & 0 & * \\ & 1 & 0 \end{pmatrix}$ (where d is the flat connection on the trivialized torsor $X \times G$). In fact any choice of matrix with non vanishing sub-diagonal will do, however we shall show that locally every oper is isomorphic to a unique oper with this form (which of course depends on the choice of coordinate, dx).

Remark 1.3. If $(\mathcal{F}_B, \nabla)$ is a G-oper then for any $\eta \in \Gamma(X, \mathfrak{b}_{\mathcal{F}} \otimes \Omega_X)$ we can form a new oper $\nabla + \eta$.

When G is of non-adjoint type we give a definition which differs from that in [1]. In their definition a G-oper (for an arbitrary reductive, connected G) has $Z(G)$ as it's automorphism group, the effect of our change is to eliminate these as well as restrict the admissible opers.

When G is of adjoint type the oper conditions imply ∇ induces an isomorphism
$$\mathcal{F}_H \times_H \mathfrak{g}^1/\mathfrak{g}^2 \xrightarrow{\tilde{\phi}} \Omega_X^{\oplus r} \cong \Omega_X \times_{\check{\rho}} H \times_H \mathfrak{g}^1/\mathfrak{g}^2$$
here Ω_X is identified with the underlying $\mathbb{G}_m$ torsor, and $\check{\rho}$ is the co-character associated sum of the fundamental co-weights (which is a character since G is of adjoint type). In the adjoint case there exists a unique isomorphism
$$\mathcal{F}_H \xrightarrow{\phi} \mathcal{F}_H^{can} := \Omega_X \times_{\check{\rho}} H$$
inducing $\tilde{\phi}$. When G is not necessarily of adjoint type $\check{\rho}$ might not be a co-character, but $2\check{\rho}$ is and if we fix a square root of the canonical bundle $\Omega_X^{1/2}$ then ∇ would still induce
$$\mathcal{F}_H \times_H \mathfrak{g}^1/\mathfrak{g}^2 \xrightarrow{\tilde{\phi}} \Omega_X^{\oplus r} \cong \Omega_X^{1/2} \times_{2\check{\rho}} H \times_H \mathfrak{g}^1/\mathfrak{g}^2$$

[3] recall every G-bundle on a smooth curve admits a reduction to B.

Definition 1.4. Fix a square root of the canonical bundle, $\Omega_X^{1/2}$. For G not necessarily adjoint, in addition to the contents of definition 1.1, we require the data of an isomorphism

$$\mathcal{F}_H \xrightarrow{\phi} \Omega_X^{1/2} \times_{2\check{\rho}} H$$

which is compatible with $\tilde{\phi}$ as above. We call ϕ the *marking*.

As mentioned above, for G of adjoint type such a ϕ exists and is unique. In general the data of 1.1 does not imply the existence of ϕ, hence relative to [1] we are indeed limiting our collection of opers. Moreover, $\tilde{\phi}$ determines ϕ only up to an automorphism of the underlying torsor given by multiplication with a central element; fixing ϕ eliminates these automorphisms.

Example 1.5. In the case of Gl_n or SL_n, an oper an be described in terms of a rank n vector bundle as follows: it consists of the data $(\mathcal{E}, (\mathcal{E}_i), \nabla, \phi)$ where $\mathcal{E}$ is a rank n vector bundle over X, $(\mathcal{E}_i)_{i=1}^n$ is a complete flag for $\mathcal{E}$, $\phi : \mathcal{E}_1 \xrightarrow{\cong} \Omega^{\otimes(n-1)/2}$ is an isomorphism, and finally $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_X$ is connection which satisfies.

(1) $\nabla(\mathcal{E}_i) \subseteq \mathcal{E}_{i+1} \otimes \Omega_X$, thus induces morphisms between the invertible sheaves $\mathrm{gr}_i\mathcal{E} \to \mathrm{gr}_{i+1}\mathcal{E} \otimes \Omega_X$.
(2) For each i the morphism above is an isomorphism.

2. Opers as a functor

Next we proceed to collect opers into a functor on D_X-schemes. Indeed if $Y \xrightarrow{f} X$ is a D_X-scheme, then the notion of a *G-torsor over Y with connection along X* makes sense: it is a G-torsor $\mathfrak{F}_G \xrightarrow{p} Y$ for which $\mathfrak{F}_G$ is compatibly a G-equivariant D_X-scheme, i.e. the map p is horizontal. For a *G-oper on Y along X* we additionally provide the data of a reduction of the torsor structure to the Borel, as well as an isomorphism $\mathfrak{F}_H \xrightarrow{\phi} f^*\Omega_X^{1/2} \times_{2\check{\rho}} H$ which satisfy:

