Outline. Today we quantize. First step is the local picture: we'll recall from Andrei's first talk the local Hitchin map and its interpretation in terms of the action of $G(\widehat{\mathcal{K}}_x)$ on $Bun_{G,x}$; then we will do the local quantization using the central extension and determinant line bundle of my first talks and the twisted Harish-Chandra machinery of Sam's third talk. But this local quantization is really just a stepping stone to the global quantization, and there's where the crystallography of Jacob's talks comes in: it turns out that the global story falls out once we do the local story "in families", letting x move around: we will get a crystal of stories over X whose fiber at x is the local story and whose flat sections are the global story.

These notes are organized as follows: the first half consists of literal notes — a fairly faithful transcript of what I said in the talk (or maybe, rather, what I planned to say in the talk); but then the second half delivers the details and technicalities which I dodged during the course of the talk by repeatedly promising that they would be in the "notes".

I. Literal Notes.

The Local Hitchin Map. Let $x \in X$ be our pointed curve, and G our connected reductive group (and throw in whatever assumptions we really need...) Recall that there is an action of $G(\widehat{\mathcal{K}}_x)$ on $Bun_{G,x}$ by changing the gluing data in the formal punctured disk; on the level of Lie algebras, this gives a map

$$\mathfrak{g} \otimes \widehat{\mathcal{K}}_x \longrightarrow \Gamma(Bun_{G,x}, T_{Bun_{G,x}}),$$

inducing

$$Sym(\mathfrak{g} \otimes \widehat{\mathcal{K}}_x) \longrightarrow \Gamma(T^* Bun_{G,x}, \mathcal{O}),$$

which descends to

$$h_x^{cl} : Sym(\mathfrak{g} \otimes \widehat{\mathcal{K}}_x / \mathfrak{g} \otimes \widehat{\mathcal{O}}_x)^{G(\widehat{\mathcal{O}}_x)} \longrightarrow \Gamma(T^* Bun_G, \mathcal{O}),$$

since Bun_G is the quotient of $Bun_{G,x}$ by the action of $G(\widehat{\mathcal{O}}_x)$. And recall from Andrei's talk that h_x^{cl} is nothing but the local Hitchin map: we can identify

$$\mathfrak{z}_x^{cl} := Sym(\mathfrak{g} \otimes \widehat{\mathcal{K}}_x / \mathfrak{g} \otimes \widehat{\mathcal{O}}_x)^{G(\widehat{\mathcal{O}}_x)}$$

with the functions on

$$Hitch_x := \Gamma(Spec(\widehat{\mathcal{O}}_x), C_\omega) = (\mathfrak{g}^* \otimes \omega_{\widehat{\mathcal{O}}_x}) /\!/ G(\widehat{\mathcal{O}}_x),$$

essentially because $\widehat{\mathcal{K}}_x / \widehat{\mathcal{O}}_x$ is dual (in the sense of llcvs), under the residue pairing, to the canonical bundle $\omega_{\widehat{\mathcal{O}}_x}$. Recall that $C = \mathfrak{g}^* /\!/ G$, and that C_ω stands for the ω-twist of the constant $\mathbb{G}_m$-space $C \times X$ over X, or ditto with X replaced by $Spec(\widehat{\mathcal{O}}_x)$.

The Local Quantization. Now we will quantize the above, replacing symmetric algebras by universal enveloping algebras and functions on the cotangent space by (twisted) differential operators. Recall the central extension $\widetilde{G(\mathcal{K}_x)}$ of $G(\widehat{\mathcal{K}}_x)$ and its action on the line bundle $p^*\omega$, where $p : Bun_{G,x} \to Bun_G$ is the quotient by $G(\widehat{\mathcal{O}}_x)$. Then, just as above, we have

$$U' \longrightarrow \Gamma(Bun_{G,x}, \mathcal{D}_{p^*\omega}),$$

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where U' is the quotient of the universal enveloping algebra of $\widetilde{\mathfrak{g} \otimes \mathcal{K}_x}$ by setting the central $\mathbb{C}$ to equal the scalars $\mathbb{C}$, and $\mathcal{D}_{p^*\omega}$ stands for differential operators acting on the line bundle $p^*\omega$. And again, just as above, this induces

$$(U'/(U' \cdot \mathfrak{g} \otimes \widehat{\mathcal{O}_x}))^{G(\widehat{\mathcal{O}_x})} \longrightarrow \Gamma(Bun_G, \mathcal{D}_\omega)$$

