CLASSICAL VS. GEOMETRIC LANGLANDS

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Notes by Sam Raskin.

1. Introduction

So far in the seminar there has been no discussion of the arithmetic aspects of the Langlands program which originally motivated the development of a geometric program. Therefore, we will give an introduction to the arithmetic setting and how it relates to the geometric setting we have been studying this semester.

The Langlands correspondence gives an equivalence between two very different pictures, and some things which are very difficult to resolve on one side are easily resolved on the other.

This lecture is intended to be an informal overview, so technical analytic and algebraic conditions are disregarded.

2. Class field theory

2.1.   We will begin with a discussion of local class field theory. Note that even though a local story may sometimes seem more natural, sometimes in relating the local with the global the former is made more clear.

2.2.   First, at the "0-level," let us recall that for $k = \mathbb{F}_q$, the absolute Galois group $\mathrm{Gal}(\overline{k}/k) = \widehat{\mathbb{Z}}$ and is generated by the Frobenius $x \mapsto x^q$.

2.3.   Now let us describe what the abelianized Galois group is for F a non-archimedean local field with ring of integers $\mathscr{O}_F$ a complete DVR with maximal ideal $\mathfrak{M}_F$ and residue field $\mathscr{O}_F/\mathfrak{M}_F = k = \mathbb{F}_q$. For a fixed separable closure $\overline{F}$ of F, there exists a maximal unramified extension F_{un} giving rise to $\mathrm{Gal}(\overline{F}/F) \twoheadrightarrow \mathrm{Gal}(F_{un}/F) = \mathrm{Gal}(\overline{k}/k) = \widehat{\mathbb{Z}}$. We have the isomorphism $F^\times/\mathscr{O}_F^\times \xrightarrow{\overline{\kappa}} \mathbb{Z}$ given by the valuation ν_F.

The main theorem of local class field theory says that there is a lift κ which is almost an isomorphism:

$$
\begin{array}{ccc}
F^\times & \xrightarrow{\ \ \kappa\ \ } & \mathrm{Gal}(\overline{F}/F)_{ab} \\
\downarrow & & \downarrow \\
F^\times/\mathscr{O}_F^\times & \xrightarrow{\ \ \overline{\kappa}\ \ } & \mathrm{Gal}(\overline{k}/k)
\end{array}
$$

For the same reason that $\mathbb{Z} \neq \widehat{\mathbb{Z}}$, κ is not quite an isomorphism. We will ignore such things in the future, and forget for example that $\mathbb{Z} \neq \widehat{\mathbb{Z}}$.

2.4.   As stated, this is too unpolished. We must know how to characterize κ uniquely and how to construct κ. Both of these questions admit several answers, some of which stay entirely within the local world. We will answer the first through the global picture, giving the first instance of the general remark made at the beginning of this section.

2.5.   Let K be a global field and let $\mathfrak{p}_K$ be its set of its (finite or infinite) places. For $v \in \mathfrak{p}_K$, we have the completion K_v. The adèles $\mathbb{A}_K$ are the restricted direct product $\prod_{v \in \mathfrak{p}_K}' K_v$ which comes equipped with the natural diagonal embedding $K \hookrightarrow \mathbb{A}_K$ realizing K as a discrete subring. This induces $\mathbb{G}_m(K) \hookrightarrow \mathbb{G}_m(\mathbb{A}_K)$. We denote by $\mathbb{A}_K^{int}$ the integral adèles, which are just the direct product over the finite places $\prod_{v \in \mathfrak{p}_K^{finite}} \mathscr{O}_{K_v}$. For $\mathbb{A}_K^\infty$ the product of the completions of K at its infinite places, $\mathbb{G}_m(\mathbb{A}_K^\infty)^o$ denotes the connected component of $\mathbb{G}_m(\mathbb{A}_K^\infty)$.
   Global class field theory gives an isomorphism:

$$\mathbb{G}_m(\mathbb{A}_K^{int})\mathbb{G}_m(\mathbb{A}_K^\infty)^o \backslash \mathbb{G}_m(\mathbb{A}_K)/\mathbb{G}_m(K) \xrightarrow{\kappa_K} \mathrm{Gal}(\overline{K}/K)_{ab}$$

This isomorphism is more easily characterized uniquely than its local counterpart. One requires that for every L/K a finite abelian extension and a (finite) place $v \in \mathfrak{p}_K$ unramified in this extension that the invertible adèle $(\ldots, 1, t_v, 1, \ldots)$ which is 1 away from K_v and a uniformizer there is sent to the Frobenius Fr_v at v in $\mathrm{Gal}(L/K)$. Cebotarev density immediately implies that if such an isomorphism exists, then it is unique.

