The Hecke category
(part I—factorizable structure)

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In this lecture and the next, we will describe the "Hecke category", namely, the thing which acts on D-modules on Bun_G and with respect to which action the notion of Hecke eigensheaves is defined. In fact, almost none of this content actually concerns Bun_G, so before we move into talking about something apparently completely different, we will give a general description of the goal and indicate why the context must change. Throughout this lecture, our D-modules are assumed to be holonomic.

The Hecke stack; motivation

Back in the very first lecture, Dennis described some particular examples of Hecke functors for $\mathrm{Bun}_n = \mathrm{Bun}_{GL_n}(X)$ (as always, X is the smooth projective curve we are using). They all concerned diagrams

$$\mathrm{Bun}_n \xleftarrow{\overleftarrow{h}_x^{\mathrm{std}}} \mathscr{H}_x^{\mathrm{std}} \xrightarrow{\overrightarrow{h}_x^{\mathrm{std}}} \mathrm{Bun}_n$$

where $x \in X(\mathbb{C})$ and the middle object is a stack

$$\mathscr{H}_x^{\mathrm{std}}(S) = \left\{ (\mathcal{V}_1, \mathcal{V}_1, \alpha) \,\middle|\, \begin{array}{c} \mathcal{V}_i \text{ are vector bundles of rank } n \text{ on } X_S = S \times X \\ \alpha\colon \mathcal{V}_1 \to \mathcal{V}_2 \text{ is an injective map of coherent sheaves} \\ \operatorname{coker}(\alpha) \text{ is flat over } S \text{, supported on } \{x\}_S \text{, and has length 1} \end{array} \right\}.$$

He also gave other examples of possible "Hecke stacks" with progressively elaborate conditions on α, and defined corresponding "Hecke functors"

$$H_x^{\mathrm{std}}\colon \mathbf{D}\text{-}\mathbf{mod}(\mathrm{Bun}_n) \to \mathbf{D}\text{-}\mathbf{mod}(\mathrm{Bun}_n), \quad H_x^{\mathrm{std}}(\mathcal{F}) = (\overrightarrow{h}_x^{\mathrm{std}})_!(\overleftarrow{h}_x^{\mathrm{std}})^*\mathcal{F}[n-1]$$

(or with other shifts, for the other stacks). He then indicated that we would need to consider the *category of Hecke functors* in order to properly state the eigensheaf condition. Given the above, seemingly ad-hoc description, it would appear impossible to give a reasonable description of this category. In fact, however, such a description exists and is very naturally given in terms of the affine grassmannian Gr_G, which we will review and generalize in this lecture. Recall that for any group G, the affine grassmannian is the functor, defined on affine schemes $S = \operatorname{Spec} R$ as:

$$\mathrm{Gr}_G(S) = \left\{ (\mathcal{T}, t) \,\middle|\, \begin{array}{c} \mathcal{T} \text{ is a } G\text{-torsor on the formal disk } \operatorname{Spec} R[[z]] = \mathbb{D}_S \\ t\colon \mathcal{T}^0 \to \mathcal{T} \text{ is a trivialization of } \mathcal{T} \text{ on } \mathbb{D}_S \setminus \{0\}_S = \mathbb{D}_S^\circ \end{array} \right\} \tag{1}$$

and that the group ind-scheme $G(\widehat{\mathcal{K}}_x)$ acts on it by changing the trivialization. ($\{0\}_S$ means the closed subscheme $S \subset \mathbb{D}_S$ corresponding to $z = 0$.) Here and hereafter, $\widehat{\mathcal{K}}_x$ is the complete local field of X at x, the fraction field of the completed local ring $\widehat{\mathcal{O}}_x$, and we will often identify $\widehat{\mathcal{O}}_x \cong \mathbb{C}[[z]]$ and $\widehat{\mathcal{K}}_x \cong \mathbb{C}((z))$

by choosing a uniformizing parameter z near x; when this happens, we will just write $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{O}}$. In this notation, then,

$$\mathrm{Gr}_G = G(\widehat{\mathcal{K}})/G(\widehat{\mathcal{O}}). \tag{2}$$

For any group G, not just GL_n, we define the *Hecke stack at* x, where we simply let α be anything at all so long as it is an isomorphism away from x.

$$\mathscr{H}_x(S) = \left\{ (\mathcal{T}_1, \mathcal{T}_2, \alpha) \,\middle|\, \begin{array}{c} \mathcal{T}_i \text{ are } G\text{-torsors on } X_S \\ \alpha\colon \mathcal{T}_1 \to \mathcal{T}_2 \text{ is an isomorphism on } X_S \setminus \{x\}_S \end{array} \right\}.$$

There is again a convolution diagram

$$\mathrm{Bun}_G \xleftarrow{\overleftarrow{h}_x} \mathscr{H}_x \xrightarrow{\overrightarrow{h}_x} \mathrm{Bun}_G.$$

For any point of $\mathrm{Bun}_G(\mathbb{C})$ (that is, a G-torsor $\mathcal{T}_1$ on X), the fiber of $\overleftarrow{h}$ over $\mathcal{T}_1$ is noncanonically identified with Gr_G. Indeed, if we (noncanonically) pick a trivialization of $\mathcal{T}_1$ on $\mathbb{D}$, then $\mathcal{T}_2$, restricted to $\mathbb{D}^\circ$, can vary over all possible G-torsors and α over all trivializations, since $\mathcal{T}_1$ is now trivial (this is the Beauville–Laszlo theorem, which says that we can always glue on $\mathcal{T}_1$ away from x to complete $\mathcal{T}_2$).

It is not hard to show (using this same logic) that $\mathscr{H}_x$ is actually a Gr_G-bundle over Bun_G, where the structure group is in fact $G(\widehat{\mathcal{O}}_x)$; we will return to this more precisely next time. Therefore, the $G(\widehat{\mathcal{O}}_x)$-orbits on Gr_G induce a global stratification of $\mathscr{H}_x$; it turns out that their various closures are exactly the strange Hecke stacks considered before.

