Groups Acting on Categories

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The main purpose of this talk is to introduce the abstract formalism of groups acting on categories, and certain related notions: weak and strong equivariance, the adjoint localization/globalization functors and convolution.

We will often find ourselves in the following general setting: suppose we have a fixed base field k and two $k-$linear abelian categories $\mathcal{C}$ and $\mathcal{C}'$. How do we make sense of the category $\mathcal{C} \otimes \mathcal{C}'$ [1]? In general, this category is *generated* by objects $C \otimes C'$ for $C \in \mathcal{C}$ and $C' \in \mathcal{C}'$. The actual meaning of the word "generated" is in some general triangulated sense, and we won't bother ourselves with this general definition.

However, in all of our examples we only encounter tensor products of the form $A\text{-mod} \otimes \mathcal{C}$, where A is a $k-$algebra and $\mathcal{C}$ is a k linear category. This has a simpler description: objects of this tensor category are by definition **objects of $\mathcal{C}$ with an A-module structure**. In other words,

$$A\text{-mod} \otimes \mathcal{C} = \{(C, \phi)|\ C \in \mathcal{C}, \phi : A \longrightarrow \text{End}_k(C)\}.$$

The above definition can be naturally generalized to the case when A-mod is replaced by QCoh or D-mod.

Example 1 *We have* k-mod $=$ Vect. *Since* $\mathcal{C}$ *is abelian, there is a natural identification* k-mod $\otimes\ \ \mathcal{C} = \mathcal{C}$.

Example 2 *Let B be another $k-$algebra. Then by the above definition:*

$$A\text{-mod} \otimes B\text{-mod} = \{(C, \phi)|\ C \in B\text{-mod}, \phi : A \longrightarrow \text{End}_B(C)\} =$$

[1] All tensor products are over the fixed base field k

$$= \{(C, \phi, \phi') |\ C \in \mathrm{Vect}, \phi : A \longrightarrow \mathrm{End}_k(C), \phi' : B \longrightarrow \mathrm{End}_k(C)$$
$$\text{such that } \phi \text{ and } \phi' \text{ commute}\} = (A \otimes B)\text{-mod}.$$

Example 3 *Given $M \in A$-mod and $C \in \mathcal{C}$, let us show how to form the object $M \otimes C \in A\text{-mod} \otimes \mathcal{C}$. By definition, M is a vector space endowed with a morphism of algebras $A \longrightarrow \mathrm{End}_k(M)$. Then we simply form the object $M \otimes C \in \mathrm{Vect} \otimes \mathcal{C} = \mathcal{C}$, and let the action map be:*

$$A \longrightarrow \mathrm{End}_k(M) \otimes \mathrm{Id}_C \subset \mathrm{End}_k(M \otimes C).$$

Let G be a group scheme and let $\mathcal{C}$ be an abelian category. A **weak** action of G on $\mathcal{C}$ is a functor:

$$\mathrm{act}^* : \mathcal{C} \longrightarrow \mathrm{QCoh}_G \otimes \mathcal{C},$$

which is *unital*:

$$\mathrm{act}^*|_{1 \hookrightarrow G} = \mathrm{Id} : \mathcal{C} \longrightarrow \mathcal{C}, \tag{1}$$

and *multiplicative*, in the sense that we are given an identification between the two compositions of the following diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathrm{act}^*} & \mathrm{QCoh}_G \otimes \mathcal{C} \\ {\scriptstyle \mathrm{act}^*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Id} \otimes \mathrm{act}^*} \\ \mathrm{QCoh}_G \otimes \mathcal{C} & \xrightarrow{\mathrm{mult}^* \otimes \mathrm{Id}} & \mathrm{QCoh}_G \otimes \mathrm{QCoh}_G \otimes \mathcal{C}. \end{array} \tag{2}$$

We can generalize the above to the case when G is a group ind-scheme. A **weak** action of G on $\mathcal{C}$ consists of functors:

$$\mathrm{act}^*|_S : \mathcal{C} \longrightarrow \mathrm{QCoh}_S \otimes \mathcal{C}, \tag{3}$$

functorially in $S \xrightarrow{\varphi} G$. We further ask act^* to be unital and multiplicative, by analogy with (1) and (2). Alternatively, the datum of (3) is the same as asking for an $S-$linear action of $G(S)$ on $\mathrm{QCoh}_S \otimes \mathcal{C}$, functorially in $S \xrightarrow{\varphi} G$.

