# The Hecke category
# (part II—Satake equivalence)

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In last week's lecture, we discussed the *Hecke category* **Sph** of spherical, or $G(\widehat{O})$-equivariant D-modules on the affine grassmannian $\mathrm{Gr}_G$ and constructed a convolution product on it. We also introduced the *factorizable monoidal categories* $\mathbf{Sph_n}$ of spherical D-modules on the relative grassmannians $\mathrm{Gr}_{G,X^n} = \mathrm{Gr}_n$, and stated the *geometric Satake equivalence*:

**Theorem 1.** *The convolution $*$ admits a commutativity constraint making **Sph** into a rigid tensor category. There exists a faithful, exact tensor "fiber" functor* $\mathrm{Sat} \colon \mathbf{Sph} \to \mathbf{Vect}$ *inducing an equivalence (modulo a sign in the commutativity constraint) of* **Sph** *with* $\mathbf{Rep}(^L G)$ *as tensor categories, where* $^L G$ *is the* Langlands dual group *of the reductive group $G$, whose weights are the coweights of $G$ and vice versa.*

*More generally, there is an equivalence* $\mathrm{Sat}_n \colon \mathbf{Sph_n} \to \mathbf{Rep_n}(^L G)$ *which is monoidal for each $n$, respects the factorizable structures on both sides as $n$ varies (including the $S_n$-equivariance), and for $n = 1$, agrees with* $\mathrm{Sat}$ *when restricted to each point of $X$.*

We proved the claim that $*$ sends D-modules to D-modules and constructed its commutativity constraint by establishing that it coincided with the *fusion product* on the relative grassmannian $\mathrm{Gr}_{G,X}$. This time we will describe the structure of **Sph** as a tensor category, identifying its identity object and duals and showing that it is in fact semisimple, so that every object is a direct sum of irreducibles, and we will describe those irreducible objects. Finally, we will construct the fiber functor and deduce Theorem 1 from the "Tannakian formalism". Before proceeding to this proof, we describe its significance for the Hecke operations.

## Hecke eigensheaves

Recall the Hecke stack $\mathscr{H}_n$ having two projections:

$$\mathrm{Bun}_G \xleftarrow{\overleftarrow{h}_n} \mathscr{H}_n \xrightarrow{\overrightarrow{h}_n} X^n \times \mathrm{Bun}_G$$

described as the stack parametrizing pairs of $G$-torsors on $X$ with an isomorphism away from $n$ points of $X$. It is a bundle over $\mathrm{Bun}_G$ with fiber $\mathrm{Gr}_{G,X^n}$, the relative grassmannian introduced last time. As we did for the $\mathrm{Gr}_n$-bundle $\widetilde{\mathrm{Gr}}_{2n}$ over $\mathrm{Gr}_n$ in the appendix to last week's notes, we may construct a twisted product of $\mathcal{M} \in \mathbf{D\text{-mod}}(\mathrm{Bun}_G)$ and $\mathcal{F} \in \mathbf{Sph_n}$, giving a D-module $\mathcal{M} \widetilde{\boxtimes} \mathcal{F}$ on $\mathscr{H}_n$. Then we may define

$$H_n^{\mathcal{F}}(\mathcal{M}) = (\overrightarrow{h}_n)_!(\mathcal{M} \widetilde{\boxtimes} \mathcal{F}).$$

Thus, we have an action of the monoidal category $\mathbf{Sph_n}$ sending $\mathbf{D\text{-mod}}(\mathrm{Bun}_G)$ to $\mathbf{D\text{-mod}}(X^n \times \mathrm{Bun}_G)$. Individually, each action is associative for the convolution product on the former; taken together, they are compatible with the factorizable structure in the following sense:

- For any partition $p$ of $n$ into $m$ parts, and denoting $\Delta_*$ the pushforward along the diagonal determined by $p$, we have for $\mathcal{M} \in \mathbf{Sph_m}$ an isomorphism in $\mathbf{D\text{-mod}}(X^n \times \mathrm{Bun}_G)$:

$$H^{\Delta_* \mathcal{F}}(\mathcal{M}) \cong \Delta_* H^{\mathcal{F}}(\mathcal{M}).$$

These isomorphisms are of course compatible with refinement of $p$.

- Suppose for simplicity that $p$ is the partition $n = n_1 + n_2$, determining a complement of divisors $X^n_{n_1,n_2} \subset X^n$. Recall the category $\mathbf{Sph_{n_1,n_2}}$ of $(G(\widehat{\mathcal{O}})_{n_1} \times G(\widehat{\mathcal{O}})_{n_2})$-equivariant D-modules on $\mathrm{Gr}_n |X^n_{n_1,n_2}$. There are two additional actions

$$\mathbf{Sph_{n_1}} \times \mathbf{Sph_{n_2}} \times \mathbf{D\text{-}mod}(\mathrm{Bun}_G) \to \mathbf{D\text{-}mod}(X^n \times \mathrm{Bun}_G)$$

$$\mathbf{Sph_{n_1,n_2}} \times \mathbf{D\text{-}mod}(\mathrm{Bun}_G) \to \mathbf{D\text{-}mod}(X^n_{n_1,n_2} \times \mathrm{Bun}_G)$$

  defined similarly to the action of $\mathbf{Sph_n}$, and all three actions agree after restriction to $X^n_{n_1,n_2}$ on the right and using the factorization maps

$$\mathbf{Sph_n} \to \mathbf{Sph_{n_1,n_2}} \leftarrow \mathbf{Sph_{n_1}} \times \mathbf{Sph_{n_2}}.$$

- $S_n$ acts on $\mathbf{Sph_n}$ via the equivariance of $\mathrm{Gr}_n$, and acts on $\mathbf{D\text{-}mod}(X^n \times \mathrm{Bun}_G)$ via the natural equivariance of $X^n$; the action map is then $S_n$-equivariant for these structures.

Let $\mathcal{E}$ be an $^LG$-local system on $X$; i.e. an $^LG$-torsor on $X$ with a connection or, equivalently, a crystal of $^LG$-torsors over $X$. We will associate to it a functor $\mathbf{Rep_n}(^LG) \to \mathbf{D\text{-}mod}(X^n)$ as follows. Recall that an $^LG$-torsor is the same as an exact tensor functor $\mathbf{Rep}(^LG) \to \mathbf{Vect}$, and so an $^LG$-local system on $X$ sends $\mathbf{Rep_1}(^LG)$ to $\mathbf{D\text{-}mod}(X)$.

Now let $\mathcal{F} \in \mathbf{Rep_2}(^LG)$; here is how to get the associated D-module $\mathcal{F}_{\mathcal{E}} \in \mathbf{D\text{-}mod}(X^2)$.

- On $X^2 \backslash \Delta$, we define $\mathcal{F}_{\mathcal{E}}$ to be the bundle with fiber $\mathcal{F}$ associated to the $(^LG)^2$-torsor $\mathcal{E}^2 = \mathrm{pr}_1^* \mathcal{E} \times \mathrm{pr}_2^* \mathcal{E}$, where by definition of $\mathbf{Rep_2}(^LG)$, there is an action of this group on $\mathcal{F}$ over $X^2 \setminus \Delta$.

- Now let $U$ be a tubular neighborhood of $\Delta$ (in the analytic topology; we leave to the imagination the algebraic analogue of this construction), and $c \colon U \to X$ the contraction map. We define $\mathcal{F}_{\mathcal{E}}|_U$ to be the bundle with fiber $\mathcal{F}|_U$ associated to the $^LG$-torsor $c^* \mathcal{E}$, which makes sense since $^LG$ acts on $\mathcal{F}$ on all of $X^2$.

- On $U \cap X^2 \setminus \Delta$, $\mathcal{E}^2$ is the induction of $c^* \mathcal{E}$ from $^LG$ to $(^LG)^2$ along the diagonal inclusion, since the cross-section of $U$ about $\Delta$ is contractible. Thus, we have a natural isomorphism of $\mathcal{F}_{\mathcal{E}}|_{X^2 \backslash \Delta}$ with $\mathcal{F}_{\mathcal{E}}|_U$ on the intersection, and we glue.