(1) $c(\nabla)$, which is the composition $\nabla : f^*TX \to \mathcal{E}_{\mathfrak{F}_G} \to (\mathfrak{g}/\mathfrak{b})_{\mathfrak{F}}$, actually lands in $(\mathfrak{g}^{-1}/\mathfrak{b})_{\mathfrak{F}}$ (note these are all vector bundles over Y).
(2) For each negative simple root α, the composition $f^*TX \to (\mathfrak{g}^{-1}/\mathfrak{b})_{\mathfrak{F}} \to (\mathfrak{g}^{-1}/\mathfrak{b})_{\mathfrak{F}}^{\alpha}$ is an isomorphism. Thus we have an induced map $(\mathfrak{g}^1/\mathfrak{g}^2)_{\mathfrak{F}} \xrightarrow{\nabla} (f^*\Omega_X)^{\oplus r}$.
(3) The map induced by ϕ ,

$$(\mathfrak{g}^1/\mathfrak{g}^2)_{\mathfrak{F}} = \mathfrak{F}_H \times_H \mathfrak{g}^1/\mathfrak{g}^2 \xrightarrow{\phi} f^*\Omega_X^{1/2} \times_{2\check{\rho}} H \times_H \mathfrak{g}^1/\mathfrak{g}^2 = (f^*\Omega_X)^{\oplus r}$$

agrees with the map induced by ∇ above.

For any two such objects the notion of an isomorphism is evident and we can form the groupoid $\mathcal{O}p_G(Y)$, whose objects are G-opers over $Y \to X$. Lastly, pullback of such opers along horizontal maps is defined (and has the same underlying G-local system as the pullback in that category) and we end up with a functor on D_X-schemes, $\mathcal{O}p_G$, which is a sheaf at least in the Zariski topology.

In particular, we may endow $Y = S \times X$ with the flat connection and obtain the notion of an *S family of opers on X.* Abusing notation we use $\mathcal{O}p_G(X)$ to denote the functor on schemes whose S-points are $\mathcal{O}p(S \times X)$; thus $\mathcal{O}p_G(X)$ is a functor over schemes.

In the next secion we shall show that $\mathcal{O}p_{\mathfrak{g}}$ is representable by a D_X-scheme which is (non-canonically) isomorphic to the Hitchin jet bundle $\mathcal{J}ets_X(\mathfrak{c}_{{}^L\mathfrak{g}} \times_{\mathbb{G}_m} \Omega_X)$. It's horizontal sections, $\mathcal{O}p_{\mathfrak{g}}(X)$ will thus be represented by $\mathrm{Hitch}_{{}^L\mathfrak{g}}(X)$, hence naturally be an affine space whenever X is complete.

For now we show that $\mathcal{O}p_G$ is fibered over D_X-schemes in sets, i.e. opers have no automorphisms.

Proposition 2.1. *A G-oper, $(\mathfrak{F}_B, \nabla, \phi)$, over a D_X-scheme $Y \to X$ has no (non trivial) automorphisms.*

Proof. This is a local question so we may assume that $\mathfrak{F}_B = Y \times B$ is trivialized, and X has a local coordinate dx. We can write $\nabla = d + \eta \otimes dx$, where d is the "flat connection" on $Y \times B \to X$ induced by the connection on Y and the trivialization, and $\eta : Y \to \mathfrak{g}^{-1}$ has non-vanishing entries for each simple negative root. Considering the underlying torsor $\mathfrak{F}_B = Y \times B$ as a right torsor, an automorphism consists of left translation by a map $\sigma : Y \to B$. Since σ must preserve ϕ it must act trivially on $\mathcal{F}_H$, this forces the image of σ to actually fall in N, so $\sigma : Y \to N$. Conversely any such automorphism will be trivial mod N, have no effect on $\mathcal{F}_H$ and will preserve ϕ. The effect of such a map on the connection is

$$\sigma.(d + \eta) = d + d\sigma \cdot l_{\sigma^{-1}} + Ad(\sigma)\eta \otimes dx$$

Assume σ induces an automorphism of the oper, i.e. preserves the connection. Note that $d\sigma \cdot l_{\sigma^{-1}} \in \Gamma(X, \mathfrak{g}^1 \otimes \Omega_X)$ (recall $\mathfrak{n} = \mathfrak{g}^1$) so

$$Ad(\sigma)\eta \equiv \eta \ \mathrm{mod} \mathfrak{g}^1$$

In N we can take logarithms so let $\sigma = \log n$ for some $n \in \mathfrak{n}$, let $\mathfrak{g}^i$ be the smallest term of the filtration containing n, we show $i = r$ and hence $\sigma = 1$. Indeed $d\sigma \cdot l_{\sigma^{-1}} = d \log \sigma = dn \in \Gamma(X, \mathfrak{g}^i \otimes \Omega_X) = \Gamma(X, \mathfrak{g}^1 \otimes \Omega_X)$ so that

$$Ad(\sigma)\eta \equiv \eta \ \mathrm{mod} \mathfrak{g}^{\mathfrak{i}}.$$

Choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{b}$, and write $\eta = \eta^{-1} + \beta$ with $\eta^{-1} : Y \to \mathfrak{g}^{-1}$ having non vanishing coordinates in each simple negative root space, and $\beta : Y \to \mathfrak{b}$.

$$\begin{aligned} \mathfrak{g}^i &\ni Ad(\sigma)(\eta^{-1} + \beta) - (\eta^{-1} + \beta) = \\ &= Ad(exp(n))(\eta^{-1} + \beta) - (\eta^{-1} + \beta) \\ &= [n, \eta^{-1} + \beta] + 1/2[n, [n, \eta^{-1} + \beta]] + \cdots \\ &\in [n, \eta^{-1}] + \mathfrak{g}^i \end{aligned}$$

hence $[\eta^{-1}, n] \in \mathfrak{g}^{\mathfrak{i}}$.