Now, both sides here are filtered, the LHS by PBW and the RHS by order of the differential operator, and the map respects the filtration. This gives a quantization of h_x^{cl} in the sense that the induced map on associated gradeds embeds into h_x^{cl}; but unfortunately there is a pretty big kink in the LHS making this quantization uninteresting: that natural embedding

$$gr\left((U'/(U' \cdot \mathfrak{g} \otimes \widehat{\mathcal{O}_x}))^{G(\widehat{\mathcal{O}_x})}\right) \hookrightarrow \left(gr(U'/(U' \cdot \mathfrak{g} \otimes \widehat{\mathcal{O}_x}))\right)^{G(\widehat{\mathcal{O}_x})} = \mathfrak{Z}_x^{cl},$$

is far from an isomorphism: the LHS is one-dimensional. To fix this we need to twist: instead of requiring a central $t \in \mathbb{C}$ to act as the scaclar t, we make it act by $-\frac{1}{2}t$, and instead of differential operators on ω, we need to consider differential operators on $\omega^{-1/2}$, which makes formal sense since we can twist the ring of differential operators on a line bundle by arbitrary scalars.

If we make this change, then a theorem of Feigin and Fenkel, whose proof we're postponing till the spring, shows that the above embedding is actually an isomorphism:

Theorem. Let $\mathfrak{Z}_x = (U'/(U' \cdot \mathfrak{g} \otimes \widehat{\mathcal{O}_x}))^{G(\widehat{\mathcal{O}_x})}$, where here now U' stands for the quotient of the universal enveloping algebra of $\widetilde{\mathfrak{g} \otimes \mathcal{K}_x}$ by setting the central $\mathbb{C}$ to equal $-1/2$ of the scalars $\mathbb{C}$. Then the natural map

$$gr(\mathfrak{Z}_x) \hookrightarrow \mathfrak{Z}_x^{cl}$$

is an isomorphism, and hence the above-constructed map

$$h_x : \mathfrak{Z}_x \to \Gamma(Bun_G, \mathcal{D}_{\omega^{-1/2}})$$

quantizes the local Hitchin map in the fullest sense: we have $gr(h_x) = h_x^{cl}$. (The fact that $gr(\Gamma(Bun_G, \mathcal{D}_{\omega^{-1/2}}) \hookrightarrow \Gamma(T^*Bun_G, \mathcal{O})$ is an isomorphism, while not automatic, follows from the Feigin-Fenkel claim and the fact that h_x^{cl} is surjective on components of Bun_G.)

In-Families. Now we let $x \in X$ move in families. Let's start with the classical picture, i.e. the Hitchin fibration. We can make the definition of $Hitch_x$ work in families, and in fact we already have: it is simply $Jets(C_\omega)$, the jet construction applied to the X-scheme C_ω. This is a crystal of affine schemes over X whose fiber at x is $Hitch_x$ and whose flat sections are the global $Hitch = \Gamma(X, C_\omega)$, by the adjointness property of the jet construction. Then we can define

$$\mathfrak{Z}^{cl} = \mathcal{O}(Jets(C_\omega));$$

it is a crystal of (trivial) poisson algebras over X, and the Hitchin map can be interpreted as a map of such

$$h^{cl} : \mathfrak{Z}^{cl} \longrightarrow \Gamma(T^*Bun_G, \mathcal{O}) \otimes \mathcal{O}(X),$$

the RHS being a constant crystal of poisson algebras. Again, this recovers the local Hitchin map as the fiber at any $x \in X$ and the global Hitchin map as flat sections.