Recall the definition of equivariance of a D-module with respect to the action of a group on the underlying space; in the case of $G(\widehat{\mathcal{O}}_x)$ acting on Gr_G, it means that the two pullbacks

$$G(\widehat{\mathcal{O}}_x) \times \mathrm{Gr}_G \underset{\mathrm{pr}}{\overset{a}{\rightrightarrows}} \mathrm{Gr}_G$$

are isomorphic, with the isomorphism subject to some natural conditions. Any such D-module $\mathcal{F}$ can be extended along $\mathscr{H}_x$ to a "twisted pullback" $\widetilde{\mathcal{F}}$; for $\mathcal{M} \in \textbf{D-mod}(\mathrm{Bun}_G)$, set

$$\mathcal{M} \mathbin{\widetilde{\boxtimes}} \mathcal{F} = (\overleftarrow{h}_x)^* \mathcal{M} \otimes \widetilde{\mathcal{F}} \qquad\qquad H_x^{\mathcal{F}}(\mathcal{M}) = (\overrightarrow{h}_x)_!(\mathcal{M} \mathbin{\widetilde{\boxtimes}} \mathcal{F});$$

this is the uniform definition of the Hecke functors. We see, therefore, that the *Hecke category* of Hecke functors is simply the category $\textbf{D-mod}^{G(\widehat{\mathcal{O}}_x)}(\mathrm{Gr}_G)$ or, as we will call it later, **Sph**.

One further modification is possible. If $x \in X(\mathbb{C})$ is not fixed but allowed to vary, or to multiply to several points, then there arise relative and higher Hecke stacks $\mathscr{H}_{X^n} = \mathscr{H}_n$ defined by

$$\mathscr{H}_n(S) = \left\{ (\vec{x}, \mathcal{T}_1, \mathcal{T}_2, \alpha) \,\middle|\, \begin{array}{c} \vec{x} \in X^n(S), \mathcal{T}_i \text{ are } G\text{-torsors on } X_S \\ \alpha\colon \mathcal{T}_1 \to \mathcal{T}_2 \text{ is an isomorphism on } X_S \setminus \bigcup \Gamma(x_i) \end{array} \right\}.$$

Here, $\Gamma(x_i)$ is the graph of $x_i\colon S \to X$ inside X_S. There are diagrams

$$\mathrm{Bun}_G \xleftarrow{\overleftarrow{h}_n} \mathscr{H}_n \xrightarrow{\overrightarrow{h}_n} \mathrm{Bun}_G \times X^n$$

and the fiber of $\overleftarrow{h}_n$ is something we have not seen yet but which we will introduce presently: the "factorizable" grassmannian.

The factorizable grassmannian

Recall the "global" version of Gr_G: for a fixed choice of $x \in X(\mathbb{C})$, we have:

$$\mathrm{Gr}_G(S) = \left\{ (\mathcal{T}, t) \,\middle|\, \begin{array}{c} \mathcal{T} \text{ is a } G\text{-torsor on } X_S \\ t \text{ is a trivialization of } \mathcal{T} \text{ on } X_S \setminus \{x\}_S \end{array} \right\}.$$

This has the same relationship to x as does $\mathscr{H}_x$, and the dependency problem is solved in the same way by defining unrestricted and relative versions:

$$\mathrm{Gr}_{G,X^n}(S) = \mathrm{Gr}_n(S) = \left\{ (\vec{x}, \mathcal{T}, t) \middle| \begin{array}{c} \mathcal{T} \text{ is a } G\text{-torsor on } X_S \\ t \text{ is a trivialization of } \mathcal{T} \text{ on } X_S \setminus \bigcup \Gamma(x_i) \end{array} \right\}.$$

Note that X can be any smooth curve in this definition, not necessarily complete (or, indeed, even algebraic). These are all ind-proper schemes over X^n, and they have a number of relationships comprising the *factorizable structure*:

- For $n, m \in \mathbb{N}$, let p be a partition of $[1, n]$ into m parts and Δ_p be the corresponding copy of X^m inside X^n. Then there are isomorphisms
$$\mathrm{Gr}_n|_{\Delta_p} \cong \mathrm{Gr}_m$$
which are compatible with refinement of the partition p;
- Let p be a partition as above and suppose its parts p_i have sizes n_i; let U_p be the open subset of X^n consisting of coordinates $(x_1, \ldots, x_n)$ such that if $x_i = x_j$, then i, j are in the same part of p. Then there are isomorphisms
$$\mathrm{Gr}_n|_{U_p} \cong \Big(\prod \mathrm{Gr}_{n_i}\Big)\Big|_{U_p}$$
compatible with refinement of the partition p (together with, of course, further restrictions to finer U_p's). Furthermore, these isomorphisms are compatible with those above when restricting both *to* some diagonal, and *away from* others, in either order.
- For any n, an equivariance structure for the action of the symmetric group S_n on X^n which is compatible with both of the above classes of isomorphisms.

It is possible to give a precise statement of the nature of these compatibilities, but as it provides rather little reward for the necessary work, it is in the appendix. The proofs are simple:

- We construct the factorization maps along the diagonals. If we have coordinates $x_1 = \cdots = x_{n_1}, \ldots, x_{n-n_m+1} = \cdots = x_n$, then we may set $x_{i_j} = y_i$ for all $j \in p_i$, where $\vec{y}\colon S \to X^m$ has just the distinct coordinates. Then $\bigcup_j \Gamma(x_j) = \bigcup_i \Gamma(y_i)$; since X_S is the same in both cases, the possible G-torsors on X_S are the same in both cases, so $\mathrm{Gr}_n|_{\Delta_p} = \mathrm{Gr}_m$.
- Suppose we again single out a partition, but this time, none of the maps x_i in different parts intersect. Thus, the $D_i = \bigcup_{j \in p_i} \Gamma(x_j)$ are disjoint, for $i = 1, \ldots, n$; denote by U_i the complement of all the D_j other than D_i and $V_i = X_S \setminus D_i$. Then U_i and V_i are an open cover of X_S and we may define a G-torsor $\mathcal{T}_i$ on X_S by gluing $\mathcal{T}|_{U_i}$ to the trivial torsor $\mathcal{T}^0|_{V_i}$ along the isomorphism t on $U_i \cap V_i = X_S \setminus \bigcup \Gamma(x_i)$. $\mathcal{T}_i$ has a natural trivialization t_i on $X \setminus U_i$ and the triple $((x_j)_{j \in p_i}, \mathcal{T}_i, t_i)$ is in $\mathrm{Gr}_{n_i}(S)$.