We show this implies $n \in \mathfrak{g}^{i+1}$ which is a contradiction to the choice of i unless $n = 0$ (implying $\sigma = 1$). Let $\{e_0, h_0, f_0\}$ the standard basis of $\mathfrak{sl}_2$, let $\mathfrak{b}_{\mathfrak{o}} = \text{span}\{e_0, h_0\}$ denote the standard Borel. At every point of Y the map η^{-1} determines a principal embedding [4] $\mathfrak{sl}_2 \to \mathfrak{g}$, which carries $\mathfrak{b}_{\mathfrak{o}}$ to $\mathfrak{b}$ and for which η^{-1} is a scalar multiple of the image of f_0. Considering $\mathfrak{g}$ as an $\mathfrak{sl}_2$ representation via this principle embedding, our filtration corresponds to the weight space filtration and for a positive weight i, hence $ad_{f_0} : \mathfrak{g}^i/\mathfrak{g}^{i+1} \hookrightarrow \mathfrak{g}^{i-1}/\mathfrak{g}^{\mathfrak{i}}$. □

Opers were introduced as living within local systems, this is true to the following extent. The forgetful functor from $\mathcal{O}p_G(X) \to$ loc-sys$_G(X)$ is always faithful. However, under our definition an oper cannot have non-trivial automorphism while the underlying local system might the issue is that a local system isomorphism may not preserve the flag or the marking). Nonetheless less we do have the following.

Proposition 2.2. *Let X be a complete curve of genus $g \geq 2$, G semi-simple of adjoint type, and $(\mathfrak{F}_G, \nabla)$ a G-local system on X which admits an oper structure. Then the oper structure is unique.*

We use the Harder Narasimhan flag:

Lemma 2.3. *Let $\mathfrak{F}_G$ be a be torsor over a complete curve X of genus $g \geq 2$. Let $\mathfrak{F}_B$ be a reduction to the Borel such that for each simple positive root α, the $\mathbb{G}_m$-torsor $\mathfrak{F}_H \times_\alpha \mathbb{G}_m$ has positive degree for. Then $\mathfrak{F}_B$ is unique.*

proof of 2.2. Let $\mathfrak{F}_B$ be a reduction to the Borel which makes $(\mathfrak{F}_G, \nabla)$ an oper, we show that $\mathfrak{F}_B$ is the Harder Narasimhan flag, hence is unique. Indeed for every positive simple root $\mathfrak{F}_H \times_\alpha \mathbb{G}_m \cong \Omega_X$;since $g \geq 2$, it has positive degree. □

[4]A principal embedding is one in which e_0, f_0 map to principal nilpotents in $\mathfrak{g}$; such an embedding determines a Cartan subalgebra relative to which e_0 and f_0 map to sums of basis elements of the positive/negative root spaces respectively.

Corollary 2.4. *When X is a complete curve and $g \geq 2$ and $\mathfrak{g}$ is semi-simple then $\mathcal{O}p_{\mathfrak{g}}(X)$ is a full and faithful subcategory of loc-sys$_G(X)$ (where G is the adjoint group of $\mathfrak{g}$).*

In the non-adjoint case the ambiguity in the oper structure on $(\mathfrak{F}_G, \nabla)$ is only due to the choice of marking (i.e. the flag is uniquely determined). It follows the the fiber of $\mathcal{O}\mathfrak{p}_G(X) \to \text{loc-sys}_G(X)$ is a naturally torsor for the 'automorphisms of ϕ over ∇" (see 1.4); evidently this is $Z =$the center of G.

3. A description of the space of $\mathfrak{g}$-opers

We now restrict ourselves to the adjoint case of $\mathfrak{g}$-opers with $\mathfrak{g}$ semi-simple, and show that $\mathcal{O}p_{\mathfrak{g}}$ is a torsor for (the jet scheme of) a certain vector bundle over X. Passing to global sections we obtain an isomorphism of $\mathcal{O}p_{\mathfrak{g}}(X)$ with an affine space which we shall identify with $\text{Hitch}_{^L\mathfrak{g}}(X)$. Recall that in Andrei's talk D-algebras were heuristically introduced as being to non-linear PDE's what D-mod's are to linear PDE's. That opers are a torsor for a vector bundle corresponds to the fact that solutions to a non-linear PDE are a torsor for the solutions of it's homogeneous counterpart.

It turns out that for any $\mathfrak{g}$ this torsor structure may be obtained from the torsor structure of $Op_{\mathfrak{sl}_2}$, and we start by considering this case.

3.1. $\mathfrak{sl}_2$-opers.

Lemma 3.1. *$\mathcal{O}p_{\mathfrak{sl}_2}$ is naturally a torsor for*[5] $\Omega_X^{\otimes 2}$.