Now we will quantize h^{cl} by simply redoing the local quantization in families. This requires defining the central extension of $G(\widehat{\mathcal{K}_x})$ in families, that is, defining a canonical $\mathbb{G}_m$-extension $\widetilde{L_G}$ of the loop group $L_G \to X$ introduced in Jacob's talk. For this one proceeds as in my first talk; what's required is to make sense of the notion of a family of llcvs over an arbitrary scheme, and redo the determinant construction in this generality. See the notes for the details. But once this is done, we get an action of $\widetilde{L_G}$ on the line bundle $p^*\omega$, where $p : Bun_{G,\underline{x}} \to Bun_G \times X$ is the forgetful map from the in-families version of $Bun_{G,x}$ to the constant scheme Bun_G over X.

Then we can run the Harish-Chandra formalism as above, and the result is that we have a filtered associative algebra $\mathfrak{Z}$ over X whose fiber at x is the $\mathfrak{Z}_x$ from above, and a map of filtered associative algebras over X

$$h : \mathfrak{Z} \longrightarrow \Gamma(Bun_G, \mathcal{D}_{\omega^{-1/2}}) \otimes \mathcal{O}(X).$$

But moreover, since these constructions only depended on the formal neighborhood of any X-points, the above is actually canonically a map of filtered crystals of associative algebras (on the right constant once again); and we claim that this quantizes the above map h^{cl} of crystals of Poisson algebras. For this we just need that the map $gr(\mathfrak{Z}) \to \mathfrak{Z}^{cl}$ is an isomorphism; but this is actually something we can check on closed points, so it follows from the above Feigin-Fenkel theorem.

Global Quantization. Since the global Hitchin map was obtained from $h^{cl} : \mathfrak{Z}^{cl} \to \Gamma(T^*Bun_G, \mathcal{O}) \otimes \mathcal{O}(X)$ by taking flat sections, to get its quantization we ought to take flat sections of $h : \mathfrak{Z} \to \Gamma(Bun_G, \mathcal{D}_{\omega^{-1/2}}) \otimes \mathcal{O}(X)$. But for this to make sense we need for $\mathfrak{Z}$ and $\Gamma(Bun_G, \mathcal{D}_{\omega^{-1/2}})$ to be commutative. Fortunately, this is true; we will argue for it following BD.

First of all, it will suffice to see that $\mathfrak{Z}_x$ is commutative for every $\mathbb{C}$-point $x \in X$; indeed, we can argue that this implies that $\mathfrak{Z}$ is commutative, but moreover, since the local Hitchin map is surjective components of Bun_G, so is its quantized version, and therefore the commutativity of $\Gamma(Bun_G, \mathcal{D}_{\omega^{-1/2}})$ follows from the commutativity of $\mathfrak{Z}_x$ for any $x \in X$.

Now, somewhat perversely, to prove this local fact we use the global map $h : \mathfrak{Z} \to \Gamma(T^*Bun_G, \mathcal{D}_{\omega^{-1/2}}) \otimes \mathcal{O}(X)$: as a first step, we claim that the map $[h_x(-), h_x(-)] : \mathfrak{Z}_x \otimes \mathfrak{Z}_x \to \Gamma(T^*Bun_G, \mathcal{D}_{\omega^{-1/2}})$ is trivial, and for this we note that the existence of the global map implies that the set of $y \in X$ for which the map $[h_x(-), h_y(-)]$ is trivial is closed; therefore it will suffice to show that if $x \neq y$ then $[h_x, h_y] = 0$. This can be proved as follows: there is a scheme $Bun_{G,x,y}$ parametrizing G-bundles on X with formal trivialization at x and y, and a forgetful map Bun_G, which is just a quotient by $G(\widehat{\mathcal{O}_x}) \times G(\widehat{\mathcal{O}_y})$, and we can rerun the local quantization in this situation to get a map $\mathfrak{Z}_x \otimes \mathfrak{Z}_y \to \Gamma(T^*Bun_G, \mathcal{D}_{\omega^{-1/2}})$ restricting to h_x and h_y on the factors; and this implies that the images commute, as desired.