  Conversely, given such a collection, let $\mathcal{T}$ be the torsor obtained by gluing $\mathcal{T}_i|_{U_i}$ over the open cover $\{U_i\}$ of X_S, where on $U_i \cap U_j$, $\mathcal{T}_i$ is identified with $\mathcal{T}_j$ via $t_j \circ t_i^{-1}$, which obviously satisfy the cocycle condition on triple intersections. Then $\mathcal{T}$ has a natural trivialization t on $\bigcap U_i = X_S \setminus \bigcup \Gamma(x_i)$ coming from the t_i, and $(\vec{x}, \mathcal{T}, t) \in \mathrm{Gr}_n(S)$.
- For the S_n-equivariance, it is clear that in a point $(x, \mathcal{T}, \phi)$, both $\mathcal{T}$ and ϕ are independent of the order of the coordinates of x.

Just like Gr_G, there is a description of Gr_n as a quotient of some "loop group" by some "arc group", both of them now in factorizable forms. Namely, they are

$$G(\widehat{\mathcal{O}})_n(S) = \left\{ (\vec{x}, g) \middle| \vec{x} \in X^n(S), g \in G(\widehat{X}_{S,x}) \right\}$$
$$G(\widehat{\mathcal{K}})_n(S) = \left\{ (\vec{x}, g) \middle| \vec{x} \in X^n(S), g \in G\big(\widehat{X}_{S,x} \setminus \bigcup \Gamma(x_i)\big) \right\}$$

where $\widehat{X}_{S,x}$ refers to the schemy formal neighborhood of $D = \bigcup \Gamma(x_i)$,

$$\widehat{X}_{S,x} = \operatorname{Spec}_{X_S}(\widehat{\mathcal{O}}_{X_S,D}).$$

Then $\mathrm{Gr}_n = G(\widehat{\mathcal{K}})_n / G(\widehat{\mathcal{O}})_n$, and thus both groups act on Gr_n; this follows, as for the affine grassmannian, from the Beauville–Laszlo theorem. This is so similar to Gr_G that one is entitled to ask what the relationship is, and the answer is simply that Gr_1 is a Gr_G-bundle over X, where the structure group is the group $\mathrm{Aut}(\widehat{\mathcal{O}})$. Indeed, if we choose on some Zariski-open subset U of X a regular function z which is a local parameter at every point, then z identifies each $\mathcal{O}_x$ with $\widehat{\mathcal{O}} = \mathbb{C}[\![z]\!]$ and thus identifies $G(\widehat{\mathcal{O}})_1$ and $G(\widehat{\mathcal{K}})_1$ with $G(\widehat{\mathcal{K}}) \times U$ and $G(\widehat{\mathcal{O}}) \times U$, and thus their quotient with $\mathrm{Gr}_G \times U$. The transition maps are obviously given by elements of $\mathrm{Aut}(\widehat{\mathcal{O}})$. This is a useful conceptual notion, but its most practical form is that if X is, as we will take it sometimes, a small complex disk (in the analytic topology), then $\mathrm{Gr}_1 \cong \mathrm{Gr}_G \times X$.

The relative loop and arc groups $G(\widehat{\mathcal{O}})_n$ and $G(\widehat{\mathcal{K}})_n$ are factorizable in the same way as Gr_n (as made precise in the appendix).

Convolution and the geometric Satake equivalence

Now we introduce the main object of study: the Hecke category.

Definition 1. The *n'th big Hecke category*, denoted $\mathbf{Sph_n}$, is the category of *spherical*, or $G(\widehat{\mathcal{O}})_n$-equivariant D-modules on Gr_n; the regular *Hecke category* $\mathbf{Sph}$ is the category of $G(\widehat{\mathcal{O}})$-equivariant D-modules on Gr_G.

We will generally talk just about $\mathbf{Sph_1}$ and $\mathbf{Sph}$, and in the end we will state (without proof) the appropriate generalizations to $\mathbf{Sph_n}$. The most important property of these categories is that they have *convolution products*, which are obtained by certain *convolution diagrams*. The most natural way of defining convolution is to do it on $G(\widehat{\mathcal{K}})$ (or, indeed, $G(\widehat{\mathcal{K}})_n$), via the multiplication map

$$G(\widehat{\mathcal{K}}) \times G(\widehat{\mathcal{K}}) \xrightarrow{m} G(\widehat{\mathcal{K}}). \tag{3}$$

For any complexes $A^\bullet, B^\bullet$ of D-modules on $G(\widehat{\mathcal{K}})$, the formula

$$A^\bullet * B^\bullet = m_!(A^\bullet \boxtimes B^\bullet)$$

is the geometric analogue of convolution of functions, $(f_1 * f_2)(g) = \int_G f(h) g(h^{-1}g)\, dh$. Unfortunately, this definition is not amenable to analysis since $G(\widehat{\mathcal{K}})$ is so wild. But suppose that we have sheaves $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{Sph}$, and denote $q\colon G(\widehat{\mathcal{K}}) \to \mathrm{Gr}_G$; then $q^*(\mathcal{F}_1) * q^*(\mathcal{F}_2)$ can be computed on a much better space. Indeed, the $q^*\mathcal{F}_i$ are $G(\widehat{\mathcal{O}})$-equivariant on both the left and the right (which are different since G is not, in general, commutative) and thus (3), along with the objects on it, descends to the diagram:

$$\mathrm{Conv}_G = G(\widehat{\mathcal{K}}) \times_{G(\widehat{\mathcal{O}})} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G\,. \tag{4}$$

Conv_G is called the "convolution diagram". There is *one* projection $\mathrm{pr}\colon \mathrm{Conv}_G \to \mathrm{Gr}_G$; it and m are defined by the formulas (referring to definitions (2) and (1)):

$$\mathrm{pr}(g, (\mathcal{T}, t)) = g \bmod G(\widehat{\mathcal{O}}) \qquad\qquad m(g, (\mathcal{T}, t)) = (\mathcal{T}, g \cdot t).$$

These maps in fact express Conv_G as the product $\mathrm{Gr}_G \times \mathrm{Gr}_G$, but we will not want this identification. Rather, for $\mathcal{F} \in \mathbf{Sph}$, we define $\widetilde{\mathcal{F}}$ to be the descent of $\mathrm{pr}_2^*(q^*\mathcal{F})$ from the left-hand side of (3) to Conv_G, and for $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{Sph}$,

$$\mathcal{F}_1 \mathbin{\widetilde{\boxtimes}} \mathcal{F}_2 = \mathrm{pr}^* \mathcal{F}_1 \otimes \widetilde{\mathcal{F}}_2. \tag{5}$$

Then the convolution of $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{Sph}$ is

$$\mathcal{F}_1 * \mathcal{F}_2 = m_*(\mathcal{F}_1 \mathbin{\widetilde{\boxtimes}} \mathcal{F}_2). \tag{6}$$

Note that, *a priori*, this is merely a complex of D-modules and, indeed, makes sense for any equivariant complexes in the derived category. Later, we will show that it indeed sends $\mathbf{Sph} \times \mathbf{Sph}$ to $\mathbf{Sph}$.