2.6.   With this, let us return to the local setting. For E/F a finite abelian extension, should have $\kappa_{E/F} : F^\times \longrightarrow \mathrm{Gal}(E/F)$. Choose L/K an extension of global fields with $v \in \mathfrak{p}_K$ having a unique place $w \in \mathfrak{p}_L$ over it and such that L_w/K_v is isomorphic to E/F. Then $\kappa_{E/F}$ should make the following diagram commute:

$$
\begin{array}{ccc}
K_v^\times = F^\times & \xrightarrow{\ \ \kappa_{E/F}\ \ } & \mathrm{Gal}(E/F) \\
\downarrow & & \downarrow \simeq \\
\mathbb{G}_m(\mathbb{A}_K^{int})\backslash \mathbb{G}_m(\mathbb{A}_K)/\mathbb{G}_m(K) & \longrightarrow & \mathrm{Gal}(L/K)
\end{array}
$$

where the map $\mathrm{Gal}(E/F) \hookrightarrow \mathrm{Gal}(L/K)$ should be the realization of $\mathrm{Gal}(E/F)$ as the decomposition group at v of $\mathrm{Gal}(L/K)$. Requiring this diagram to commute evidently defines $\kappa_{E/F}$. If one can show that $\kappa_{E/F}$ is independent of the choice of L/K, this implies that one has a construction and unique characterization of the local class field theory isomorphism. One can prove this independence by formulating and proving a purely local characterization of the isomorphism.

Note that now we can formulate our problem in its entirety, even if we can't prove it yet!

Also, observe that as formulated, it is unclear that the local isomorphism

2.7. A serious problem is how to construct the maximal abelian extensions F_{ab}/F and K_{ab}/K explicitly, along with the reciprocity map. For F, this problem was solved by Lubin and Tate. For $K = \mathbb{Q}$ this is given by the Kronecker-Weber theorem and for $K = \mathbb{Q}[\sqrt{-a}]$, $(a > 0)$, this is given by the theory of complex multiplication. However, already for $\mathbb{Q}[\sqrt{2}]$ there is no clear solution. This problem is also solved geometrically when the characteristic of $K > 0$. For example, if K is the function field of a projective curve X over a finite field k, then the unramified part of the abelianized Galois group is given explicitly by $\mathrm{Pic}(X)(k)$.[1]

3. L-FUNCTIONS

3.1. Before proceeding, we need to discuss L-functions of Galois representations in the local and global settings.

3.2. First, suppose F is non-archimedean and let $\rho : \mathrm{Gal}(F_{un}/F) \longrightarrow \mathrm{Aut}(V)$ be an unramified representation for V a finite dimensional $\overline{\mathbb{Q}_\ell}$-vector space.[2] Then let $L_{\rho,F}(s) = \det(1 - q^s \rho(\mathrm{Fr}))^{-1}$.

Next, suppose F as before and let $\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{Aut}(V)$ a possibly ramified representation. Then $\mathrm{Gal}(F_{un}/F)$ acts on $V^{\mathrm{Gal}(\overline{F}/F_{un})}$, and we define $L_{\rho,F}$ to be the L-function for this unramified representation.

For F archimedean local, there is a notion of L-function which we do not discuss.

Finally, for $\rho : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Aut}(V)$, we set $L_\rho(s) = \prod_{v \in \mathfrak{p}_K} L_{\rho_v}(s)$ where ρ_v is given by composition with $\mathrm{Gal}(\overline{K_v}/K_v) \longrightarrow \mathrm{Gal}(\overline{K}/K)$.

Example 3.1. If $K = \mathbb{Q}$ and ρ is the trivial representation, then L_ρ is the Riemann ζ-function (or the completed ζ-function if one includes the infinite primes).

3.3. Many important questions in number theory can be reduced to an understanding of the analytic behaviour of $L_\rho(s)$. For example, we know of Riemann's old work relating the distribution of the prime numbers to the zeros of $\zeta(s)$. Langlands' correspondence is compelling in its description of L-functions of Galois representations.