Likewise, but in a more complicated way, we may twist any object of $\mathbf{Rep_n}(^LG)$ by an $^LG$-local system. This construction has the important property that it is a *monoidal* functor $\mathbf{Rep_n}(^LG) \to \mathbf{D\text{-}mod}(X^n)$ and is also compatible with the structures of factorizable categories on both sides. Employing the Satake equivalence, each $^LG$-local system induces a factorizable monoidal functor

$$\mathrm{Sat}_{\mathcal{E},n} \colon \mathbf{Sph_n} \to \mathbf{D\text{-}mod}(X^n).$$

Now we make the following definition.

**Definition 2.** Let $\mathcal{E}$ be a $^LG$-local system. A *Hecke eigensheaf* with eigenvalue $\mathcal{E}$ is a D-module $\mathcal{M} \in \mathbf{D\text{-}mod}(\mathrm{Bun}_G)$ together with natural isomorphisms

$$H_n^{\mathcal{F}}(\mathcal{M}) \cong \mathrm{Sat}_{\mathcal{E},n}(\mathcal{F}) \boxtimes \mathcal{M}$$

with the following properties:

- The isomorphisms are compatible with the monoidal structure: we have a commutative diagram

$$
\begin{array}{ccc}
H_n^{\mathcal{F}_1 * \mathcal{F}_2}(\mathcal{M}) & \longrightarrow & \mathrm{Sat}_{\mathcal{E},n}(\mathcal{F}_1 * \mathcal{F}_2) \boxtimes \mathcal{M} \\
\downarrow & & \downarrow \\
H_n^{\mathcal{F}_1} H_n^{\mathcal{F}_2}(\mathcal{M}) & \longrightarrow & (\mathrm{Sat}_{\mathcal{E},n}(\mathcal{F}_1) \otimes \mathrm{Sat}_{\mathcal{E},n}(\mathcal{F}_2)) \boxtimes \mathcal{M}
\end{array}
$$

and another one expressing the trivial action of the unit objects in **Sph_n**. This works as well in the derived category of spherical sheaves, but for D-modules themselves it is an application of the fusion product to the factorizable structure:

- For any partition $p$ with corresponding diagonal $\Delta$, this isomorphism is compatible with application of $\Delta_*$ to both sides.

- For a partition $n = n_1 + n_2$ with corresponding open subset $X^n_{n_1,n_2} \subset X^n$, there is a functor $\mathrm{Sat}_{\mathcal{E},n_1,n_2}$, analogous to $\mathrm{Sat}_{\mathcal{E},n}$, on **Sph_{n_1,n_2}**, and isomorphisms on $X^n_{n_1,n_2}$ filling in the bottom rows of commutative diagrams involving $\mathcal{F} \in \mathbf{Sph_n}$, $\mathcal{F}_i \in \mathbf{Sph_{n_i}}$,

$$
\begin{array}{ccc}
H^{\mathcal{F}}_n(\mathcal{M})|X^n_{n_1,n_2} & \longrightarrow & \mathrm{Sat}_{\mathcal{E},n}(\mathcal{F})|X^n_{n_1,n_2} \boxtimes \mathcal{M} \\
\downarrow & & \downarrow \\
H^{\mathcal{F}|X^n_{n_1,n_2}}_n(\mathcal{M}) & \longrightarrow & \mathrm{Sat}_{\mathcal{E},n_1,n_2}(\mathcal{F}|X^n_{n_1,n_2}) \boxtimes \mathcal{M}
\end{array}
$$

- The isomorphisms are compatible with the $S_n$-equivariance structures up to the sign in the Satake equivalence.

Now we proceed to the proof of Theorem 1 (for $n = 0$ only).

# Structure of the Hecke category

We begin by stating the rigidity of the convolution product. Let $1 \in \mathrm{Gr}_G$ be the image of $G(\widehat{\mathcal{O}}) \subset G(\widehat{\mathcal{K}})$, and thus obviously a closed orbit of $G(\widehat{\mathcal{O}})$, and in the following definition we denote by $\mathrm{inv} \colon G(\widehat{\mathcal{K}}) \to G(\widehat{\mathcal{K}})$ the inversion morphism.

**Definition 3.** The *delta function* $\delta_1$ is $1_*(\mathbb{C})$, the skyscraper sheaf (D-module) supported at 1. For $\mathcal{F} \in \mathbf{Sph}$, its *contragradient* $\mathcal{F}^\vee$ is defined by the equation $q^* \mathcal{F}^\vee = \mathbb{D} \, \mathrm{inv}^*(q^* \mathcal{F})$. Note that the right $G(\widehat{\mathcal{O}})$-equivariance of $q^* \mathcal{F}$ induces right equivariance of $q^* \mathcal{F}^\vee$, so that it does indeed descend along $q$, justifying the notation and the definition.

We leave to the appendix the verification that these are, indeed, identity and dual objects in **Sph** with respect to the convolution product, meaning that for any $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathbf{Sph}$, we have

$$
\delta_1 * \mathcal{F} \cong \mathcal{F} \cong \mathcal{F} * \delta_1 \qquad\qquad \mathrm{Hom}(\mathcal{G} * \mathcal{F}, \mathcal{H}) \cong \mathrm{Hom}(\mathcal{G}, \mathcal{F}^\vee * \mathcal{H}).
$$

We turn now to the question of desribing the irreducible objects in **Sph**, in preparation both for proving that it is semisimple, and for describing the structure of $^L G$. By general principles, every such object $\mathcal{F}$ is of the form $j_{!*}\mathcal{L}$, where $j$ is the inclusion of a smooth, locally closed subspace of $\mathrm{Gr}_G$ and $\mathcal{L}$ is an irreducible, locally free $D$-module on it. Since for us, $\mathcal{F}$ must be $G(\widehat{\mathcal{O}})$-equivariant, the support of $\mathcal{L}$ is a union of $G(\widehat{\mathcal{O}})$-orbits. However, in $\mathrm{Gr}_G$, every finite-dimensional subscheme has only finitely many orbits in it, so that one of them must be open in the support of $\mathcal{L}$; we may assume, therefore, that $\mathcal{L}$ is supported on a single orbit of $G(\widehat{\mathcal{O}})$.

Further analysis of the irreducibles requires more discussion of the orbits themselves. These have a nice description based on the representation theory of $G$. We simply assert the properties we will need; in them, we fix for the first time a maximal torus $T$ of $G$ and a Borel subgroup $B$ containing $T$, with $W$ the Weyl group of $G$ with respect to $T$. As usual, $X_*(T)$ and $X^*(T)$ are the coweights and weights of $G$. We specify simple roots $\alpha_i$ and coroots $\check{\alpha}_i$, and let $2\rho$ be the sum of the $\alpha_i$. We denote the partial ordering on $X_*(T)$ corresponding to the $\check{\alpha}_i$ by $\leq$.

**Proposition 4.** There is a bijection between the orbits of $G(\widehat{\mathcal{O}})$ in $\mathrm{Gr}_G$ and the set of $W$-orbits in $X_*(T)$, and each is denoted $\mathrm{Gr}^{\check{\lambda}}_G$, where $\check{\lambda}$ is the dominant weight in each orbit. They have the following properties:

1. Their dimensions are $\dim \mathrm{Gr}_G^{\check\lambda} = \langle 2\rho, \check\lambda \rangle$. $\mathrm{Gr}_G^{\check\lambda}$ contains the image in $\mathrm{Gr}_G$ of $t^{\check\lambda}$, where identifying $\widehat{\mathcal{K}} \cong \mathbb{C}((t))$ and $\check\lambda \colon \mathbf{G_m} \to G$, we have $t^{\check\lambda} = \check\lambda(t) \in G(\widehat{\mathcal{K}})$.