The program established above is easily generalized to Gr_1 and to the Gr_n in general. Using the same words, the product on $G(\widehat{\mathcal{K}})_n$,

$$G(\widehat{\mathcal{K}})_n \times_{X^n} G(\widehat{\mathcal{K}})_n \xrightarrow{m} G(\widehat{\mathcal{K}})_n \tag{3'}$$

descends to the double quotient by actions of $G(\widehat{\mathcal{O}})_n$ to a map from the convolution diagram

$$G(\widehat{\mathcal{K}})_n \times_{G(\widehat{\mathcal{O}})_n} \mathrm{Gr}_n \xrightarrow{m} \mathrm{Gr}_n\,. \tag{4'}$$

The left-hand side is denoted Conv_n and admits, as before, one projection $\mathrm{pr}\colon \mathrm{Conv}_n \to \mathrm{Gr}_n$. When $n = 1$, this map is naturally identified with that of (4) over every point of X. For $\mathcal{F} \in \mathbf{Sph}_n$ (or indeed, any equivariant complex), there is a twisted pullback $\widetilde{\mathcal{F}}$ on Conv_n, and we set

$$\mathcal{F}_1 \mathbin{\widetilde{\boxtimes}} \mathcal{F}_2 = (\mathrm{pr}^* \mathcal{F}_1 \otimes \widetilde{\mathcal{F}}_2)[-n] \qquad\qquad \mathcal{F}_1 * \mathcal{F}_2 = m_*(\mathcal{F}_1 \mathbin{\widetilde{\boxtimes}} \mathcal{F}_2). \tag{5', 6'}$$

As for (6), these are merely complexes of D-modules for now; we will return later to the question of how these convolutions are related to that of (6). Returning to the ordinary grassmannian Gr_G, the theorem which is the subject of these lectures is the *geometric Satake equivalence*:

Theorem 2. *The convolution $*$ admits a commutativity constraint making* **Sph** *into a rigid tensor ("Tannakian") category. There exists a faithful, exact tensor functor* $\mathbf{Sph} \to \mathbf{Vect}$ *inducing an equivalence (modulo a sign in the commutativity constraint) of* **Sph** *with* $\mathbf{Rep}(^LG)$ *as tensor categories, where ^LG is the* Langlands dual group *of the reductive group G, whose weights are the coweights of G and vice versa.*

Once the equivalence $\mathbf{Sph} \cong \mathbf{Rep}(^LG)$ is established as categories, the convolution becomes less important, and is replaced by another form of factorizability related to convolution on the $\mathbf{Sph_n}$. We will digress from the proof in order to formulate a generalization of the above theorem.

Just as the Gr_n are factorizable, the categories $\mathbf{Sph_n}$ on them have a factorizable structure as well. Imprecisely, this structure consists of the following data:

- For any partition p of $[1, n]$ into m parts, there is a direct image functor

  $$\Delta_*\colon \mathbf{Sph_m} \to \mathbf{Sph_n}$$

  corresponding to the identification of the restriction of Gr_n along Δ_p with Gr_m. This functor is right-exact and in fact has a right adjoint $\Delta^!$ in the derived category.

- Suppose (for simplicity) that p is the partition $n = n_1 + n_2$ and that U_p is the corresponding open set. Over U_p there is a category $\mathbf{Sph_p}$ of D-modules on U_p which are equivariant with respect to the action of $(G(\widehat{\mathcal{O}})_{n_1} \times G(\widehat{\mathcal{O}})_{n_2})|_{U_p}$, and a diagram

  $$\mathbf{Sph_n} \to \mathbf{Sph_p} \leftarrow \mathbf{Sph_{n_1}} \times \mathbf{Sph_{n_2}}.$$

  Here the first map is restriction from X^n to U_p. For $\mathcal{F}_i \in \mathbf{Sph_{n_i}}$, their image under the second map is $(\mathcal{F}_1 \boxtimes \mathcal{F}_2)|_{U_p}$. As before, these maps admit right adjoints and, when $n = 2$, are actually exact.

- There is a version of the above point for finer partitions, and both of these maps are compatible with convolution in $\mathbf{Sph_n}$.

The factorizable structure of the $\mathbf{Sph_n}$ corresponds to a factorizable notion of ^LG-representation. To separate the notion from ^LG, let H be any group. If $\mathcal{F} \in \mathbf{D\text{-}mod}(X^n)$, then we say that H *acts factorizably* on $\mathcal{F}$ if for every partition p of n into m parts, there is an action of H^m on $\mathcal{F}|_{U_p}$, and these actions are consistent with refinement of p. This consistency is exemplified by the following situation: let $n = 3$, and say that p is the partition $3 = 2 + 1$ (in that order); then $U_p = X^3 \setminus (\Delta_{13} \cup \Delta_{23})$. Let q be the complete partition $3 = 1 + 1 + 1$, so that U_q is the complement of all the diagonals. On U_q, H^3 acts on $\mathcal{F}$, and on U_p, H^2 acts on $\mathcal{F}$; we require that restricted to U_q, the *first* factor of H^2 act as the diagonal of the *first two* factors of H^3, while the last factors act identically. We will denote by $\mathbf{Rep_n}(H)$ the category of such factorizable representations of H in $\mathbf{D\text{-}mod}(X^n)$.

The categories $\mathbf{Rep_n}(H)$ have the same factorizable structure as the $\mathbf{Sph_n}$: a direct image along diagonals, and restriction and product maps away from the diagonals. Finally, we can state the big Satake equivalence:

Theorem 3. *There are equivalences of categories identifying all the $\mathbf{Sph_n}$ with the $\mathbf{Rep_n}({}^L G)$; this equivalence respects their factorizable structures as well as convolution.*

We will only prove Theorem 2; Theorem 3 follows in a totally formal manner from it.