Proof. Let $(\mathcal{F}_B, \nabla, \phi)$ be any oper on a D_X-scheme $Y \xrightarrow{f} X$. We start by noting in our case $\mathfrak{n} = \mathfrak{g}^1/\mathfrak{g}^2$ so the that by ∇ induces $\mathfrak{n}_{\mathcal{F}} \cong f^*\Omega_X$, thus $\mathfrak{n}_{\mathcal{F}} \otimes f^*\Omega_X \cong f^*\Omega_X^{\otimes 2}$ and $\Omega_X^{\otimes 2}$ both a-priori act on $Op_{\mathfrak{sl}_2}(Y)$ and $Op_{\mathfrak{sl}_2}$ (respectively) by modifying the connection.

To prove the action is simply transitive we may work locally. Replacing X and Y by appropriate Zariski opens we may assume that X has a local coordinate dx, and $\mathcal{F}_B \cong Y \times B$ is trivial (since B is solvable $\mathfrak{F}_B$, is Zariski locally trivial). As in the proof of lemma 2.1, a choice of coordinate and trivialization gives

$$\nabla = d + dx \otimes \eta$$

[5]There is some abuse of notation going on here, we are actually considering $\Omega_X^{\otimes 2}$ as a functor on D_X-schemes (rather than X-schemes). We could say instead that $\mathcal{O}p_{\mathfrak{sl}_2}$ is a torsor for $\mathcal{J}ets_X(\Omega_X^{\otimes 2})$, since this is the D_X-scheme representing the functor.

where d is the flat connection w.r.t the trivialization and $\eta : Y \to \mathfrak{sl}_2$ does not vanish mod $\mathfrak{b}$. Recall that a change of the trivialization given by left multiplication by $\sigma : Y \to B$ has the effect of modifying the connection to

$$d + d\sigma\sigma^{-1} + Ad(\sigma)\eta$$

Computing directly (this is where PSL_2 being of adjoint type comes in) we see that every coordinate determines a unique *canonical trivialization* which brings the connection to *canonical form* (both relative to the choice of coordinate)

$$\nabla = d + dx \otimes \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$$

thus the action of $f^*\Omega_X \otimes \mathfrak{n}_{\mathcal{F}}(Y)$ on $\mathcal{O}p_{\mathfrak{sl}_2}(Y)$ is free and transitive. Note also the the uniqueness implies that the underlying B torsor of any oper is trivialized whenever we have a coordinate. $\square$

Corollary 3.2. *All $\mathfrak{sl}_2$- torsors all have isomorphic underlying B-torsors. Denote this torsor $\mathfrak{F}^0_{B_0}$.*

Remark 3.3. For a complete curve X $\mathfrak{sl}_2$-opers always exist, hence $\mathcal{O}p_{\mathfrak{sl}_2}$ is a trivial torsor for $\Omega_X^{\otimes 2}$:

When $g \geq 2,$ this is implied by the fact that $H^1(X, \Omega_X^{\otimes 2}) = 0$.
When $g = 1$ there is a global coordinate and the trivialized torsor $X \times B$ with $\nabla = d + dx \otimes \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$ is an oper.
When $g = 0$ one can either compute directly and see there is a unique oper (uniqueness is implied by the fact that $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^{\otimes 2}) = 0$). Alternatively, note that an oper is equivalent to the data of a $\mathbb{P}^1$ bundle with connection and a section with non-vanishing co-variant derivative; take $\mathbb{P}^1 \times \mathbb{P}^1$ with the trivial connection and diagonal section.

3.2. $\mathfrak{g}$-opers.

Let $\mathfrak{g}$ be an arbitrary semi-simple lie algebra, with a chosen Borel $\mathfrak{b}$.

Construction 3.4. Let $\{e_0, h_0, f_0\}$ the standard basis of $\mathfrak{sl}_2$, let $\mathfrak{b}_\circ = \mathrm{sp}\{e_0, h_0\}$ denote the standard Borel, and choose a principal embedding $\mathfrak{sl}_2 \to \mathfrak{g}$, which carries $\mathfrak{b}_\circ$ to $\mathfrak{b}$. To the map of Lie algebras corresponds a map of groups $PSL_2 \xrightarrow{\iota_G} G$, where G is the adjoint group of $\mathfrak{g}$.

We now describe how ι_G may be used to construct a $\mathfrak{g}$-opers out of an $\mathfrak{sl}_2$-opers. Let $(\mathfrak{F}^0_{B_0}, \nabla_0)$ be an $\mathfrak{sl}_2$-oper on $Y \xrightarrow{f} X$ [6]. Let $\mathfrak{F}_B$, $\mathfrak{F}_G$ respectively be the B , G torsors which $\mathfrak{F}^0_{B_0}$ induces via ι_G. The connection ∇_0 naturally induces a connection on

[6]Recall that for a $\mathfrak{g}$-oper the isomorphism ϕ is uniquely determined by the rest of the data, hence we ignore it.

$\mathfrak{F}_G$ which satisfies the oper condition because the map ι_G is principal. In fact, if locally in an appropriate trivialization

$$\nabla_0 = d + dx \otimes \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$$

then

$$\nabla = d + dx \otimes (f + \alpha e))$$

In particular global $\mathfrak{g}$-opers always exist.

We wish to present $\mathfrak{g}$-opers as a torsor for an appropriate vector bundle.