So, by that first step we have that if $a \in \mathfrak{Z}_x$ is a commutator, then $h_x(a) = 0$. Now, h_x is not injective, but it does have a slightly weaker property: if we let d denote the degree of a, then we necessarily have $a = 0$ from $h_x(a) = 0$ provided that the natural map $H^0(X; \omega_X) \to \omega_{\widehat{\mathcal{O}_x}}/\mathfrak{m}_x^d \omega_{\widehat{\mathcal{O}_x}}$ is surjective. Indeed, we can view the symbol σa of a as a function on $\mathfrak{g}^* \otimes (\omega_{\widehat{\mathcal{O}_x}}/\mathfrak{m}_x^d \omega_{\widehat{\mathcal{O}_x}})$, and the fact that $h_x^{cl}(\sigma a) = 0$ implies that a vanishes on the image of $H^0(X; \mathfrak{g}^* \otimes \omega_X) \to \mathfrak{g}^* \otimes (\omega_{\widehat{\mathcal{O}_x}}/\mathfrak{m}_x^d \omega_{\widehat{\mathcal{O}_x}})$.

Of course, this surjectivity condition won't always be satisfied, but we can nonetheless finish the proof by the following feint: since the formal neighborhoods of any points on any curves are isomorphic and $\mathfrak{Z}_x$ depends only on this formal neighborhood, we are free to replace $x \in X$ by any pointed curve we like, in particular one for which the condition holds, and we conclude as desired. Thus $\mathfrak{Z}_x$ is commutative, and hence so are both $\mathfrak{Z}$ and $\Gamma(Bun_G, \mathcal{D}_{\omega^{-1/2}})$, as explained above.

Thus h is actually a map of filtered crystals of commutative algebras, and we can take flat sections (I mean conformal blocks, what would be flat sections on Spec), getting

$$\Gamma_{flat}(h) : \Gamma_{flat}(\mathfrak{Z}) \longrightarrow \Gamma(Bun_G, \mathcal{D}_{\omega^{-1/2}}).$$

Now, recall that if $\mathcal{A}$ is a crystal of commutative algebras, the unit map $\mathcal{A} \to \Gamma_{flat}(\mathcal{A}) \otimes \mathcal{O}(X)$ is surjective (geometrically, a section being flat is a closed condition); therefore, given a filtration on $\mathcal{A}$ we get an induced one on $\Gamma_{flat}(\mathcal{A}) \otimes \mathcal{O}(X)$; but since a subcrystal of a constant crystal is constant, this is equivalent to having a filtration on $\Gamma_{flat}(\mathcal{A})$ itself. Moreover we have a canonical surjection

$$\Gamma_{flat}(gr(\mathcal{A})) \twoheadrightarrow gr(\Gamma_{flat}(\mathcal{A}))$$

by taking $\Gamma_{flat} \circ gr$ of the above unit map; thus, to prove that $\Gamma_{flat}(h)$ is a global quantization, we just need to show that this surjection is also an injection for $\mathcal{A} = \mathfrak{Z}$. But this is simple: in that case, the map h itself shows that the surjection $\Gamma_{flat}(gr(\mathfrak{Z})) \twoheadrightarrow gr(\Gamma_{flat}(\mathfrak{Z}))$ is a factoring of the classical global Hitchin map $\Gamma_{flat}(gr(\mathfrak{Z})) \to \Gamma(T^*Bun_G, \mathcal{O})$, which we already know to be injective.

So we've done what we promised: h is a map of filtered crystals of commutative algebras whose gr is h^{cl}; moreover this story gives a quantization of the local Hitchin map on fibers and of the global Hitchin map on flat sections.