The fusion product

The convolution product (6′) is in fact a generalization of that (6) on Gr_G, at least as long as the objects being convolved are D-modules rather than complexes. The connection is via a local computation on X: suppose that X is a small complex disk with center denoted x, so that $\mathrm{Gr}_1 \cong \mathrm{Gr}_G \times X$. For $\mathcal{F} \in \mathbf{Sph}$, let

$$\mathcal{F}' = \mathrm{pr}_{\mathrm{Gr}_G}^* \mathcal{F}[1]$$

be its extension, along this product, to Gr_1. (It should be noted that the product decomposition of Gr_1 is canonical only up to the action of $\mathrm{Aut}(\widehat{\mathcal{O}})$. However, it can be shown, in a manner not depending on the fusion product, that any element of $\mathbf{Sph}$ has a unique structure of $\mathrm{Aut}(\widehat{\mathcal{O}})$-equivariance, so that in fact this does not interfere with the arguments.)

In this section, we will show that convolution on $\mathbf{Sph}$ has values again in $\mathbf{Sph}$ and that it has a natural commutativity constraint. The key is the following claim, which establishes convolution in $\mathbf{Sph}$ as a *fusion product*, so called because convolution at a point $x \in X$ is obtained via tensor product over two points $y, z \in X$ which come together (or "fuse") at x.

Lemma 4. Let $\mathcal{F}_i \in \mathbf{Sph}$, and $\mathcal{F}_i'$ their extensions as above to Gr_1. Let $j\colon X^2 \setminus \Delta \to X^2$ be the inclusion, and identify $j^* \mathrm{Gr}_2 \cong j^*(\mathrm{Gr}_1 \times \mathrm{Gr}_1)$ by factorization. Then

$$\mathcal{F}_1 * \mathcal{F}_2 = \Big(\Delta^* j_{!*} j^*(\mathcal{F}_1' \boxtimes \mathcal{F}_2')[-1]\Big)\Big|_x [-1].$$

Note that this product depends *only* on the factorization structure of Gr_2.

One of the properties of $j_{!*}$ is that if $\Delta\colon X \to X^2$ is the inclusion of the diagonal, then $\Delta^* j_{!*}(\mathcal{M})[-1]$ is a D-module for any D-module $\mathcal{M}$ (rather than, as it is *a priori*, a complex on X). This immediately implies that $\mathcal{F}_1 * \mathcal{F}_2$ is a D-module. It also gives a commutativity constraint for $*$, coming from the isomorphism

$$\mathrm{sw}^*(\mathcal{F}_1' \boxtimes \mathcal{F}_2') = \mathrm{sw}^*(\mathrm{pr}_1^* \mathcal{F}_1' \otimes \mathrm{pr}_2^* \mathcal{F}_2') = \mathrm{pr}_2^* \mathcal{F}_1' \otimes \mathrm{pr}_1^* \mathcal{F}_1' \cong \mathrm{pr}_1^* \mathcal{F}_2' \otimes \mathrm{pr}_2^* \mathcal{F}_1' = \mathcal{F}_2' \boxtimes \mathcal{F}_1'$$

where $\mathrm{sw}\colon X^2 \to X^2$ swaps the coordinates and, of course, $\mathrm{sw} \circ \Delta = \Delta$, so the above isomorphism indeed gives an isomorphism of $\mathcal{F}_1 * \mathcal{F}_2$ with $\mathcal{F}_2 * \mathcal{F}_1$. Lemma 4 shows why it is necessary to work in the abelian category $\mathbf{Sph}$, rather than the derived category in which the definitions of convolution also make sense: the operation $j_{!*}$ is only a functor on D-modules.

Thus, we need only prove Lemma 4. In order to set up the core theoretical argument, we introduce the convolution grassmannian $\widetilde{\mathrm{Gr}}_2$. Once again, we give a quick (though correct) definition here and defer a technical development to the appendix. Recalling (4′), let $\widetilde{\mathrm{Gr}}_2$ be the closed subscheme of pairs $((\vec{x}, g), (\vec{x}, \mathcal{T}, t))$ in Conv_2 with the following properties:

- As an element of $G\big(\widehat{X}_{S,x} \setminus (\Gamma(x_1) \cup \Gamma(x_2))\big)$, g extends to $\widehat{X}_{S,x} \setminus \Gamma(x_1)$;

- The trivialization t, defined on $X_S \setminus (\Gamma(x_1) \cup \Gamma(x_2))$, extends to $X_S \setminus \Gamma(x_2)$.

Both of these conditions are invariant under multiplication by $G(\widehat{\mathcal{O}})_2$, so do in fact define a subfunctor. It is evident from this definition that over $X^2 \setminus \Delta$, there is a natural identification of $\widetilde{\mathrm{Gr}}_2$ with $\mathrm{Gr}_1 \times \mathrm{Gr}_1$, and that $\widetilde{\mathrm{Gr}}_2|_\Delta \cong \mathrm{Conv}_1$. Furthermore, the map m of (4′) induces a map, likewise called m, from $\widetilde{\mathrm{Gr}}_2$ to Gr_2.

There is a "cheap" inclusion $X \times \mathrm{Gr}_1 \hookrightarrow \mathrm{Gr}_2$, sending a pair $(x, (y, \mathcal{T}, t))$ to $((x, y), \mathcal{T}, t)$; likewise, there is an inclusion of $\mathrm{Gr}_1 \times X$ in Gr_2. Using them, we construct a twisted product $\mathcal{F}_1 \,\widetilde{\boxtimes}\, \mathcal{F}_2$ for any $\mathcal{F}_i \in \mathbf{Sph_1}$ in the following way:

- Let $\mathcal{F}_i' = \mathrm{pr}_{\mathrm{Gr}_1}^* \mathcal{F}_i[1]$ on $\mathrm{Gr}_1 \times X$ and $X \times \mathrm{Gr}_1$ respectively, considered as objects of $\mathbf{Sph_2}$;

- The tensor product $(q^*\mathcal{F}_1' \boxtimes q^*\mathcal{F}_2')[-2]$ on $G(\widehat{\mathcal{K}})_2 \times_{X^2} G(\widehat{\mathcal{K}})_2$ is $G(\widehat{\mathcal{O}})_2$-biequivariant and so descends to Conv_2;

- The descended D-module $\mathcal{F}_1 \,\widetilde{\boxtimes}\, \mathcal{F}_2$ happens to live on $\widetilde{\mathrm{Gr}}_2$.