2. We have $\overline{\mathrm{Gr}}_G^{\check\lambda} = \bigcup_{\check\mu < \check\lambda} \mathrm{Gr}_G^{\check\mu}$ (these are not smooth).

3. The stabilizer of $t^{\check\lambda}$ in $G(\widehat{\mathcal{O}})$ is connected.

4. $q^{-1}\overline{\mathrm{Gr}}_G^{\check\lambda}$ admits the following representation-theoretic description: $g \in q^{-1}\overline{\mathrm{Gr}}_G^{\check\lambda}$ if and only if for any dominant weight $\mu$ and its highest-weight representation $V^\mu$, and for any $\vec{v} \in V^\mu \otimes \widehat{\mathcal{O}}$, we have

$$t^{\langle \mu, \check\lambda \rangle} g(\vec{v} \otimes 1_{\widehat{\mathcal{K}}}) \subset V^\mu \otimes \widehat{\mathcal{O}}.$$

The category of $G(\widehat{\mathcal{O}})$-equivariant D-modules on any $G(\widehat{\mathcal{O}})$-orbit is equivalent to the category of representations of the group of connected components of the stabilizer of any point, which by (3) is trivial. Thus, there is a unique $G(\widehat{\mathcal{O}})$-equivariant, irreducible local system on $\mathrm{Gr}_G^{\check\lambda}$. It follows from this and the comments preceding the proposition that every irreducible object of **Sph** is of the form

$$\mathcal{J}(\check\lambda) = j_{!*}\mathcal{O}$$

where $j$ is the inclusion of $\mathrm{Gr}_G^{\check\lambda}$ and $\mathcal{O}$ is the trivial D-module on it. Its support is the singular, finite-dimensional space $\overline{\mathrm{Gr}}_G^{\check\lambda}$. Note that what we have denoted $\delta_1$ is also $\mathcal{J}(0)$. By point (2) of the above proposition, the connected components of $\mathrm{Gr}_G$ are identified with $X_*(T)/\Lambda_r = \pi_1(G)$, where $\Lambda_r$ is the coroot lattice of $G$; the identity component is that all of whose $G(\widehat{\mathcal{O}})$-orbits are indexed by $\Lambda_r$.

Finally, we state the following structure theorem, whose proof is deferred to the appendix.

**Proposition 5.** **Sph** is semisimple.

Based on this, the following proposition makes sense; it is easy to verify by direct computation in $G(\widehat{\mathcal{K}})$:

**Proposition 6.** The product $\mathcal{J}(\check\lambda) * \mathcal{J}(\check\mu)$ is a sum of $\mathcal{J}(\check\nu)$'s with $\check\nu \leq \check\lambda + \check\mu$ and $\check\lambda + \check\nu$ appearing exactly once.

# The fiber functor and weights

We have already shown that **Sph** formally resembles the category of representations of a reductive group: it is a rigid tensor category which is also a semisimple abelian category. According to the Tannakian duality theorem, in order to actually produce such a reductive group all we need is a faithful, exact tensor functor $F \colon \mathbf{Sph} \to \mathbf{Vect}$. We will produce this using the cohomology functors and $\mathrm{Gr}_T$. The central diagram here is:

$$\mathrm{Gr}_G \xleftarrow{b} \mathrm{Gr}_B \xrightarrow{t} \mathrm{Gr}_T \tag{1}$$

in which the arrows are induced by the inclusion $B \to G$ and projection $B \to T$ (we do not ever consider $T$ as a subset of $G$ for this purpose). We will, effectively, take $F$ to be $t_! b^*$, with some important technical modifications. The first is to describe the target.

**Lemma 7.** Let $\mathbf{Sph}_T$ be the Hecke category for $T$. Then there is a natural equivalence of $\mathbf{Sph}_T$ with the category of graded vector spaces $\mathbf{Vect}^{X_*(T)}$, sending convolution to tensor product.

*Proof.* Topologically, $\mathrm{Gr}_T \cong X_*(T)$ as a discrete group, so a D-module is identified merely with its component on each $\mathrm{Gr}_T^{\check\lambda}$ for $\check\lambda \in X_*(T)$ (since $T$ is a torus, every weight is dominant), which is identified with just a vector space (this is equally true for $T(\widehat{\mathcal{O}})$-equivariant D-modules, since we have already classified those supported on a single orbit).

For convolution, the action of $T(\widehat{\mathcal{O}})$ on $\mathrm{Gr}_T$ is trivial since $T$ is commutative, so $\mathrm{Conv}_T \cong \mathrm{Gr}_T \times \mathrm{Gr}_T$ and $\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2 = \mathcal{F}_1 \boxtimes \mathcal{F}_2$ on it. Given that $\mathrm{Gr}_T = X_*(T)$, this box product is just the tensor product, and the multiplication map is the identity on each component. $\square$

There is a technicality associated to the grading here which is related to the way the fiber functor is defined and we will need to impose a supersymmetry on the tensor product in $\mathbf{Sph}_T$ in order to make things work out, but that is for later.

Observe that $t$ induces a bijection between connected components of $\mathrm{Gr}_B$ and of $\mathrm{Gr}_T$, so that the former are again indexed by $X_*(T)$. For any coweight $\check{\lambda}$, let $S^{\check{\lambda}}$ be the corresponding component. It is obvious that $b$ is an injection of topological spaces, so $S^{\check{\lambda}}$ is a subset of $\mathrm{Gr}_G$. These have the following properties:

**Proposition 8.** $S^{\check{\lambda}}$ is an orbit for $N(\widehat{\mathcal{K}})$, where $N$ is the unipotent radical of $B$, and is characterized by containing $t^{\check{\lambda}}$. In addition:

1. $S^{\check{\lambda}}$ has infinite dimension, but for any dominant $\check{\mu}$, each irreducible component of $S^{\check{\lambda}} \cap \mathrm{Gr}_G^{\check{\mu}}$ has dimension $\langle \rho, \check{\lambda} + \check{\mu} \rangle$, with the intersection empty if and only if the number is negative or larger than $\dim \mathrm{Gr}_G^{\check{\mu}}$ (see Proposition 4(1)). The intersection $S^{\check{\lambda}} \cap \mathrm{Gr}_G^{\check{\lambda}}$ is open and dense in the latter, and $S^{w_0(\check{\lambda})} \cap \mathrm{Gr}_G^{\check{\lambda}}$ is a single point.

2. We have $\overline{S}^{\check{\lambda}} \subset \overline{S}^{\check{\mu}}$ if and only if $\check{\lambda} \leq \check{\mu}$, and $S^{\check{\mu}} = \overline{S}^{\check{\mu}} \setminus \bigcup_{\check{\lambda} < \check{\mu}} S^{\check{\lambda}}$.

3. The $\overline{S}^{\check{\lambda}}$ have the following representation-theoretic description: for any dominant $\mu$ and denoting by $\ell^\mu$ the highest-weight line in $V^\mu$, we have $g \in q^{-1}\overline{S}^{\check{\lambda}}$ if and only if for all $\mu$, we have

$$t^{\langle \check{\lambda}, -\mu \rangle} g\big(\ell^\mu \otimes 1_{G(\widehat{\mathcal{K}})}\big) \subset V^\mu \otimes \widehat{\mathcal{K}}.$$

**Definition 9.** For any $\mathcal{F} \in \mathbf{Sph}$, let

$$F(\mathcal{F})^{\check{\lambda}} = F(\mathcal{F})\Big|_{\mathrm{Gr}_T^{\check{\lambda}}} = H^{\langle 2\rho, \check{\lambda} \rangle}\left(t_! b^*(\mathcal{F})\Big|_{\mathrm{Gr}_T^{\check{\lambda}}}\right).$$

Then $F \colon \mathbf{Sph} \to \mathbf{Sph}_T$ is the *fiber functor*.

**Proposition 10.** $F$ is an exact monoidal functor $F \colon \mathbf{Sph} \to \mathbf{Sph}_T$, compatible with the braidings on each side. Furthermore, each $F(\mathcal{F})^{\check{\lambda}}$ is the *top* cohomology of the corresponding complex.