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[1] The Lang isogeny for $\mathrm{Pic}(X)$ defines a Galois cover of $\mathrm{Pic}^0(X)$ with group $\mathrm{Pic}(X)(k)$ and so a map $\pi_1(\mathrm{Pic}^0(X)) \longrightarrow \mathrm{Pic}(X)(k)$. A choice of base-point in X gives us an Abel-Jacobi map $X \longrightarrow \mathrm{Pic}^0(X)$. Since the unramified part of the abelianized Galois group of K is given by $\pi_1(X)_{ab}$, it is clear at least how to define the map.

[2] If we were being more precise, we might worry that ℓ is equal to the residue characteristic of F.

4. Langlands' conjectures

4.1.   We restrict to the case $G = GL_n$ throughout. Let us remind the reader that we are ignoring technical aspects, so many of the statements below are untrue as stated.

4.2.   First, let us give a moral understanding of Langlands' conjecture in the local setting. There should be something of an equivalence between n-dimensional representations of $\mathrm{Gal}(\overline{F}/F)$ and irreducible representations of $GL_n(F)$. For example, in the class field theory setting $n = 1$, we saw that characters of the Galois group were equivalent to characters of $F^\times = \mathbb{G}_m(F)$, which are exactly the irreducible representations because the group is commutative.

4.3.   Now let us discuss the local equivalence in more detail. When discussing class field theory before, we proceeded from the unramified local setting to the general local setting by way of the global theory. We proceed similarly here. An irreducible representation $GL_n(F) \longrightarrow \mathrm{Aut}(W)$ of $GL_n(F)$ is *unramified* if $W^{GL_n(\mathscr{O}_F)} \neq 0$.

The unramified local equivalence tells us to expect an equivalence between unramified semisimple[3] representations of $\mathrm{Gal}(\overline{F}/F_{un})$ and irreducible unramified representations $\pi : GL_n(F) \longrightarrow \mathrm{Aut}(W)$. Such an equivalence is provided by the Satake equivalence, which we will briefly describe.

The Satake equivalence can be constructed explicitly as follows. It is clear that the data of unramified semisimple Galois representations are classified by unordered n-tuples $\overline{z} = (z_1, \ldots, z_n) \in \overline{\mathbb{Q}_\ell^\times}/S_n$, given by taking the eigenvalues of the image of Frobenius. Let $B \subset GL_n(F)$ be the Borel subgroup of upper triangular matrices and let T be the torus $B/[B, B]$ of diagonal matrices. To such $\overline{z}$, we set $\chi_{\overline{z}}$ be the character of the Borel defined by taking the character $\mathrm{diag}(a_1, \ldots, a_n) \mapsto \prod z_i^{\nu_F(a_i)}$ of T and pulling it back to B via $B \twoheadrightarrow T$. Here $\mathrm{diag}(a_1, \ldots, a_n)$ is the diagonal matrix with entries $\{a_i\}$. Then $\mathrm{Ind}_B^{GL_n(F)} \chi_{\overline{z}}$ has a unique quotient which is an irreducible unramified representation, and this is our representation π.

4.4.   Next, we will discuss the global setting. In fact, the local Langlands correspondence may be characterized in a purely local way, though the proof of the local Langlands correspondence for general n requires global methods. In any case, we will prefer to characterize it globally.

First, let us remark that any irreducible representation π of $GL_n(\mathbb{A}_K) = \prod'_{v \in \mathfrak{p}_K} GL_n(K_v)$ admits a factorization $\otimes'_{v \in \mathfrak{p}_K} \pi_v$ where π_v is unramified for all but finitely many $v \in \mathfrak{p}_K$.

We say that an irreducible representation $\pi : GL_n(\mathbb{A}_K) \to \mathrm{Aut}(W)$ is *automorphic* if there exists a $GL_n(\mathbb{A}_K)$-equivariant embedding $W \hookrightarrow L^2(GL_n(\mathbb{A}_K)/GL_n(K))$. The strong multiplicity one theorem (which is special to $G = GL_n$) says that such an embedding is unique up to scaling.

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[3]That is, where $\mathrm{Fr} \in \mathrm{Gal}(\overline{F}/F_{un})$ is sent to a semi-simple matrix.