Definition 5. The *outer convolution* of $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{Sph_1}$ is $\mathcal{F}_1 *_o \mathcal{F}_2 = m_*(\mathcal{F}_1 \,\widetilde{\boxtimes}\, \mathcal{F}_2)$.

Clearly, $\mathcal{F}_1 * \mathcal{F}_2 = \Delta^*(\mathcal{F}_1 *_o \mathcal{F}_2)[-1]$, so to prove Lemma 4, it suffices to prove (going back to $\mathcal{F}_i \in \mathbf{Sph}$)

$$\mathcal{F}_1' *_o \mathcal{F}_2' = j_{!*} j^*(\mathcal{F}_1' \boxtimes \mathcal{F}_2'). \tag{7}$$

To do this, we introduce a catalyst in the form of the *unipotent nearby and vanishing cycles functors*; rather than giving a detailed discussion of them, we refer the reader to the notes [2] on Beilinson's paper [1]. Here, only the following properties are important (once again, the D-modules are holonomic):

- For any scheme Y and Cartier divisor $D \subset Y$ with open complement U, there is a functor of unipotent nearby cycles around D, $\Psi_D^{\mathrm{un}} \colon \mathbf{D\text{-}mod}(U) \to \mathbf{D\text{-}mod}(D)$, together with an endomorphism (unipotent on each $\Psi_D^{\mathrm{un}}(\mathcal{F})$) called the *monodromy*. There is likewise a functor $\Phi_D^{\mathrm{un}} \colon \mathbf{D\text{-}mod}(Y) \to \mathbf{D\text{-}mod}(D)$ of *unipotent vanishing cycles*.

- Let $j \colon U \to Y$ be the inclusion. Suppose that $\mathcal{F} \in \mathbf{D\text{-}mod}(Y)$ and that $\Psi_D^{\mathrm{un}}(j^*\mathcal{F})$ has trivial monodromy; then a necessary and sufficient condition that $\mathcal{F} \cong j_{!*}(j^*\mathcal{F})$ is that $\Phi_D^{\mathrm{un}}(\mathcal{F}) = 0$. If $\mathcal{F}$ is a free $\mathcal{O}_Y$-module, then it has both of these properties. When this happens, then $i^*\mathcal{F}[-1] = i^!\mathcal{F}[1] = \Psi_D^{\mathrm{un}}(\mathcal{F})$, where i is the inclusion of D. (This is the only one of these facts that relies on the theory from Beilinson's paper.)

- Ψ_D^{un} is *local on D* in that for any open set V and $\mathcal{F} \in \mathbf{D\text{-}mod}(U)$, we have $\Psi_D^{\mathrm{un}}(\mathcal{F})|_V \cong \Psi_D^{\mathrm{un}}(\mathcal{F}|_V)$, and this isomorphism respects the monodromy. This is likewise true for Φ_D^{un} and $\mathcal{F} \in \mathbf{D\text{-}mod}(Y)$.

- Nearby and vanishing cycles respect products, as follows: let $Z = Y \times F$, set $E = \mathrm{pr}_Y^{-1}(D)$, and let $\mathcal{F}_Y \in \mathbf{D\text{-}mod}(Y)$, $\mathcal{F}_F \in \mathbf{D\text{-}mod}(F)$. Then we have $\Psi_E^{\mathrm{un}}(\mathcal{F}_Y \boxtimes \mathcal{F}_F) \cong \Psi_D^{\mathrm{un}}(\mathcal{F}_Y) \boxtimes \mathcal{F}_F$ and likewise for Φ^{un}, and this isomorphism respects the monodromy.

- If $p \colon Z \to Y$ is a proper morphism and $E = p^{-1}(D)$, then $p \circ \Psi_E^{\mathrm{un}} \cong \Psi_D^{\mathrm{un}} \circ p$ (nearby cycles commute with proper direct image) and this isomorphism respects the monodromy. Likewise, vanishing cycles commute with proper direct image.

The glue that makes this all stick together is the following easy lemma:

Lemma 6. Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{D\text{-}mod}(\mathrm{Gr}_G)$, $\mathcal{F}_i' \in \mathbf{D\text{-}mod}(\mathrm{Gr}_1)$ their extension to Gr_1. Then $\mathcal{F}_1' \boxtimes \mathcal{F}_2'$ has no vanishing cycles and its nearby cycles have trivial monodromy.

Proof. We continue to identify $\mathrm{Gr}_1 \cong \mathrm{Gr}_G \times X$, and we write pr to mean (in this proof) the projection $\mathrm{Gr}_1 \times \mathrm{Gr}_1 \to (\mathrm{Gr}_G)^2$. Then we have

$$\mathcal{F}_1' \boxtimes \mathcal{F}_2' = \mathrm{pr}^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)[2].$$

Take $Y = X^2$, $D = \Delta$, and $F = \mathrm{Gr}_2$ in the statement that the cycles functors respect products, and let $\mathcal{F}_Y$ be $\mathcal{O}_{X^2}$ with the trivial D-module structure. Then it has no vanishing cycles or nearby-cycles monodromy; thus, the same is true of the tensor product (which, to be precise, we take to be $\mathcal{F}_F$). □

The proof of (7) is now just chaining together the above properties. It turns out (one can argue directly, or see the appendix; either way, this is analogous to the fact that $\mathrm{Conv}_G \cong \mathrm{Gr}_G \times \mathrm{Gr}_G$) that $\widetilde{\mathrm{Gr}}_2$ is locally isomorphic to $\mathrm{Gr}_1 \times \mathrm{Gr}_1$. Thus, Lemma 6 applies, so $\Phi^{\mathrm{un}}_\Delta(\mathcal{F}'_1 \mathbin{\widetilde{\boxtimes}} \mathcal{F}'_2) = 0$ and the monodromy action on $\Psi^{\mathrm{un}}_\Delta(\mathcal{F}'_1 \mathbin{\widetilde{\boxtimes}} \mathcal{F}'_2)$ is trivial. Since m is a proper map, m_* preserves these properties, so the same is true of $\mathcal{F}'_1 *_o \mathcal{F}'_2$, and the criterion for it to equal the minimal extension of its own restriction applies. To complete the proof, we note that $j^*(\mathcal{F}'_1 *_o \mathcal{F}'_2) = j^*(\mathcal{F}'_1 \boxtimes \mathcal{F}'_2)$ by the factorizability of $\widetilde{\mathrm{Gr}}_2$ away from Δ.