*Proof.* That $F$ is exact is tautological since $\mathbf{Sph}$ is semisimple and $F$ is additive (maps in a semisimple category are pure linear algebra, which $F$ preserves).

To show that $F$ is a tensor functor, we generalize the fiber functor slightly. Replacing the diagram (1) by its analogue with $\mathrm{Gr}_{G,X^n}$, etc. in place of $\mathrm{Gr}_G$, we define "fiber functors"

$$F_n \colon \mathrm{Sph}_{\mathrm{G,n}} \to \mathrm{Sph}_{\mathrm{T,n}}$$

in the obvious notation, with the same form as Definition 9. Some care must be devoted to the proof that they preserve D-modules, but this is true.

These functors are compatible with the factorizable structures on both categories in that (in the specific cases relevant for us)

$$F_2(\mathcal{F})|_\Delta[-1] \cong F_1(\mathcal{F}|_\Delta[-1]) \qquad F_2|_{X^2 \setminus \Delta}\Big((\mathcal{F}_1 \boxtimes \mathcal{F}_2)|_{X^2 \setminus \Delta}\Big) = (F_1(\mathcal{F}_1) \boxtimes F_2(\mathcal{F}_2))|_{X^2 \setminus \Delta}.$$

where $\mathcal{F} \in \mathbf{Sph_2}$ and $\mathcal{F}_i \in \mathbf{Sph_1}$. These two expressions are connected by the fusion product of $\mathcal{F}_1$ and $\mathcal{F}_2$, where

$$\mathcal{F}_1 * \mathcal{F}_2 = \Delta^* j_{!*}(\mathcal{F}_1 \boxtimes \mathcal{F}_2)|_{X^2 \setminus \Delta}[-1].$$

To show that $F_1$ (and hence $F$, by restriction to a single point of $x$) is compatible with $*$, therefore, it suffices to prove the assertion

$$F_2\Big(j_{!*}(\mathcal{F}_1 \boxtimes \mathcal{F}_2)|_{X^2 \setminus \Delta}\Big) = j_{!*}F_2|_{X^2 \setminus \Delta}\Big(\mathcal{F}_1 \boxtimes \mathcal{F}_2)|_{X^2 \setminus \Delta}\Big)$$

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and restrict to $\Delta$. This claim can be established either by showing that the sheaf on the left is acyclic about $\Delta$ (as the outer convolution was when we proved the fusion product in the previous lecture notes, but the same proof does not apply) or by directly showing that $\Delta^* F_2(\dots)[-1]$ and $\Delta^! F_2(\dots)[1]$ are D-modules, which characterizes the minimal extension. The former can be justified by introducing a more general form of acyclicity which $F_2$ does preserve, and the latter by direct computation on the convolution grassmannian $\widetilde{\mathrm{Gr}}_2$.

By Proposition 5, it suffices to show the last claim when $\mathcal{F} = \mathcal{J}(\check{\mu})$ for some $\check{\mu}$. Since $S^{\check{\lambda}}$ is an $N(\widehat{\mathcal{K}})$-orbit, it is ind-affine, and its intersection $I$ with $\overline{\mathrm{Gr}}_G^{\check{\mu}}$ is a closed subspace, hence also affine. If $b^{\check{\lambda}}$ is the inclusion of $I$ in $\mathrm{Gr}_G^{\check{\mu}}$, then $(b^{\check{\lambda}})^*[-\operatorname{codim} I]$ is right-exact, and since $I$ is affine, $(t|_I)_![\dim I]$ is right-exact. Thus, $t_! b^*[\dim I - \operatorname{codim} I]$ is right-exact on $\mathrm{Gr}_G^{\check{\lambda}}$. Here $\dim I - \operatorname{codim} I = 2 \dim I - \dim \mathrm{Gr}_G^{\check{\mu}} = \langle 2\rho, \check{\lambda} \rangle$ by Proposition 8(1). $\qquad\square$

Because we define $F$ by ignoring the degree of the cohomology, the commutativity constraint in $\mathbf{Sph}_T$ must be modified by a sign in order for the functor of Lemma 7 to preserve the braiding.

$F$ does not depend on the choice of $B$, since those are permuted by conjugation by $G$, and $G \subset G(\widehat{\mathcal{O}})$, with respect to which $\mathbf{Sph}$ is equivariant. It is also obviously faithful: if $\check{\lambda}$ is dominant, then $w_0(\check{\lambda})$ is anti-dominant and in particular, it follows from Proposition 8(1) that $S^{w_0(\check{\lambda})} \cap \overline{\mathrm{Gr}}_G^{\check{\lambda}}$ is just one point and is contained in $\mathrm{Gr}_G^{\check{\lambda}}$. By definition of $\mathcal{J}(\check{\lambda})$, it has a nonzero stalk at any point there, so its cohomology $F(\mathcal{J}(\check{\lambda}))^{w_0(\check{\lambda})}$ does not vanish.

We conclude the following theorem from the Tannakian formalism:

**Proposition 11.** $F$ induces an equivalence of tensor abelian categories between $\mathbf{Sph}$ and $\mathbf{Rep}(\check{G})'$ for some reductive group $\check{G}$, where the prime means the modified commutativity constraint on tensor products of vector spaces. $\check{G}$ has a maximal torus $\check{T} \cong {}^L T$, where $X^*({}^L T) = X_*(T)$, and a Borel subgroup $\check{B}$ containing $\check{T}$ with respect to which the dominant coweights of $G$ are identified with the dominant weights of $\check{G}$.

That the dominant weights of $\check{G}$ are the dominant coweights of $G$ follows from Proposition 6 and a classical criterion on Borel subgroups: to specify a Borel subgroup of a reductive group $H$ is the same as to specify a highest-weight line in each irreducible representation of $H$, such that these lines are compatible with tensor product (these are the "Plücker relations"). Here, that line is $F(\mathcal{J}(\check{\lambda}))^{\check{\lambda}}$, as justified by the computation showing that $F$ is faithful and Proposition 6, together with the fact that $F$ is a tensor functor.

To identify the maximal torus of $\check{B}$, we apply the following description: if $H$ is a reductive group and $C$ a Borel subgroup specified as above, and if the forgetful functor $\mathbf{Rep}(H) \to \mathbf{Vect}$ factors (as a tensor functor) through $\mathbf{Vect}^\Lambda$ for a finitely-generated "weight" lattice $\Lambda$, then the torus $S$ with $X^*(S) = \Lambda$ is a closed subgroup of $H$ if and only if every weight occurs, is contained in $C$ if and only if the weights of the "highest-weight lines" specified for $C$ are compatible with tensor products, and is maximal in $C$ if and only if those weights have multiplicity one in their corresponding representation. By construction, this is true with $\Lambda = X_*(T) = X^*({}^L T)$.

# Identifying the dual group

In this section we describe the structure of the dual group $\check{G}$ and show that it is isomorphic to ${}^L G$. We already know it is reductive. To see that it is connected, by the Tannakian formalism it suffices to show that the abelian (not monoidal) category generated by finitely many objects of $\mathbf{Sph}$ is not closed under convolution (such a tensor category would be the representations of a finite quotient group of $\mathbf{Sph}$). We may assume the objects are of the form $\mathcal{J}(\check{\lambda})$, in which case their convolutions all contain summands of the form $\mathcal{J}(\sum \check{\lambda}_i)$ by Proposition 6, whereas their direct sums only contain the $\check{\lambda}_i$ individually. It remains only to identify the root systems. Our strategy for doing this will rely on the following group-theoretic lemma:

**Lemma 12.** Let $H$ and $H'$ be reductive groups with maximal tori $U, U'$ that are isomorphic; suppose further that $C$ and $C'$ are Borel subgroups containing these tori, the choice of which identifies the dominant weights

in $X^*(U)$ with those in $X^*(U')$ under this isomorphism. Suppose that for every simple root $\alpha$ of $H$, with corresponding Levi factor $L$ (whose only simple root is $\alpha$), there is a commutative diagram

$$
\begin{array}{ccc}
L & \longrightarrow & H' \\
\uparrow & & \uparrow \\
U & =\!\!= & U'
\end{array}
\tag{2}
$$

Then there is a unique isomorphism $H \to H'$ extending these maps.