4.5. Let $\pi = \otimes'_{v \in \mathfrak{p}_K} \pi_v$ be an automorphic representation "motivic at its infinite places" (this corresponds to modding out by $\mathbb{G}_m(\mathbb{A}_K^\infty)^o$ in Section 2.5) and let $S \subset \mathfrak{p}_K$ be a finite set such that π_v is unramified for $v \notin S$. Then the global Langlands correspondence predicts that there should exist $\rho_\pi : \mathrm{Gal}(\overline{K}/K) \longrightarrow GL_n(\overline{\mathbb{Q}_\ell})$ such that for $v \notin S$, $\rho_{\pi,v}$ is unramified and corresponds to π_v under the Satake equivalence. Cebotarev density tells us that such a representation is unique because characteristic 0 representations are determined by their characters. Similarly, to any $\rho : \mathrm{Gal}(\overline{K}/K) \longrightarrow GL_n(\overline{\mathbb{Q}_\ell})$, there should correspond such an automorphic representation paired by Satake away from a finite set S containing the ramified places.

4.6. Now let us return to the local setting, permitting ramification. This is formulated now exactly as for class field theory.

Given $\tau : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{Aut}(V)$ with V n-dimensional, we want to construct $\pi_\tau : GL_n(F) \longrightarrow \mathrm{Aut}(W)$ irreducible. Choose L/K an infinite Galois extension with K a global field with places $w \in \mathfrak{p}_L$ over $v \in \mathfrak{p}_K$ so that L_w/K_v is isomorphic to $\overline{F}/F$. Then $\mathrm{Gal}(L/K) = \mathrm{Gal}(\overline{F}/F)$, so that we have the Galois representation ρ for the group K arising as the composition:

$$\mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Gal}(L/K) \simeq \mathrm{Gal}(\overline{F}/F) \xrightarrow{\tau} \mathrm{Aut}(V)$$

Global Langlands predicts the existence of some corresponding automorphic representation $\pi_\rho = \otimes'_{v \in \mathfrak{p}_K} \pi_{\rho,v}$. Then our representation of $GL_n(F)$ corresponding to τ should be $\pi_{\rho,v}$. Note that one should show that this is independent of the choice of L/K, which as in the class field theory setting may be proved by giving a purely local characterization of the correspondence.

4.7. What is known about the local and global Langlands correspondences (for $G = GL_n$)? The local Langlands conjectures are known by the work of Harris and Taylor. Global Langlands is known for function fields[4] A great deal is known for $K = \mathbb{Q}$ and $n = 2$, though it cannot be said that the Langlands conjectures are known.

5. Applications

5.1. Let us describe the relationship to L-functions as introduced above. First, we need to explain the notion of a L-function attached to a representation. Let π_F be an unramified representation of $GL_n(F)$ corresponding to $\overline{z} = (z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n/S_n$ via Satake. Then one defines:

$$L_{\pi_F}(s) = \frac{1}{(1 - (qz_1)^{-s}) \ldots (1 - (qz_n)^{-s})}$$

If π is ramified, there still exists a good notion of $L_\pi(s)$ which we do not discuss. For $\pi = \otimes'_{v \in \mathfrak{p}_K}$, we set $L_\pi(s) = \prod_{v \in \mathfrak{p}_K} L_{\pi_v}(s)$.

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[4]As formulated above, i.e., non-categorically.

5.2. If π is an automorphic representation corresponding to ρ by the global Langlands correspondence, then $L_\pi(s)$ and $L_\rho(s)$ are essentially equal. In particular, one admits a holomorphic or meromorphic extension to all of $\mathbb{C}$ if and only if the other does. But one knows:

Theorem 5.1. *$L_\pi(s)$ is a meromorphic function of s. Moreover, $L_\pi(s)$ is holomorphic if π is "cuspidal" (which implies that it lives in the subspace of functions rapidly decreasing at infinity).*

This theorem is due to Godement-Jacquet and it is proved by imitating Tate's proof of the holomorphicity of L-functions attached to Hecke characters. It could conceivably have been given in the course of the lecture, but the talk went in a different direction.

It follows from class field theory that $L_\rho(s)$ admits a meromorphic continuation to the whole complex plane. The Langlands conjectures would imply Artin's conjecture that for ρ irreducible they admit holomorphic continuation to the whole complex plane.