II. Details and Technicalities.

Let X be a smooth, proper, connected curve over an algebraically closed field k of characteristic zero, and G a connected reductive group over k.

Let $\mathcal{S}$ denote the category of sheaves of groupoids on Aff_k in the etale topology. Recall the following fact:

Lemma 1. *If $A \in Aff_k$ with largest reduced quotient A^{red}, then the functor $B \mapsto B \otimes_A A^{red} = B^{red}$ establishes an equivalence between the etale site over A and the etale site over A^{red}.*

This implies in particular that the deRham space X^{dR} from Jacob's talk lies in $\mathcal{S}$. Recall also the map $\pi : X \to X^{dR}$ and the functors $\pi^* : \mathcal{S}/X^{dR} \to \mathcal{S}/X$ (underlying space) and $\pi_* : \mathcal{S}/X \to \mathcal{S}/X^{dR}$ (jet construction), as well as the map $p : X^{dR} \to *$ and the functors $p^* : \mathcal{S} \to \mathcal{S}/X^{dR}$ (constant crystal) and $p_* : \mathcal{S}/X^{dR} \to \mathcal{S}$ (flat sections).

Level Structures. First step: define $Bun_{G,x}$ in families (over X).

Definition 1. *Let $Bun_{G,lvl}$ denote the pullback*

$$
\begin{array}{ccc}
Bun_{G,lvl} & \longrightarrow & p^*Bun_G \\
\downarrow & & \downarrow \\
\pi_* X & \longrightarrow & \pi_*(BG \times X),
\end{array}
$$

and let $G_{lvl} = \pi_*(G \times X)$; here we consider everything as living in $\mathcal{S}/X^{dR}$.

Theorem 1. *We have:*

(1) *There is an action of G_{lvl} on $Bun_{G,lvl} \to p^*Bun_G$ making it a G_{lvl}-torsor;*
(2) *G_{lvl} and $Bun_{G,lvl}$ are both relative schemes over X^{dR}.*

Proof. For (1), it suffices to show the same for the bottom map $\pi_*X \to \pi_*(BG \times X)$ of the diagram defining $Bun_{G,lvl}$. There the action comes by functoriality from the action of G on $pt \to BG$, and to show the torsor claim it suffices to show that π_* preserves torsors. This follows formally from the fact that π_* commutes with taking B of a group sheaf, which is a consequence of the fact that π_* (of presheaves) commutes with sheafification, which comes from Lemma 1.

For (2), the fact that G_{lvl} is schematic follows from Jacob's lecture: he showed that π_* sends schemes to schemes from X being smooth. For $Bun_{G,lvl}$, we first reduce to $G = GL_r$: represent G as a closed subgroup of some GL_r, say $G \to GL_r$. Then in the commutative diagram

$$
\begin{array}{ccc}
Bun_{G,lvl} & \longrightarrow & Bun_{r,lvl} \\
\downarrow & & \downarrow \\
p^*Bun_G & \longrightarrow & p^*Bun_r,
\end{array}
$$

we have by (1) that the left map is schematic and the right map is separated; furthermore the bottom map is schematic since $Bun_G \to Bun_r$ is schematic (as we showed in the course proving representability of Bun_G), and it then follows that the top horizontal map is schematic as well, allowing for the reduction.

To handle $Bun_{r,lvl}$, note first that by Jacob's talk it is sufficient to establish schematicity of $\pi^*Bun_{r,lvl} \to X$, or just of $\pi^*Bun_{r,lvl}$. We will do this as in Nir's talk, using the map $f : \pi^*Bun_{r,lvl} \to Bun_r \times X$. Here are the steps:

(1) Define the system $\ldots \to Bun_{r,lvl_2} \to Bun_{r,lvl_1} \to Bun_{r,lvl_0}$ with inverse limit $\pi^*Bun_{r,lvl}$, affine transition maps, and compatible maps $f_n : Bun_{r,lvl_n} \to Bun_r \times X$ for all n, each inducing f (we will actually do this for any G);
(2) Define the suitable open cover $U_{m,m'}$ of $Bun_r \times X$ with the property that for every m, m' we can argue that $f_N^*(U_{m,m'}) \subseteq Bun_{r,lvl_N}$ is a scheme for some N.