Example 4 *The trivial weak action of G on $\mathcal{C}$ is*

$$\mathrm{triv}^*|_S : \mathcal{C} \longrightarrow \mathrm{QCoh}_S \otimes \mathcal{C}, \quad C \longrightarrow \mathcal{O}_S \otimes C.$$

To get an idea of how the object $\mathcal{O}_S \otimes C$ looks like, see Example 3.

Example 5 *Let A be a topological associative algebra, acted on by G. This induces a weak G action on the category A-mod of discrete A modules. In particular, when $A = U(\mathfrak{g})$ this gives a weak action of G on $\mathfrak{g}$-mod.*

Example 6 *Let $\mathcal{Y}$ be an ind-scheme, acted on by G. This induces a weak G action on the category $\mathrm{QCoh}^*(\mathcal{Y})$. If $\mathcal{Y} = \mathrm{Spec}\ A$ is affine, this reduces to the previous example.*

Given a weak action of G on a category $\mathcal{C}$, we say that an object $C \in \mathcal{C}$ is **weakly equivariant** if it comes equipped with an isomorphism:

$$\mathrm{act}^*(C) \cong \mathrm{triv}^*(C) \tag{4}$$

This should be perceived as a functorial family of isomorphisms $\mathrm{act}^*(C)|_S \cong \mathrm{triv}^*(C)|_S$ for all $\varphi : S \longrightarrow G$, respecting the unit and associativity constraints. We write $\mathcal{C}^{w,G}$ for the category of weakly equivariant objects.

Example 7 *For the trivial action of G on Vect, the category $\mathrm{Vect}^{w,G}$ is precisely the category of $G-$representations (vector spaces V equipped with a map $G \longrightarrow GL_k(V)$).*

Example 8 *In the setting of Example 5, we have*

$$\mathrm{Ob}(A\text{-mod})^{w,G} = \{(M \in A\text{-mod},\ \xi : G \longrightarrow \mathrm{Aut}(M)\}, \tag{5}$$

such that:

$$\begin{array}{ccc} M & \xrightarrow{a} & M \\ {\scriptstyle \xi(g)}\downarrow & & \downarrow{\scriptstyle \xi(g)} \\ M & \xrightarrow{g \cdot a} & M \end{array} \tag{6}$$

for any $a \in A, g \in G$. This map ξ is precisely what one needs to trivialize the module M over G.

For a group ind-scheme G, we let $G^{(1)} = \mathrm{Spf}(\mathbb{C} \oplus \varepsilon \cdot \mathfrak{g}^*)$ denote the first infinitesimal neighborhood of the unit $1 \in G$, and we let $\widehat{G}_1$ be the formal completion of G at the unit. A weak action of G on $\mathcal{C}$ is called **strong** if either of the following **equivalent** conditions are satisfied:

- We are given have functorial isomorphisms between the functors (3) for any pair of infinitesimally close points $\varphi, \varphi' : S \longrightarrow G$, satisfying certain compatibility conditions.
- We are given functorial trivializations of the functor (3) for any $\varphi : S \longrightarrow \widehat{G}_1$, respecting the unit, the multiplication, and the adjoint action of G on $\widehat{G}_1$.

Remark 1 *The second condition above is actually equivalent to a weaker version. It is enough to be given a trivialization not on $\widehat{G}_1$, but on $G^{(1)}$:*

$$\mathrm{act}|_{G^{(1)}} \cong \mathrm{triv}|_{G^{(1)}}, \tag{7}$$

*which is compatible with the unit and the Lie algebra structure. This is why strong actions are sometimes called **infinitesimally trivial**.*

In this strong case, if we have a weakly equivariant object $C \in \mathcal{C}^{w,G}$, then (4) and (7) give us two isomorphisms:

$$\mathrm{act}^*(C)|_{G^{(1)}} \cong \mathrm{triv}^*(C)|_{G^{(1)}}.$$

If these isomorphisms coincide, then we call C a **strongly equivariant** object. We denote the subcategory of strongly equivariant objects by $\mathcal{C}^G$.