To prove this, we need an even smaller lemma on algebraic groups, which proves itself.

**Lemma 13.** Let $K, L$ be reductive groups with maximal tori $S, U$. Let $f\colon L \to H$ be an algebraic group homomorphism such that $f|_S$ is an isomorphism of $S$ with $U$. Let $\alpha$ be a root of $L$ and in the Lie algebra $\mathfrak{l}$, let $v$ be a weight vector for the adjoint action of $L$, with weight $\alpha \in X^*(U) = X^*(S)$. Then $df(v)$ is a weight vector with weight $\alpha$ for the adjoint action of $K$ on $\mathfrak{k}$, so $\alpha$ is a root of $K$. $\qquad\square$

*Proof of Lemma 12.* We apply Lemma 13 to $H'$ and $L$, concluding that $\alpha$ is a root of $H'$ for any simple root $\alpha$ of $H$. The collection of all the $\alpha$ determine the set of dominant weights in $X^*(U) = X^*(U')$ as those weights $\lambda$ such that $\langle \lambda, \check{\alpha} \rangle \geq 0$ for every $\alpha$ which is a simple root of $H$. But that means that $\{\alpha\}$ determines the Weyl chamber of weights corresponding to $C'$, and therefore to the basis determined by $C'$. Since $\{\alpha\}$ is a basis for the weight lattice, it is in fact the basis for the root sytem corresponding to $C'$.

Thus, $H$ and $H'$ have the same simple roots; we claim that they have the same coroots as well. Indeed, each Levi $L$ corresponding to $\alpha$ has $\check{\alpha}$ as a simple coroot, and the map $L \to H'$ sends $\check{\alpha}$ to some coroot of $H'$ (which is of course equal to $\check{\alpha}$, since the tori in $L$ and $H'$ are identified). By definition of the simple reflections, $\check{\alpha}$ is negated by $s_\alpha$, which means that it is a multiple of the coroot $\check{\beta}$ of $H'$ dual to $\alpha$, and since $\langle \alpha, \check{\alpha} \rangle = 2 = \langle \alpha, \check{\beta} \rangle$, that multiple must be 1.

Thus, $H$ and $H'$ have the same weights, the same coweights, and inside them the same roots and coroots, with the simple roots identified and the simple coroots identified. Since a reductive group is determined by this "root data" up to isomorphism, there is an isomorphism of $H$ with $H'$ identifying the Levi factors for the simple roots. This isomorphism is unique, since in fact any automorphism of a reductive group which fixes a maximal torus is a diagram automorphism, determined only by its induced automorphism of the Dynkin diagram, which by the assumption that it fixes the simple roots, is the identity map. $\qquad\square$

By Proposition 11, we have already established all the ingredients necessary to apply Lemma 12 other than the maps of Levi factors. To obtain these, we make a further generalization of (1). Let $P$ be any parabolic subgroup of $G$ and $L$ its Levi quotient, yielding a diagram

$$
\mathrm{Gr}_G \overset{p}{\leftarrow} \mathrm{Gr}_P \overset{l}{\to} \mathrm{Gr}_L
\tag{3}
$$

with respect to which we may define a fiber functor $F_L$ (not depending on the containing $P$) in the same way as $F$ (one must be careful about which degree to take). The relevant properties are:

**Proposition 14.** Each $F_P$ is a monoidal functor from **Sph** to $\mathbf{Sph}_L$. Furthermore, if $P_1 \subset P_2$ there is a natural factorization of $F_{P_1}$ through $F_{P_2}$.

*Proof.* The first statement is proven in precisely the same way as for $F$, and the second is a simple argument based on diagram (3). $\qquad\square$

By Tannakian formalism, this means that $F_P$ induces maps $\check{L} \to \check{G}$ which factor through each other according to the inclusions of the Levi quotients $L$. In particular, they all contain the maximal torus $\check{T} = {}^L T$ of $\check{G}$, so that diagram (2) commutes. Let $\alpha$ be a simple root of $G$, $\check{\alpha}$ its dual simple coroot, hence a simple root of ${}^L G$, and let $M_\alpha$ and ${}^L M_\alpha$ be the corresponding Levi factors, which have semisimple rank 1. In order to apply Lemma 12, we must show that $\check{M}_\alpha \cong {}^L M_\alpha$, for which the following lemma suffices:

**Lemma 15.** When $G$ has semisimple rank 1, then so does the dual group $\check{G}$; the roots of $\check{G}$ in $X_*(T) = X^*(^L T)$ are the coroots of $G$. Therefore $\check{G} \cong {}^L G$.

*Proof.* The D-module $\mathcal{J}(\check{\lambda})$ corresponds to a highest-weight representation whose highest weight is in the root lattice if and only if its weight space of weight 0 is nonempty, and this certainly can only occur if $\mathrm{Gr}_G^{\check{\lambda}}$ is in the connected component, so that $\check{\lambda}$ is in the coroot lattice of $G$. Thus, $\check{G}$ has semisimple rank at most 1. Its rank is not zero, because (for example) $\mathcal{J}(\check{\alpha})$ has at least two nonzero weight spaces, of weights $\check{\alpha}$ and $-\check{\alpha}$, and is irreducible (of course, it actually has the weight 0 as well).

To identify the root lattices precisely, we consider first the groups $\mathrm{SL}_2$ and $\mathrm{PGL}_2$. The map $\mathrm{SL}_2 \to \mathrm{PGL}_2$ induces a map $\mathrm{Gr}_{\mathrm{SL}_2} \to \mathrm{Gr}_{\mathrm{PGL}_2}$ identifying the former with the connected component of the identity in the latter. The corresponding map $\mathrm{SL}_2(\widehat{\mathcal{O}}) \to \mathrm{PGL}_2(\widehat{\mathcal{O}})$ is again surjective (the one with $\widehat{\mathcal{K}}$-coordinates is not, though) and if $\mathcal{F} \in \mathbf{Sph}_{\mathrm{SL}_2}$, then the equivariance of $\mathcal{F}$ for $\mathrm{SL}_2(\widehat{\mathcal{O}})$ is trivial on the kernel of this map, because it is central. Therefore, any D-module which is supported on the connected component and invariant for the former is also *a fortiori* invariant for the latter. There also exist on $\mathrm{Gr}_{\mathrm{PGL}_2}$ elements of $\mathbf{Sph}$ which are not supported on the connected component, so that the inclusion $\mathbf{Sph}_{\mathrm{SL}_2} \to \mathbf{Sph}_{\mathrm{PGL}_2}$ induces a map $\mathrm{P\check{G}L}_2 \to \mathrm{\check{S}L}_2$ whose corresponding map on weights, in the other direction, has index 2. Since it must send the simple root of one to that of the other, we find that the only possibility is that the simple root of the former is 2 and of the latter is 1 (identifying their weight spaces each with $\mathbb{Z}$), which are exactly the simple coroots of $\mathrm{PGL}_2$ and $\mathrm{SL}_2$, respectively.

Now let $G$ be any group of semisimple rank 1 and let $\overline{G} = G/Z(G)^0$ be the semisimple quotient. Then the induced map $\mathrm{Gr}_G \to \mathrm{Gr}_{\overline{G}}$ identifies the former with the product of the latter and $\mathrm{Gr}_{Z(G)^0}$, and we identify $\mathrm{Gr}_{\overline{G}}$ as a subset of $\mathrm{Gr}_G$ via the zero section. Since $G \to \overline{G}$ is the quotient by a central subgroup, the structure of $\overline{G}(\widehat{\mathcal{O}})$-equivariance on a D-module on $\mathrm{Gr}_{\overline{G}}$ is equivalent to a $G(\widehat{\mathcal{O}})$-equivariance structure, so we identify $\mathbf{Sph}_{\overline{G}} \subset \mathbf{Sph}_G$ as those objects supported on $\mathrm{Gr}_{\overline{G}}$. As in the previous paragraph, this inclusion is a tensor functor and so induces a map $\check{G} \to \check{\overline{G}}$.