Appendix: the convolution grassmannian

In this appendix, we discuss the convolution grassmannian more formally. There are in fact many variations, but we only need one:

$$\widetilde{\mathrm{Gr}}_2(S) = \left\{ (x_1, x_2, \mathcal{T}_1, \mathcal{T}_2, t, \alpha) \;\middle|\; \begin{array}{c} x_i \in X(S), \mathcal{T}_i \text{ are } G\text{-torsors on } X_S \\ t \text{ is a trivialization of } \mathcal{T}_1 \text{ on } X_S \setminus \Gamma(x_1) \\ \alpha \text{ is an isomorphism } \mathcal{T}_1 \cong \mathcal{T}_2 \text{ on } X_S \setminus \Gamma(x_2) \end{array} \right\}.$$

The reason for its existence is that it admits the diagram (4′):

$$\mathrm{Gr}_1 \xleftarrow{\mathrm{pr}} \widetilde{\mathrm{Gr}}_2 \xrightarrow{m} \mathrm{Gr}_2\,.$$

Clearly, $\widetilde{\mathrm{Gr}}_2$ resembles a product of Gr_1 with itself, but that product does not admit a map such as m. The existence of m is evident from the definition of $\widetilde{\mathrm{Gr}}_2$, though: just set

$$m(x_1, x_2, \mathcal{T}_1, \mathcal{T}_2, t, \alpha) = ((x_1, x_2), \mathcal{T}_2, \alpha \circ t).$$

Likewise, pr sends such a point to $(x_1, \mathcal{T}_1, t)$. Just like the Gr_n, $\widetilde{\mathrm{Gr}}_2$ is ind-proper, hence m is a proper map.

Although it is not actually the product $\mathrm{Gr}_1 \times \mathrm{Gr}_1$, the projection map pr is in fact a Gr_1-bundle over Gr_1. To see this, we define the following functor:

$$\widetilde{G}(\widehat{\mathcal{O}})_1(S) = \left\{ (x_1, x_2, \mathcal{T}, t_1, t_2) \;\middle|\; \begin{array}{c} (x_1, \mathcal{T}, t_1) \in \mathrm{Gr}_1(S) \\ t_2 \text{ is a trivialization of } \mathcal{T} \text{ on } \widehat{X}_{S,x_2} \end{array} \right\}.$$

It is easy to see that $\mathrm{Gr}_1 \times G(\widehat{\mathcal{O}})_1$ acts, over $\mathrm{Gr}_1 \times X$, on this by altering t_2, and that this action is a torsor. The claim is that $\widetilde{\mathrm{Gr}}_2$ is the bundle associated to this torsor with fiber Gr_1. This means that there is a map:

$$\widetilde{G}(\widehat{\mathcal{O}})_1 \times_X \mathrm{Gr}_1 \to \widetilde{\mathrm{Gr}}_2. \tag{8}$$

To construct it, suppose we have a pair of points

$$(x_1, x_2, \mathcal{T}, t_1, t_2) \in \widetilde{G}(\widehat{\mathcal{O}})_1(S) \qquad\qquad (x_2, \mathcal{T}', t_3) \in \mathrm{Gr}_1(S)$$

(note the equality of X-coordinates); let $\mathcal{T}_1 = \mathcal{T}$ and $t = t_1$. The restriction of t_2 to $\widehat{X}_{S,x_2} \setminus \Gamma(x_2)$ is a trivialization of $\mathcal{T}$, and the like restriction of t_3 is a trivialization of $\mathcal{T}'$; let $\mathcal{T}_2$ be the G-torsor obtained by gluing to $\mathcal{T}_1$ the restriction of $\mathcal{T}'$ to $\widehat{X}_{S,x_2}$ along $t_3 \circ t_2^{-1}$, using the Beauville–Laszlo theorem. Then by definition, $\mathcal{T}_2$ has an isomorphism α with $\mathcal{T}_1$ away from $\Gamma(x_2)$, and so

$$(x_1, x_2, \mathcal{T}_1, \mathcal{T}_2, t, \alpha) \in \widetilde{\mathrm{Gr}}_2(S).$$

This gives the map (8). To see that it is surjective, take a point such as the one above and let $\mathcal{T}'$ be $\mathcal{T}_2$ glued, via $\alpha \circ t$, to the trivial torsor away from $\Gamma(x_1) \cup \Gamma(x_2)$, thus obtaining a trivialization t_3 away from $\Gamma(x_2)$ and a point $(x_2, \mathcal{T}', t_3) \in \mathrm{Gr}_1(S)$. As before, we take $\mathcal{T} = \mathcal{T}_1$, but it is not necessarily possible to trivialize it on $\widehat{X}_{S,x_2}$. However, since $\mathcal{T}$ is a torsor, there is an open cover of S on which such trivializations exist, and we pick one t_2 (arbitrarily) on each set U of this cover and take $t_1 = t$; then $(x_1, x_2, \mathcal{T}, t_1, t_2) \in \widetilde{G}(\widehat{\mathcal{O}})_1(U)$. Thus, (8) is surjective as a map of Zariski sheaves (let alone fppf sheaves). Finally, in the course of showing this we have already identified each fiber with $G(\widehat{X}_{U,x_2})$, as desired.

Let π be the projection onto Gr_1 from the left-hand side of (8). If $\mathcal{F} \in \mathbf{Sph_1}$, then $\pi^*(\mathcal{F})$ is $G(\widehat{\mathcal{O}})_1$-equivariant and therefore descends to a D-module $\widetilde{\mathcal{F}}$ on $\widetilde{\mathrm{Gr}}_2$; as before, for $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{Sph_1}$, we define the twisted product $\mathcal{F}_1 \mathbin{\widetilde{\boxtimes}} \mathcal{F}_2 = \mathrm{pr}^* \mathcal{F}_1 \otimes \widetilde{\mathcal{F}}_2$.