These two steps will finish the job: they give that for every m and m' the functor $f^*(U_{m,m'})$ is a scheme, and moreover these guys cover $\pi^*Bun_{r,lvl}$.

For (1), we note that the definition of $Bun_{G,lvl}$, upon applying π^*, gives the pullback square

$$
\begin{array}{ccc}
\pi^*Bun_{G,lvl} & \longrightarrow & Bun_G \times X \\
\downarrow & & \downarrow \\
X & \longrightarrow & J(BG \times X),
\end{array}
$$

Now, in Jacob's talk, we learned that $J(BG \times X)$ is the inverse limit of functors $J^{(n)}(BG \times X)$, where

$$
J^{(n)}(Y) = \pi_{2*}^{(n)} \pi_1^{(n)*}(Y),
$$

denoting by $X^{(n)}$ the nth infinitesimal neighborhood of the diagonal of X and by $\pi_1^{(n)}, \pi_2^{(n)} : X^{(n)} \to X$ the two projections. To unravel a bit, just as in the proof of

(1) of this theorem we can deduce from this that

$$J^{(n)}(BG \times X) = BH^{(n)},$$

where $H^{(n)} = \pi_{2*}^{(n)}(G \times X^{(n)})$.

This then pulls back to an inverse limit expression for $Bun_{G,lvl}$ in terms of, call them Bun_{G,lvl_n}, of the desired form. And the transition maps are pullbacks of diagonals of maps $BH^{(n+1)} \to BH^{(n)}$, so to finish (1) it suffices to show that these diagonals are affine. This amounts to the following: if $A \in Aff_k$ with $Spec(A) \to X$ and $\mathcal{P}_1^{(n+1)}$ and $\mathcal{P}_2^{(n+1)}$ are $H^{(n+1)}$-bundles on $Spec(A)$, then the map $Isom(\mathcal{P}_1^{(n+1)}, \mathcal{P}_2^{(n+1)}) \to Isom(\mathcal{P}_1^{(n)}, \mathcal{P}_2^{(n)})$ is affine, where by $\mathcal{P}^{(n)}$ we mean the induced $H^{(n)}$-bundle.

We can thus finish by showing that all of these Isom's are affine over A. This follows from the groups $H^{(n)}$ being affine over X, a consequence of the general fact that restriction of scalars along a finite locally free map preserves affineness.

Now we turn to (2), specializing to $G = GL_r$. Recall from Nir's lecture the subfunctors $U_{m,m'}$ of $Bun_r \times X$: a pair $(\mathcal{E}, x) \in (Bun_r \times X)(A)$ consisting of a rank-r vector bundle $\mathcal{E}$ on $X \times Spec(A)$ and a section x of $q : X \times Spec(A) \to Spec(A)$ lies in $U_{m,m'}$ if and only if the following three conditions are satisfied:

(1) $q^* q_* \mathcal{E}(m) \to \mathcal{E}(m)$ is surjective;
(2) $Rq^1(\mathcal{E}(m)) = 0$;
(3) $q_*(\mathcal{E}(-m')) = 0$.

Here we twist with respect to the line bundle $\mathcal{O}_X(x)$, which is relatively ample over $Spec(A)$ by Riemann-Roch. The fact that these conditions in conjunction are stable under base-change (and hence that each $U_{m,m'}$ is actually a sub*functor*) follows from cohomology and base-change. Moreover, each of the above conditions is open, and any $(\mathcal{E}, x)$ satisfies them for some m, m' as a consequence of Serre vanishing; thus $\{U_{m,m'}\}$ is indeed an open cover.

... To be continued... $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

The Loop Group and Change of Trivialization.

The Central Extension and Determinant Line Bundle.

etc. etc. etc...