5.3. Let us give a reason to care about Artin's conjecture. Let E be an elliptic curve over K. Let $S \subset \mathfrak{p}_K$ be a finite set such that E has good reduction at $v \notin S$, i.e., E is defined over $\mathscr{O}_v$ and its reduction to $k_v = \mathscr{O}_v/\mathfrak{M}_v$ is smooth.

Hasse's theorem for elliptic curves, a basic instance of the Weil conjectures, says that:

$$|E_v(k_v)| = |k_v| + 1 - |k_v|^{\frac{1}{2}}(\alpha_v + \alpha_v^{-1})$$

for $|\alpha_v| = 1$. Assume $\mathrm{End}(E) = \mathbb{Z}$. Then the Sato-Tate conjecture says the set of conjugacy classes of the matrices $\mathrm{diag}(\alpha_v, \alpha_v^{-1})$ are equidistributed in $SU(2)/conjugacy$.

The Sato-Tate conjecture follows by standard arguments if one knows that the L-functions of the symmetric powers of the ℓ-adic Tate modules of E (which are easily seen to be irreducible representations) are all analytic.

Remark 5.2. The Sato-Tate conjecture is known for $K = \mathbb{Q}$, or even K totally real, by the work of Taylor et al.

6. FUNCTORIALITY

6.1. Let us briefly mention functoriality in our limited setting.

6.2. Let $\tau : GL_n \longrightarrow GL_N$ be a homomorphism and let π be an automorphic representation of $GL_n(\mathbb{A}_K)$. The Langlands conjectures predict that π gives rise to a n-dimensional Galois representation, which via τ gives a N-dimensional Galois representation, which Langlands predicts corresponds to a automorphic representation π' of $GL_N(\mathbb{A}_K)$. It is unknown how to construct such a π' directly!

7. Questions

7.1. We will give a brief probably largely unfaithful summary of some of the questions and remarks made at the end of the lecture. I should emphasize that I (Sam) poorly recorded any aspects of the dialogue and this is intended more to capture the mathematical remarks being made.

7.2. Kisin: Over function fields, one shows that Galois representations are motivic and deduces analytic continuation from this. The automorphy of the constructed representations is deduced from converse theorems and can be viewed almost as an afterthought. Perhaps this is evidence that proving automorphy of Galois representations is not the only way of proving Artin's conjecture?

Kazhdan: Over finite fields we have a good cohomology theory, but there is no glimpse of such a thing for number fields. If people found such a thing, we might have more natural, i.e. less representation-theoretic, proofs of many statements in the Langlands theory, nevermind proofs of things like the Riemann hypothesis. However, because L-functions are not rational functions, there is necessarily nontrivial analysis involved in the construction of any such theory.

7.3. Some remarks on the geometric setting: The geometric Langlands conjectures in the case of function fields relate to the arithmetic setting as follows. Grothendieck's sheaves to functions correspondence tells how to assign a function on the $\mathbb{F}_q$-points of a variety. So, e.g., in the unramified setting that this seminar is focused on, the geometric Langlands conjectures predict that for a projective curve $X/\mathbb{F}_q$ with function field K the existence of "automorphic perverse sheaves" on Bun_{GL_n} corresponding to n-dimensional unramified Galois representations (i.e., rank n-local systems on X) whose associated function on $\mathrm{Bun}_{GL_n}(\mathbb{F}_q) = GL_n(\mathbb{A}_K^{int})\backslash GL_n(\mathbb{A}_K)/GL_n(K)$ generates the automorphic representation corresponding via the arithmetic Langlands conjectures.

7.4. Applications of the categorical language over $\mathbb{C}$: for $k = \mathbb{C}$, all local systems are trivial on the disc. However, if one formulates statements correctly categorically, one can still get useful statements. A useful analogy is that one can glue trivial vector bundles to get non-trivial vector bundles: at the level of isomorphism classes, this is difficult to work with, but with categories it is clear what one is doing. Similarly, one can formulate the unramified local geometric Langlands conjectures categorically ("geometric Satake") in a way which is useful for the characterization of the global equivalence.

7.5. Etingof: Are there any applications of this geometric story to arithmetic?

Kazhdan: I don't know of any applications to the global setting. However, for example for the fundamental lemma (a local statement), logic reduces the statement to a positive characteristic statement and then Ngo tells us how to use geometric methods to prove it there.