Relative to a principal embedding, let $\mathfrak{g}^{e_0}$[7] be the stabilizer of e_0 in $\mathfrak{g}$. As in the $\mathfrak{sl}_2$ case, on a curve X with coordinate dx, lemma 3.6 will state that every $\mathfrak{g}$-oper has a canonical trivialization and that, locally, opers are a torsor for maps $Y \to \mathfrak{g}^{e_0}$. However, B doesn't act on $\mathfrak{g}^{e_0}$ so we can't a-priori twist by $\mathfrak{F}_B$ (as in the $\mathfrak{sl}_2$ case) to immediately get an action on the space of opers.

None the less, since B_0 does act on $\mathfrak{g}^{e_0}$ we can form

$$\mathfrak{g}^{e_0}_{\mathfrak{F}^0} := \mathfrak{F}^0_{H_0} \times_{H_0} \mathfrak{g}^{e_0} = \mathfrak{F}^0_{B_0} \times_{B_0} \mathfrak{g}^{e_0}$$

Using $\Omega_X^{\otimes 2} = \Omega_X \otimes \mathfrak{n}_{0\mathfrak{F}^0} \to \Omega_X \otimes \mathfrak{g}^{e_0}_{\mathfrak{F}^0}$ we construct an $\Omega_X \otimes \mathfrak{g}^{e_0}_{\mathfrak{F}^0}$-torsor

$$\mathcal{O}p_{\mathfrak{sl}_2} \times_{\Omega_X^{\otimes 2}} \left(\Omega_X \otimes \mathfrak{g}^{e_0}_{\mathfrak{F}^0}\right)$$

Lemma 3.5. *Every principal embedding gives rise to an isomorphism*

$$\mathcal{O}p_{\mathfrak{sl}_2} \times_{\Omega_X^{\otimes 2}} \left(\Omega_X \otimes \mathfrak{g}^{e_0}_{\mathfrak{F}^0}\right) \to \mathcal{O}p_{\mathfrak{g}} \tag{3.1}$$

Proof. The map is defined by sending $\left(\mathfrak{F}^0_{B_0}, \nabla; \nu\right)$ to $\left(\iota\mathfrak{F}^0_{B_0}, \iota\nabla + \nu\right)$. Where $\left(\mathfrak{F}^0_{B_0}, \nabla\right)$ is an $\mathfrak{sl}_2$-oper which induces a $\mathfrak{g}$-oper $\left(\iota\mathfrak{F}^0_{B_0}, \iota\nabla\right)$, and for $\nu \in \Omega_X \otimes \mathfrak{g}^{e_0}_{\mathfrak{F}^0}$ we define $\iota\nabla + \nu$ using $\mathfrak{g}^{e_0}_{\mathfrak{F}^0} = \mathfrak{F}^0_{B_0} \times_{B_0} \mathfrak{g}^{e_0} \to \mathfrak{F}^0_{B_0} \times_{B_0} \mathfrak{b} = \mathfrak{b}_{\mathfrak{F}^0}$. The image consists of opers whose underlying B-torsor is $\mathfrak{F}^0_B := \mathfrak{F}^0_{B_0} \times_\iota B$. To see this is an isomorphism is suffices to check locally, whence this is implied by lemma 3.6. □

Lemma 3.6. *Let X be a curve with a coordinate dx. Fix a principle embedding $\mathfrak{sl}_2 \to \mathfrak{g}$ as above. For any $\mathfrak{g}$-oper on X, the underlying B-torsor admits a unique trivialization relative to which the connection has the form*

$$\nabla = d + dx \otimes (f + \eta)$$

where $\eta : Y \to \mathfrak{g}^{e_0}$.

[7] $\mathfrak{n}_0 \subseteq \mathfrak{g}^{e_0} \subseteq \mathfrak{n}$. In the case of $\mathfrak{sl}_n$ with the standard Principal embedding, $\mathfrak{g}^e$ consists of all strictly upper triangular matrices with constant super-diagonals.

Proof. The proof uses Kostant's lemma (see [2]) and we defer it to the appendix. □

Corollary 3.7. *Global $\mathfrak{g}$-opers exist and all $\mathfrak{g}$-opers have isomorphic underlying B-torsors. Given a Principal embedding ι, this B-torsor is naturally identified with as the pushforward $\mathfrak{F}^0_B := \mathfrak{F}^0_{B_0} \times_\iota B$.*

Every principal embedding gives rise , via lemma 3.5, to a $\Omega_X \otimes \mathfrak{g}^{e_0}_{\mathfrak{F}}$-torsor structure on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}$. However, any two principal embedding are uniquely conjugate by an element of B. Thus, neither the vector space $\mathfrak{g}^{e_0}$, nor the torsor structure depend on the Principal embedding:

Lemma 3.8. *$\mathcal{O}\mathfrak{p}_{\mathfrak{g}}$ is naturally a torsor for $\Omega_X \otimes \mathfrak{g}^{e_0}_{\mathfrak{F}}$. Thus $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ is naturally a torsor for $\Gamma\left(X, \Omega_X \otimes \mathfrak{g}^{e_0}_{\mathfrak{F}}\right)$. Both are trivial torsors.*

As for the representability of the space of opers, any choice of a global oper on X gives rise to:

Corollary 3.9. *$\mathcal{O}p_{\mathfrak{g}}$ is representable by a D_X-scheme isomorphic to $\mathcal{J}ets_X(\Omega_X \otimes \mathfrak{g}^{e_0}_{\mathfrak{F}})$. When X is complete, $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ is representable by a scheme isomorphic to $\Gamma(X, \Omega_X \otimes \mathfrak{g}^{e_0})$ (for which it is naturally an affine space).*

4. Opers and the classical Hitchin space

Next we discuss the relation to the Hitchin space, which will be further developed in the next section. Identify, via the Ad action, $H_0 \cong \mathrm{Aut}\mathfrak{n}_0 = \mathbb{G}_m$ to obtain a $\mathbb{G}_m$ action on $\mathfrak{g}^{e_0}$ which we also denote Ad. $\mathcal{F}_{H_0} \cong \Omega_X$ and so

$$\Omega_X \otimes \mathfrak{g}^{e_0}_{\mathfrak{F}} = \Omega_X \otimes (\mathfrak{F}_{H_0} \times_{H_0} \mathfrak{g}^{e_0}) = \Omega_X \times_{tAd(t)} \mathfrak{g}^{e_0}$$

where $tAd(t)$ denotes the action of $t \in \mathbb{G}_m$ on $\mathfrak{g}^{e_0}$. There are natural isomorphisms

$$\mathfrak{c}_{{}^L\mathfrak{g}} := \mathrm{spec}(\mathrm{sym}{}^L\mathfrak{g})^{{}^LG} = \mathrm{spec}(\mathrm{sym}{}^L\mathfrak{h})^W = \mathrm{spec}(\mathrm{sym}\mathfrak{h}^*)^W = \mathfrak{g}//G$$

and these identifications are $\mathbb{G}_m$ equivariant w.r.t. the the action on the quotients, which is pushed forward from scalar multiplication on $\mathfrak{g}$ and ${}^L\mathfrak{g}$ respectively. Recall Kostant's section of the Chevalley map $\mathfrak{g} \to \mathfrak{g}//G$ has $f_0 + \mathfrak{g}^{e_0}$ as it's image. This section induces an isomorphism of varieties $\mathfrak{c}_{{}^L\mathfrak{g}} \to \mathfrak{g}^{e_0}$ which intertwines the $\mathbb{G}_m$ action on $\mathfrak{c}_{{}^L\mathfrak{g}}$ with the $tAd(t)$ action on $\mathfrak{g}^{e_0}$. Finally we obtain an identification schemes over X

$$\mathfrak{c}^{\Omega_X}_{{}^L\mathfrak{g}} = \Omega_X \times_{\mathbb{G}_m} \mathfrak{c}_{{}^L\mathfrak{g}} \xrightarrow{\cong} \Omega_X \otimes \mathfrak{g}^{e_0}_{\mathcal{F}} \tag{4.1}$$

The LHS, which is not a-priori a vector bundle, is now considered as such via this isomorphism.

The punch line is that corollary 3.9 now reads: every global $\mathfrak{g}$-oper over X gives rise to an isomorphism of D_X-schemes

$$\mathcal{O}p_{\mathfrak{g}} \cong \mathcal{J}ets_X(\mathfrak{c}^{\Omega_X}_{{}^L\mathfrak{g}}) = \mathrm{Hitch}_{{}^L\mathfrak{g}} = \mathrm{spec}_X\left(\mathfrak{z}^{\mathrm{cl}}_{{}^L\mathfrak{g}}\right)$$

Passing to functors of horizontal sections we get

$$\mathcal{O}p_{\mathfrak{g}}(X) \cong \mathrm{Sect}_X\left(\mathfrak{c}^{\Omega_X}_{{}^L\mathfrak{g}}\right) \cong \mathrm{Hitch}_{{}^L\mathfrak{g}}(X) \tag{4.2}$$

Next assume X is complete, then $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$ is representable by a scheme, which is naturally an affine space for $\Gamma(X, \Omega_X \otimes \mathfrak{g}^{e_0}_{\mathcal{F}})$.

Let $A_{\mathfrak{g}}$ denote the coordinate algebra of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}$, and let $A_{\mathfrak{g}}(X)$ denote the coordinate algebra of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X)$. Recall that $\mathfrak{z}^{\mathrm{cl}}_{{}^L\mathfrak{g}}$ and $\mathfrak{z}^{\mathrm{cl}}_{{}^L\mathfrak{g}}(X)$ are the coordinate rings of $\mathrm{Hitch}_{{}^L\mathfrak{g}}$ and $\mathrm{Hitch}_{{}^L\mathfrak{g}}(X)$ respectively, and that they are naturally graded. By the discussion above $A_{\mathfrak{g}}$ and $\mathfrak{z}^{\mathrm{cl}}_{{}^L\mathfrak{g}}$ are non-canonically isomorphic (as are $A_{\mathfrak{g}}(X)$ and $\mathfrak{z}^{\mathrm{cl}}_{{}^L\mathfrak{g}}(X)$), but the following is canonical:

Proposition 4.1. *There exists a canonical filtration of $A_{\mathfrak{g}}$ and a canonical isomorphism of graded rings $grA_{\mathfrak{g}} \xrightarrow{\sigma_A} \mathfrak{z}^{cl}_{{}^L\mathfrak{g}}$.*

Proof. Identify $\mathfrak{z}^{\mathrm{cl}}_{{}^L\mathfrak{g}}$ with the coordinate ring of $\mathcal{J}ets_X\left(\Omega_X \times_{t\mathrm{Ad}t} \mathfrak{g}^{e_0}\right)$ via 4.1. Note that the natural grading on $\mathfrak{z}^{\mathrm{cl}}_{{}^L\mathfrak{g}}$ corresponds to the action of $\mathbb{G}_m$ on $\Omega_X \times_{t\mathrm{Ad}t} \mathfrak{g}^{e_0}$ opposite to $t\mathrm{Ad}t$.