The inclusion of $\mathrm{Gr}_{\overline{G}}$ in $\mathrm{Gr}_G$ identifies the weight lattice of $\check{\overline{G}}$ with a sublattice of that of $\check{G}$ and the map $\check{G} \to \check{\overline{G}}$ identifies the roots of $\check{\overline{G}}$ with some (hence all, by rank 1) of those of $\check{G}$. Since $\overline{G}$ is semisimple of rank 1, the previous paragraph applies, and those roots are exactly the coroots of $\overline{G}$. But the quotient $G \to \overline{G}$ identifies the coroots of $G$ with those of $\overline{G}$, so the roots of $\check{G}$ are the coroots of $G$, as desired. $\qquad\square$

This completes the proof of the geometric Satake equivalence.

# Appendix: rigid tensor structure

In this appendix we prove that $\mathbf{Sph}$ is indeed a rigid tensor category with the identity and dual objects defined in the main text.

**Proposition 16.** The delta function is an identity for the convolution $*$ and the contragradient of $\mathcal{F}$ is its dual, in the sense that there is a natural isomorphism of functors

$$\mathrm{Hom}(\mathcal{F} * \mathcal{G}, \delta_1) \cong \mathrm{Hom}(\mathcal{G}, \mathcal{F}^\vee).$$

Furthermore, $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$, so that $\mathbf{Sph}$ is a rigid tensor category.

*Proof.* That $\delta_1$ is an identity follows easily from the definition, so we consider only the contragradient. It is clear that $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$:

$$(\mathcal{F}^\vee)^\vee = \mathbb{D}\,\mathrm{inv}^*\,\mathbb{D}\,\mathrm{inv}^*\,\mathcal{F} \cong \mathbb{D}\,\mathrm{inv}^*\,\mathrm{inv}^!\,\mathbb{D}\mathcal{F} \cong \mathbb{D}^2\mathcal{F} \cong \mathcal{F}$$

where $\mathrm{inv}^* \cong \mathrm{inv}^!$ since $\mathrm{inv}$ is an isomorphism, and where $\mathrm{inv}^2 = \mathrm{id}$ by definition. Finally, we prove that it is a dual for the convolution product. Recall that convolution is defined via the diagram

$$\mathrm{Conv}_G = G(\widehat{\mathcal{K}}) \times_{G(\widehat{\mathcal{O}})} \mathrm{Gr}_G \xrightarrow{\overline{m}} \mathrm{Gr}_G$$

and let $\pi\colon G(\widehat{\mathcal{K}}) \times G(\widehat{\mathcal{K}}) \to \mathrm{Conv}_G$ be the quotient map. The inverse image $\overline{m}^{-1}(1)$ is identified with the embedding $i\colon \mathrm{Gr}_G \to \mathrm{Conv}_G$ sending $q(g)$ to $(q(g), q(g)^{-1})$ for $g \in G(\widehat{\mathcal{K}})$, which is well-defined after applying $\pi$. It is a section of the projection map $\mathrm{pr}\colon \mathrm{Conv}_G \to \mathrm{Gr}_G$. By definition, for $\mathcal{F} \in \mathbf{Sph}$, $q^* i^* \widetilde{\mathcal{F}} = \mathrm{inv}^*(q^* \mathcal{F})$, so $\mathcal{F}^\vee = \mathbb{D} i^* \widetilde{\mathcal{F}}$. Then:

$$\mathrm{Hom}(\mathcal{G} * \mathcal{F}, \delta_1) = \mathrm{Hom}(m_*(\mathrm{pr}^* \mathcal{G} \otimes \widetilde{\mathcal{F}}), \delta_1) = \mathrm{Hom}(\mathrm{pr}^* \mathcal{G} \otimes \widetilde{\mathcal{F}}, m^! \delta_1) = \mathrm{Hom}(\mathrm{pr}^* \mathcal{G}, \mathcal{H}om(\widetilde{\mathcal{F}}, m^! \delta_1)).$$

Since $\delta_1$ is supported on 1, we have $m^! \delta_1 = i_* \mathcal{D}$, where $\mathcal{D} = m^! \mathbb{C}$ is the dualizing sheaf on the image of $i$. Then

$$\mathcal{H}om(\widetilde{\mathcal{F}}, m^! \delta_1) = \mathcal{H}om(\widetilde{\mathcal{F}}, i_* \mathcal{D}) = i_* \mathcal{H}om(i^* \widetilde{\mathcal{F}}, \mathcal{D}) = i_* \mathbb{D} i^* \widetilde{\mathcal{F}} = i_* \mathcal{F}^\vee$$

where we insert $i_*$ because, technically, the computation is on $\mathrm{Conv}_G$ and not on $\mathrm{Gr}_G$. Thus, finally,

$$\mathrm{Hom}(\mathcal{G} * \mathcal{F}, \delta_1) = \mathrm{Hom}(\mathrm{pr}^* \mathcal{G}, i_* \mathcal{F}^\vee) = \mathrm{Hom}(i^* \mathrm{pr}^* \mathcal{G}, \mathcal{F}^\vee) = \mathrm{Hom}(\mathcal{G}, \mathcal{F}^\vee)$$

since $i$ is a section of pr. $\qquad\square$

# Appendix: semisimplicity

In this appendix we prove that $\mathbf{Sph}$ is semisimple. The proof proceeds by an incremental analysis of the properties of convolutions $\mathcal{J}(\check{\lambda}) * \mathcal{J}(\check{\mu})$ of irreducibles in $\mathbf{Sph}$, beginning with the following basic observation:

**Lemma 17.** We have $\mathrm{Ext}^1(\mathcal{J}(\check{\lambda}), \mathcal{J}(\check{\lambda})) = 0$; i.e. there are no nontrivial extensions of $\mathcal{J}(\check{\lambda})$ by itself.

*Proof.* Let $\mathcal{F}$ be such an extension, and denote by $j\colon \mathrm{Gr}_G^{\check{\lambda}} \to \overline{\mathrm{Gr}}_G^{\check{\lambda}}$ the inclusion map, $i$ the inclusion of the complement. By definition, $\mathcal{J}(\check{\lambda}) = j_{!*}(\mathcal{O})$, so we have

$$H^0(i^* \mathcal{J}(\check{\lambda})) = 0 = H^0(i^! \mathcal{J}(\check{\lambda})).$$

We take the sequence $0 \to \mathcal{J}(\check{\lambda}) \to \mathcal{F} \to \mathcal{J}(\check{\lambda}) \to 0$ and apply $i^{*,!}$ to it, obtaining long exact sequences of cohomology living below and above degree 0, respectively, with the degree zero terms reading

$$\cdots \to H^0(i^* \mathcal{J}(\check{\lambda})) \to H^0(i^* \mathcal{F}) \to H^0(i^* \mathcal{J}(\check{\lambda}) \to 0$$
$$0 \to H^0(i^! \mathcal{J}(\check{\lambda})) \to H^0(i^! \mathcal{F}) \to H^0(i^* \mathcal{J}(\check{\lambda})) \to \cdots$$

and concluding that

$$H^0(i^* \mathcal{F}) = 0 = H^0(i^! \mathcal{F}).$$

This property uniquely characterizes $\mathcal{F} \cong j_{!*}(j^* \mathcal{F})$. However, $j^* \mathcal{F}$ is an extension of $\mathcal{O}$ by itself on $\mathrm{Gr}_G^{\check{\lambda}}$, and therefore $j^* \mathcal{F} = \mathcal{O} \oplus \mathcal{O}$. Applying $j_{!*}$, $\mathcal{F} = j_{!*}(\mathcal{O}) \oplus j_{!*}(\mathcal{O}) = \mathcal{J}(\check{\lambda}) \oplus \mathcal{J}(\check{\lambda})$ as well, as desired. $\qquad\square$