Appendix: factorizable structure

In this appendix, we give a rigorous description of the factorizable structure on the Gr_n. This requires some abstract nonsense with partitions of finite sets; thus, we introduce the additional notation: for any finite sets I and J (thought of as "index sets"), a *partition* of I into J parts is a surjection $p\colon I \to J$. We will write $p_j = p^{-1}(j)$ for the j'th part of this partition. We define two kinds of refinements $r\colon p' \to p$:

- For $p'\colon I' \to J$, a *first refinement* is a partition $r_1\colon I \to I'$ such that $p = p' \circ r_1$;
- For $p'\colon I \to J'$, a *second refinement* is a partition $r_2\colon J' \to J$ such that $p = r_2 \circ p'$.

Note the directions of the maps. Let **Part** be the category of partitions whose morphisms are generated by the refinements of both types. There is a natural bifunctor $\mathrm{Un}\colon \mathbf{Part} \times \mathbf{Part} \to \mathbf{Part}$ sending a pair of partitions $p_1\colon I_1 \to J_1$ and $p_2\colon I_2 \to J_2$ to their union $p\colon I_1 \cup I_2 \to J_1 \cup J_2$; this functor admits a natural commutativity constraint.

Let X be a scheme (it may as well be our curve). For an index set I, let $X^I = \prod_{i \in I} X$ be the unordered power of X corresponding to this finite set. For any partition $p\colon I \to J$, there is an induced closed immersion $i_p\colon X^J \to X^I$ sending x_j to the coordinates $(x_i \mid i \in p_j)$, with image Δ_p. There is also a corresponding *open* subset U_p of X^I (not its complement) consisting of all points (x_i) such that if $x_{i_1} = x_{i_2}$, then $p(i_1) = p(i_2)$; let $j_p\colon U_p \to X^I$ be the open immersion. For any partition $p'\colon I' \to J'$ and morphism $r\colon p \to p'$ in **Part**, there is a locally closed immersion $l_r\colon U_{p'} \to U_p$ defined as follows for the refinements:

- If $r = r_1$ is a first refinement, let $l_r = i_{r_1} \circ j_{p'}$, which clearly sends $U_{p'}$ into U_p;
- If $r = r_2$ is a second refinement, let $l_r = j_{r_2}$, which again clearly has image in U_p.

One should check that for any p, p', we have $U_{\mathrm{Un}(p,p')} \subset U_p \times U_{p'}$.

Let **PSch** ("schemes over partitions") be the category, fibered over **Part**, such that for any partition p, the fiber $\mathbf{PSch}_p$ is $\mathbf{Sch}/U_p$, the category of schemes over U_p, and let the cartesian morphisms (pullbacks along morphisms r) be given by restriction along l_r. There is again a bifunctor $\mathrm{Pr}\colon \mathbf{PSch} \times \mathbf{PSch} \to \mathbf{PSch}$ sending X_1/U_{p_1} and X_2/U_{p_2} to $(X_1 \times X_2)|U_{\mathrm{Un}(p_1,p_2)}$, admitting a natural commutativity constraint. If $\pi\colon \mathbf{PSch} \to \mathbf{Part}$ is the structure functor, then π identifies Pr with Un. In more usual terms, the two categories are *braided monoidal categories* and π is a braided monoidal functor.

Definition 7. An *sf-scheme* ("symmetric factorizable scheme") is a braided monoidal section functor F of π. This means:

1. We have $\pi \circ F = \mathrm{id}$ exactly (not up to isomorphism);
2. For every morphism $r\colon p' \to p$, there is an isomorphism of $r^* F(p)$ with $F(p')$ as schemes over $U_{p'}$, and these isomorphisms are functorial in r;

3. There is the additional datum of an isomorphism of functors making the square commute:

$$\begin{array}{ccc} \mathbf{PSch} \times \mathbf{PSch} & \xrightarrow{\mathrm{Pr}} & \mathbf{PSch} \\ {\scriptstyle F\times F}\uparrow & & \uparrow{\scriptstyle F} \\ \mathbf{Part} \times \mathbf{Part} & \xrightarrow[\mathrm{Un}]{} & \mathbf{Part} \end{array}$$

4. This isomorphism is required to be compatible with the commutativity constraints in the sense that if Sw is the functor swapping factors in either product category of the above diagram, then the following diagram of functors and natural transformations commutes:

$$\begin{array}{ccc} \mathrm{Pr} \circ \mathrm{Sw} & \longrightarrow & \mathrm{Pr} \\ \uparrow & & \uparrow \\ \mathrm{Un} \circ \mathrm{Sw} & \longrightarrow & \mathrm{Un} \end{array}$$

If for every index set I, having cardinality $\#I = n$, we have $F(I \to \{1\}) = \mathrm{Gr}_{G,X^n}$, then F is a *factorizable structure* on Gr_n, and in the main text we have described one such structure. The correspondence between the above properties and the ones given before is:

- The existence of factorization along diagonals Δ_p (the first factorization property) is a special case of (2) when $r = r_1$ is a first refinement and p is the trivial partition $I \to \{1\}$ with only one part (so $U_p = X^I$).

- Factorization on diagonal complements (the second factorization property) is a combination of (3) and the special case of (2) with $r = r_2$ a second refinement and p the trivial partition.

- The S_n-equivariance is special case of (2) in which p is the trivial partition and $p' = p$, so that r is an automorphism of I.

- Compatibility of the three structures above is the stipulation in (2) that the isomorphisms be functorial, together with the functoriality of Pr and the fact that π is a monoidal functor. The role of (4) is to ensure that the data of S_n-equivariance on Gr_n is compatible with the natural S_2-equivariance of a product $\mathrm{Gr}_{n_1} \times \mathrm{Gr}_{n_2}$ when both are identified on U_p (here p is the partition $n = n_1 + n_2$).

References

[1] A. A. Beilinson, *How to glue perverse sheaves*, K-theory, arithmetic and geometry (Moscow, 1984), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 42–51, available at http://math.harvard.edu/~ryanr/beilinson-how_to_glue_perverse_sheaves.pdf. MR **923134 (89b:**14028)

[2] Ryan Reich, *Notes on Beilinson's "How to glue perverse sheaves"*, available at arxiv:1002.1686.