As $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}$ is naturally an affine space for $\mathcal{J}ets_X\left(\Omega_X \times_{t\mathrm{Ad}t} \mathfrak{g}^{e_0}\right)$ over X, on the level of coordinate rings, the grading on the latter induces a filtration on the former which yields the desired isomorphism. $\square$

Corollary 4.2. *Let X be a complete curve. There exists a canonical filtration of $A_{\mathfrak{g}}(X)$ and a canonical isomorphism of graded rings $(grA_{\mathfrak{g}})(X) \simeq gr(A_{\mathfrak{g}}(X)) \xrightarrow{\sigma_{A(X)}} \mathfrak{z}^{cl}_{{}^L\mathfrak{g}}(X)$.*

5. The Feigin-Frenkel isomorphism

This section is the quantization of the previous, and relates opers to the quantum Hitchin space. Let X be a complete curve. Recall that $\mathfrak{z}_{{}^L\mathfrak{g}}$ (defined rather abstractly) was a filtered algebra and we had the following theorem (of Feigin and Frenkel, dicussed in Dustin's talk) and accompanying results which related the various Hitchin algeras.

Theorem 5.1 (first Feigin-Frenkel)**.** *The natural map $gr\mathfrak{z}_{\mathfrak{g}} \xrightarrow{\sigma_{\mathfrak{z}}} \mathfrak{z}^{cl}_{\mathfrak{g}}$ is an isomorphism of graded D_X-algebras.*

Proposition 5.2. *The natural surjection $\left(gr\mathfrak{z}_{\mathfrak{g}}\right)(X) \to gr\left(\mathfrak{z}_{\mathfrak{g}}(X)\right)$ is an isomorphism.*

Corollary 5.3. *We have the following isomorphisms*

$$\left(gr\mathfrak{z}_{\mathfrak{g}}\right)(X) \to gr\left(\mathfrak{z}_{\mathfrak{g}}(X)\right) \xrightarrow{\sigma_{\mathfrak{z}(X)}} \mathfrak{z}_{\mathfrak{g}}^{cl}(X)$$

As promised, the following theorem due to Feigin and Frenkel as well relates the Hitchin spaces to opers:

Theorem 5.4 (second Feigin-Frenkel)**.** *There exists an isomorphism of filtered algebras*

$$A_{\mathfrak{g}} \xrightarrow{\phi} \mathfrak{z}_{{}^L\mathfrak{g}}$$

which quantizes the map $\sigma_{A_{\mathfrak{g}}}$ of proposition 4.1, i.e. $\sigma_{\mathfrak{z}} \circ gr\phi = \sigma_{A_{\mathfrak{g}}}$.

Corollary 5.5. *ϕ induces an isomorphism of filtered algebras*

$$A_{\mathfrak{g}}(X) \xrightarrow{\phi_X} \mathfrak{z}_{{}^L\mathfrak{g}}(X)$$

which quantizes the map $\sigma_{A_{\mathfrak{g}}(X)}$ of corollary 4.2, i.e. $\sigma_{\mathfrak{z}(X)} \circ gr\phi_X = \sigma_{A_{\mathfrak{g}}(X)}$.

From the perspective of spaces we have an isomorphisms

$$\mathrm{spec}_X\left(\mathfrak{z}_{\mathfrak{g}}\right) \simeq \mathcal{O}\mathfrak{p}_{{}^L\mathfrak{g}} \qquad \text{and} \qquad \mathrm{spec}\left(\mathfrak{z}_{\mathfrak{g}}(X)\right) \simeq \mathcal{O}\mathfrak{p}_{{}^L\mathfrak{g}}(X)$$

(of D_X-schemes and schemes respectively) providing a moduli description of the quantum Hitchin space.