The properties of the minimal extension used in the above proof easily give another lemma as well:

**Lemma 18.** Let $\mathcal{F} \in \mathbf{Sph}$ have composition factors $\mathcal{J}(\check{\lambda}_i)$ for various dominant coweights $\check{\lambda}_i$; then the orbits $\mathrm{Gr}_G^{\check{\lambda}_i}$ are precisely those such that (denoting by $j$ their inclusions into $\mathrm{Gr}_G$) we have $H^0(j^* \mathcal{F}) \neq 0$. In particular, $\mathcal{F}$ has a factor supported on $\overline{\mathrm{Gr}}_G^{\check{\nu}}$ if and only if for $i\colon \{t^{\check{\nu}}\} \to \mathrm{Gr}_G$, we have

$$H^0(i^* \mathcal{F}[-\dim \mathrm{Gr}_G^{\check{\nu}}]) \neq 0. \tag{4}$$

*Proof.* The first statement, as noted, follows formally from the properties of $j_{!*}$. For the second, we know that the sheaf $H^0(j^* \mathcal{F})$ is $G(\widehat{\mathcal{O}})$-equivariant and therefore constant on $\mathrm{Gr}_G^{\check{\nu}}$, so vanishes if and only if its stalk at $t^{\check{\nu}}$ does. Accounting for the dimension Proposition 4(1), this gives (4). $\qquad\square$

We also state a technical lemma which can be proved by general reasoning. Here, to be precise, we use the convention on cohomological degrees that when $f\colon S \to \mathrm{pt}$ a proper scheme and $\mathcal{F}$ a holonomic D-module on $S$, $H^i(f_* \mathcal{F})$ vanishes for $|i| > \dim S$.

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**Lemma 19.** Let $Y$ be a proper scheme of dimension $d$, $f\colon Y \to \mathrm{pt}$ the structure map, and $A^\bullet$ a complex of (holonomic) D-modules on $Y$ such that the zeroth cohomology sheaf of $A^\bullet[-d]$ is generically nonzero. Then $H^0(f_*A^\bullet) \neq 0$ as well.

We begin to analyze the composition factors of a convolution. First, for any subvariety $V$ of $\mathrm{Gr}_G$ and dominant $\check\mu$, we define $V * \mathrm{Gr}_G^{\check\mu} \subset \mathrm{Conv}_G$ so that if $\pi\colon G(\widehat{\mathcal{K}}) \times G(\widehat{\mathcal{K}}) \to \mathrm{Conv}_G$ and $q\colon G(\widehat{\mathcal{K}}) \to \mathrm{Gr}_G$, we have

$$\pi^{-1}(V * \mathrm{Gr}_G^{\check\mu}) = q^{-1}(V) \times q^{-1}(\mathrm{Gr}_G^{\check\mu}).$$

In particular, we write

$$\mathrm{Conv}_G^{\check\lambda,\check\mu} = \mathrm{Gr}_G^{\check\lambda} * \mathrm{Gr}_G^{\check\mu}.$$

We recall the maps $\mathrm{pr}, m\colon \mathrm{Conv}_G \to \mathrm{Gr}_G$, the latter descending the multiplication map along $\pi$. Finally, to save space, we set

$$l = \dim \mathrm{Gr}_G^{\check\lambda} \qquad\qquad m = \dim \mathrm{Gr}_G^{\check\mu} \qquad\qquad n = \dim \mathrm{Gr}_G^{\check\nu}$$

when we introduce the latter orbit.

**Lemma 20.** $\mathrm{Conv}_G^{\check\lambda,\check\mu}$ is smooth and irreducible of dimension $l + m$.

*Proof.* To prove this, we use the convolution grassmannian $\widetilde{\mathrm{Gr}}_2$ over $X^2$, which is a $\mathrm{Gr}_1$-bundle over $\mathrm{Gr}_1$ and whose restriction to $\Delta$ is, at every point of $X$, isomorphic to $\mathrm{Conv}_G$. For convenience we take $X$ to be small enough that $\mathrm{Gr}_1 \cong \mathrm{Gr}_G \times X$, and let $\mathrm{Gr}_1^{\check\lambda}$ be the extension of the orbit $\mathrm{Gr}_G^{\check\lambda}$ along this product. Since these are $G(\widehat{\mathcal{O}})$-stable, there is a twisted product

$$\mathrm{Gr}_1^{\check\lambda} * \mathrm{Gr}_1^{\check\mu} = \widetilde{\mathrm{Gr}}_2^{\check\lambda,\check\mu} \subset \widetilde{\mathrm{Gr}}_2$$

which is a $\mathrm{Gr}_1^{\check\mu}$-bundle over $\mathrm{Gr}_1^{\check\lambda}$ and whose restriction to $\Delta$ is, at every point of $X$, isomorphic to $\mathrm{Conv}_G^{\check\lambda,\check\mu}$. In particular, $\mathrm{Conv}_G^{\check\lambda,\check\mu}$ is a $\mathrm{Gr}_G^{\check\mu}$-bundle over $\mathrm{Gr}_G^{\check\lambda}$, hence irreducible of dimension $l + m$. $\qquad\square$

Now we recall the definition

$$\mathcal{J}(\check\lambda) * \mathcal{J}(\check\mu) = m_*(\mathcal{J}(\check\lambda) \widetilde{\boxtimes} \mathcal{J}(\check\mu)),$$

where

$$\pi^*(\mathcal{J}(\check\lambda) \widetilde{\boxtimes} \mathcal{J}(\check\mu)) = q^*\mathcal{J}(\check\lambda) \boxtimes q^*\mathcal{J}(\check\mu).$$

Therefore, if $j$ is the inclusion of $\mathrm{Conv}_G^{\check\lambda,\check\mu}$ in $\mathrm{Conv}_G$, we have

$$\mathcal{J}(\check\lambda) \widetilde{\boxtimes} \mathcal{J}(\check\mu) = j_{!*}(\mathcal{L})$$

where $\mathcal{L}$ is a local system (= locally free D-module) on $\mathrm{Conv}_G^{\check\lambda,\check\mu}$.

**Lemma 21.** For any dominant $\check\nu$, let $F = m^{-1}(t^{\check\nu}) \cap \mathrm{Conv}_G^{\check\lambda,\check\mu}$. Then

$$\dim F \leq \frac{1}{2}(l + m - n), \tag{5}$$

with equality if and only if $\mathcal{J}(\check\nu)$ is a composition factor of $\mathcal{J}(\check\lambda) * \mathcal{J}(\check\mu)$.

*Proof.* Note that the extremal case of (5) is equivalent, using Proposition 4(1) and Lemma 20, to:

$$\mathrm{codim}\, F - \dim F = n.$$

For brevity, let $\mathcal{F} = \mathcal{J}(\check\lambda) \widetilde{\boxtimes} \mathcal{J}(\check\mu)$. To evaluate (4) for $m_*(\mathcal{F})$, we apply the proper base change theorem and then Lemma 19. Since $\mathcal{F}|_{\overline{F}}[-\mathrm{codim}\, F]$ lives in sharply nonpositive cohomological degrees on $\overline{F}$, we conclude that the above equation is the precise condition necessary for Lemma 18 to apply. If the left side were decreased, then Lemma 19 would produce positive-degree cohomology sheaves of $i^*m_*(\mathcal{F})[-n]$ and therefore of $m_*(\mathcal{F})$ and, finally, of $\mathcal{J}(\check\lambda) * \mathcal{J}(\check\mu)$, in contradiction to the fact that this is a D-module. This gives the inequality of (5). $\qquad\square$

At this point it would be desirable to insert a more elementary proof, based on the preceding results, of the following lemma:

**Lemma 22.** Every convolution $\mathcal{J}(\check{\lambda}) * \mathcal{J}(\check{\mu})$ is a direct sum of irreducibles.