Let us summarize this Hitchin quantization business so far:

$$\begin{tikzcd}
A_{{}^L\mathfrak{g}}(X) \arrow[r, "\cong"] \arrow[d, squiggly, "gr"] & \mathfrak{z}_{\mathfrak{g}}(X) \arrow[r] \arrow[d, squiggly, "gr"] & \Gamma(\mathrm{Bun}_{\mathfrak{g}}^{\gamma}X, \mathcal{D}') \arrow[d, squiggly, "gr"] \\
\mathrm{gr}A_{{}^L\mathfrak{g}}(X) \arrow[r] \arrow[d, "\cong"] & \mathrm{gr}(\mathfrak{z}_{\mathfrak{g}}(X)) \arrow[r] \arrow[d, "\cong"] & \mathrm{gr}\Gamma(\mathrm{Bun}_{\mathfrak{g}}^{\gamma}X, \mathcal{D}') \arrow[d, "\cong"] \\
A_{{}^L\mathfrak{g}}^{\mathrm{cl}}(X) \arrow[r, "\cong"] & \mathfrak{z}_{\mathfrak{g}}^{\mathrm{cl}}(X) \arrow[r, "\cong"] & \Gamma(T^*\mathrm{Bun}_{\mathfrak{g}}^{\gamma}X, \mathcal{O})
\end{tikzcd}$$

Where $A^{\mathrm{cl}}(X)$ is the coordinate ring of $\Gamma\left(X, \Omega_X \otimes \mathfrak{g}_{\mathfrak{F}^0}^{e_0}\right)$ and is introduced for the sake of symmetry. The marked isomorphisms are the ones we proved, they imply the rest of the maps are isomorphisms as well.

Finally the geometric Langlands correspondence for opers should play out as follows: starting from a ${}^L\mathfrak{g}$-local system on X, σ, which admits a (unique) oper structure we get a maximal ideal $m_\sigma \subseteq A_{{}^L\mathfrak{g}}(X)$ whose residue field we denote k_σ. Thus we get a twisted D-mod $\mathcal{D}_\sigma := \mathcal{D}' \otimes_{A_{{}^L\mathfrak{g}}(X)} k_\sigma$ (on $\mathrm{Bun}_{\mathfrak{g}}X$).This will be shown to be a Hecke Eigensheaf .

6. Appendix - Kostant's lemma and it's application

In this appendix we state a lemma of Kostant and prove lemma 3.6. This has already been discussed in the seminar to some extent, see [2]. Choose a principal embedding $\mathfrak{sl}_2 \xrightarrow{\iota} \mathfrak{g}$, and let e, h, f denote the images of the (standard) same named elements in $\mathfrak{sl}_2$. e and f are contained in unique, opposite Borels; so ι determines a Borel, Cartan and root system. Let $PSL_2 \xrightarrow{\iota G} G$ be the induced map on the group level.

Lemma 6.1. *N acts on $f + \mathfrak{b}$ freely via Ad. $f + \mathfrak{g}^e$ is a section for this action, and moreover $N^i.(f + \mathfrak{g}^e) = f + \mathfrak{g}^e + \mathfrak{n}^i$.*

With this in hand we prove lemma 3.6 :

proof of lemma 3.6. Let X have a coordinate dx, and let

$$\nabla = d + \eta^{-1} \otimes dx \quad \eta^{-1} : X \to \mathfrak{g}^{-1}$$

be the connection of a $\mathfrak{g}$-oper on a trivialized B-torsor. □

(1) We must show there exist a unique change of trivialization, $\sigma : X \to B$ such that the transformed connection

$$d + d\sigma\sigma^{-1} + Ad\,(\sigma)\,\eta^{-1} \otimes dx$$

has the desired form. Because G is of adjoint type Ad induces an isomorphism $H \to \times GL\,(\mathfrak{g}^{-\alpha})$. Since $\eta : X \to \mathfrak{g}^{-1}$ has non-vanishing entries in every simple negative root space, there exists a unique $X \xrightarrow{\sigma_H} H$ s.t. that $Ad\,(\sigma)\,\eta = f \bmod \mathfrak{b}$. $\sigma_H^{-1} d\sigma_H^{-1} \in \mathfrak{b}$ so that after changing the trivialization using σ_H it has the form

$$d + (f + \eta^0) \otimes dx \quad \eta^0 : X \to \mathfrak{b}$$

Let $\sigma_1 : X \to N$ be the unique map, promised by Konstant's lemma, such that $Ad\,(\sigma)\,\left(f + \eta^0\right) \in f + \mathfrak{g}^e$. If it weren't for the $\sigma^{-1}d\sigma$ term, which pops up when the trivialization is changed we'd be done. Nonetheless we get an approximation mod $\mathfrak{n}$, i.e. after changing the trivialization again using σ_1 the connection has the forms

$$d + (f + \eta^1) \otimes dx \quad \eta^1 : X \to \mathfrak{n}$$

Note that $\sigma_1\sigma_H$ with this property is unique mod N^2. We proceed in the same fashion inductively, noting at the i'th stage that if $f + \eta^i \in f + \mathfrak{g}^e + \mathfrak{n}^i$, then the unique σ^i such that $Ad(\sigma_i)(f + \eta^i) \in f + \mathfrak{g}^e$ in fact lies in N^{i+1} , hence $\sigma_i^{-1} d\sigma \in \mathfrak{n}^{i+1}$. After rank$\mathfrak{g}$ steps we get $\sigma_r \cdots \sigma_1 \sigma_H$ which is the required map.

References

[1] Beilinson; Drinfeld. Hitchin's integrable system. $http : //www.math.harvard.edu/ \sim gaitsgde/grad\_2009$.

[2] Dennis Gaitsgory. Seminar notes:higgs gundles, kostant section, and local triviality of g-bundles. $http : //www.math.harvard.edu/ \sim gaitsgde/grad\_2009$.