*Proof.* We have $\mathcal{J}(\check{\lambda}) * \mathcal{J}(\check{\mu}) = m_*(\mathcal{F})$ in the notation of the previous proof, where $\mathcal{F} = j_{!*}(\mathcal{L})$ with $\mathcal{L}$ a local system of rank 1 on $\mathrm{Conv}_G^{\check{\lambda},\check{\nu}}$. Therefore $\mathcal{F}$ is simple of "geometric origin" and so the decomposition theorem applies to its direct image under $m$, proving the lemma. $\square$

The preceding results allow us to prove the analogue for D-modules of the following proposition, true for representations of any reductive group $H$; in combination with the rigidity of the convolution product and Lemma 22, this provides a slick demonstration of the semisimplicity of **Sph**.

**Lemma.** If $\lambda$ is a dominant weight in the root lattice of $H$, then there is some $\mu$ such that $V^\lambda$ is a direct summand of $(V^\mu)^\vee \otimes V^\mu$.

**Lemma 23.** Let $\check{\lambda}$ be dominant and a sum of simple roots. Then there exists a $\check{\mu}$ such that $\mathcal{J}(\check{\lambda})$ is a direct summand of $\mathcal{J}(\check{\mu})^\vee * \mathcal{J}(\check{\mu})$ (in fact, this is true for any coweight $\check{\mu}$ such that $\check{\mu} - \check{\lambda}$ is dominant).

*Proof.* Since the convolution is semisimple, this is equivalent to

$$0 \neq \mathrm{Hom}(\mathcal{J}(\check{\lambda}), \mathcal{J}(\check{\mu})^\vee * \mathcal{J}(\check{\mu})) = \mathrm{Hom}(\mathcal{J}(\check{\mu}) * \mathcal{J}(\check{\lambda}), \mathcal{J}(\check{\mu}))$$

and therefore to finding a copy of $\mathcal{J}(\check{\mu})$ as a summand of $\mathcal{J}(\check{\mu}) * \mathcal{J}(\check{\lambda})$, for some $\mu$. We apply the criterion of Lemma 21 and proceed by locating an irreducible component of the fiber of $\mathrm{Conv}_G^{\check{\mu},\check{\lambda}}$ over $t^{\check{\mu}}$ with dimension at least (hence equal to) $l/2$.

To do so, we find such a component in a more amenable subspace. We claim that for $\check{\mu}$ sufficiently large, we have

$$m^{-1}(t^{\check{\mu}}) \cap (S^{\check{\mu}} * \mathrm{Gr}_G^{\check{\lambda}}) \subset m^{-1}(t^{\check{\mu}}) \cap (\mathrm{Gr}_G^{\check{\mu}} * \mathrm{Gr}_G^{\check{\lambda}}).$$

Granting this, the following additional equality is obtained by multiplying by $t^{\check{\mu}}$

$$m^{-1}(1) \cap (S^{\check{0}} * \mathrm{Gr}_G^{\check{\lambda}}) = m^{-1}(t^{\check{\mu}}) \cap (S^{\check{\mu}} * \mathrm{Gr}_G^{\check{\lambda}})$$

and identifying $m^{-1}(1) \cong \mathrm{Gr}_G$ via pr, the former is identified with $S^{\check{0}} \cap \mathrm{Gr}_G^{\check{\lambda}}$, which has pure dimension $l/2$ since $\check{\lambda}$ is in the coroot lattice, as desired.

Thus, we need only prove the claim; since $(\mathrm{pr}, m)$ identifies $m^{-1}(t^{\check{\mu}}) \cong \mathrm{Gr}_G$, we need only identify the first coordinates. We pull back via $\pi$ and consider pairs $(g, h)$ with $g \in q^{-1}(S^{\check{\mu}}) = N(\widehat{\mathcal{K}})t^{\check{\mu}}$ and $h \in q^{-1}(\mathrm{Gr}_G^{\check{\lambda}})$ such that $gh \in t^{\check{\mu}}G(\widehat{\mathcal{O}})$, or

$$g \in t^{\check{\mu}}G(\widehat{\mathcal{O}})t^{-\check{\lambda}}G(\widehat{\mathcal{O}}) \cap N(\widehat{\mathcal{K}})t^{\check{\mu}}G(\widehat{\mathcal{O}}).$$

We claim that for $\check{\mu}$ sufficiently large, this implies that $g \in q^{-1}(\mathrm{Gr}_G^{\check{\mu}})$, for which we apply Proposition 4(4). It suffices to show that $t^{\langle \omega, \check{\mu} \rangle}g(e^\nu) \in V^\omega \otimes \widehat{\mathcal{O}}$ for any weight vector $e^\nu$ in any highest-weight representation $V^\omega$ of $G$, and with $g \in t^{\check{\mu}}G(\widehat{\mathcal{O}})t^{-\check{\lambda}}$. Since $g$ is in $N(\widehat{\mathcal{K}})t^{\check{\mu}}G(\widehat{\mathcal{O}})$, we have

$$g(e^\nu) \in \sum_{\nu \leq \nu'} \widehat{\mathcal{K}}e^{\nu'}$$

with $g(e^{w(\omega)}) = t^{\langle w(\omega), \check{\mu} \rangle}e^{w(\omega)} + \dots$ for any $w \in W$. On the other hand, we have

$$t^{-\check{\lambda}}(e^\nu) = t^{\langle \nu, -\check{\lambda} \rangle}e^\nu$$

and so for $u \in G(\widehat{\mathcal{O}})$ with $u(e^\nu) = \sum_{\nu'} u_{\nu,\nu'}e^{\nu'}$ $(u_{\nu,\nu'} \in \widehat{\mathcal{O}})$, if $g = t^{\check{\mu}}ut^{-\check{\lambda}}$, we have

$$g(e^\nu) = \sum_{\nu'} t^{\langle \nu', \check{\mu} \rangle - \langle \nu, \check{\lambda} \rangle}u_{\nu,\nu'}e^{\nu'}.$$

By virtue of the previous expression we may assume $\nu' \geq \nu$, so the exponent is at least

$$\langle \nu, \tilde{\mu} - \check{\lambda} \rangle.$$

If $\tilde{\mu} - \check{\lambda}$ is dominant, this in turn is minimized when $\nu = w_0(\omega)$, when it is equal to $\langle w_0(\omega), \tilde{\mu} \rangle = -\langle \omega, \tilde{\mu} \rangle$, as desired. $\qquad\square$

Now we have all the ingredients to prove the semisimplicity of **Sph**.

*Proof of Proposition 5.* It suffices to show that there are no nontrivial extensions of the irreducible objects, so we must show that $\mathrm{Ext}^1(\mathcal{J}(\check{\lambda}), \mathcal{J}(\tilde{\mu})) = 0$ always. Since $\mathrm{Ext}^1$ is a derived functor of Hom, by the properties of the dual we have

$$\mathrm{Ext}^1(\mathcal{J}(\check{\lambda}), \mathcal{J}(\tilde{\mu})) = \mathrm{Ext}^1(\delta_1, \mathcal{J}(\check{\lambda})^\vee * \mathcal{J}(\tilde{\mu})).$$

By Lemma 22, latter D-module is semisimple, so we may assume it is just of the form $\mathcal{J}(\check{\lambda})$. If $\check{\lambda} \notin \Lambda_r$, then $\mathcal{J}(\check{\lambda})$ and $\delta_1$ are supported on different connected components of $\mathrm{Gr}_G$, so of course have no nontrivial extensions. Otherwise, Lemma 23 applies and it suffices to replace the right-hand side with $\mathcal{J}(\tilde{\mu})^\vee * \mathcal{J}(\tilde{\mu})$. Then:

$$\mathrm{Ext}^1(\delta_1, \mathcal{J}(\tilde{\mu})^\vee * \mathcal{J}(\tilde{\mu})) = \mathrm{Ext}^1(\mathcal{J}(\tilde{\mu}), \mathcal{J}(\tilde{\mu})) = 0,$$

by Lemma 17. $\qquad\square$