Example 9 *For the trivial action of G on* Vect*, the category* Vect^G *is just the category of G/G_0-representations, where G_0 is the connected component of the identity in G.*

Example 10 *Let G be a group ind-scheme acting on a topological associative algebra A via a map $G \longrightarrow \mathrm{Aut}(A)$, whose derivative is:*

$$\mathfrak{g} \longrightarrow \mathrm{Der}(A).$$

We claim that the induced action of G on A-mod *from Example 5 is strong if and only if the above Lie algebra map factors through the algebra of inner derivations via a $G-$equivariant morphism ζ:*

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathrm{Der}(A) \\ {\scriptstyle \zeta}\,\vdots & \nearrow_{\text{inner}} & \\ A & & \end{array} \tag{8}$$

The objects of $(A\text{-mod})^G$ *are pairs as in* (5) *such that the diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\xi} & \mathrm{End}(M) \\ {\scriptstyle \zeta}\downarrow & \nearrow_{\text{action}} & \\ A & & \end{array} \tag{9}$$

Example 11 *If* $A = U(\mathfrak{g})$*, then condition* (8) *holds automatically, so we can try to figure out how weak and strong equivariant* $\mathfrak{g}$*-modules look like. By Example 8, we know that weakly equivariant modules are pairs of a* $\mathfrak{g}$*-module* $a : \mathfrak{g} \longrightarrow \mathrm{End}(M)$ *and a group homomorphism* $\xi : G \longrightarrow \mathrm{Aut}(M)$*. Diagram* (6) *precisely requires that:*

$$a([x, y]) \cdot m = [d\xi(x), a(y)] \cdot m, \tag{10}$$

for all $x, y \in \mathfrak{g}$*,* $m \in M$*. Therefore,*

$$\mathrm{Ob}(\mathfrak{g}\text{-mod})^{w,G} = \{M \in \mathfrak{g}\text{-mod}, \quad \xi : G \longrightarrow \mathrm{Aut}(M)\},$$

which satisfy property (10)*. For strongly equivariant objects, relation* (9) *forces* $a = d\xi$*, and therefore:*

$$\mathrm{Ob}(\mathfrak{g}\text{-mod})^G = G\text{-mod}.$$

Example 12 *Let* G *be a group scheme acting on a smooth affine scheme of finite type* $\mathcal{Y}$*. Then* $D\text{-mod}(\mathcal{Y}) = \mathcal{D}_{\mathcal{Y}}\text{-mod}$*, and we are in the situation of Example 10. As we have seen there, the corresponding action of* G *on* $\mathcal{D}_{\mathcal{Y}}\text{-mod}$ *is strong if and only if its derivative*

$$\mathfrak{g} \longrightarrow \mathrm{Der}(\mathcal{D}_{\mathcal{Y}})$$

factors through $\mathcal{D}_{\mathcal{Y}}$*. But the factor map is precisely the derivative* $\mathfrak{g} \longrightarrow \mathcal{D}_{\mathcal{Y}}$ *of the action of* G *on* $\mathcal{Y}$ *we started with! The same claim holds in the more general setting of a group ind-scheme acting on a smooth ind-scheme, but we haven't yet defined the relevant categories.*

The above two examples are quite closely related. Indeed, in the setup of Example 12 we have the natural **localization** and **globalization** functors:

$$\mathfrak{g}\text{-mod} \xrightarrow{\text{loc}} D\text{-mod}(\mathcal{Y}), \qquad \text{loc}(V) = \mathcal{D}_{\mathcal{Y}} \otimes_{\mathfrak{g}} V,$$

$$D\text{-mod}(\mathcal{Y}) \xrightarrow{\Gamma} \mathfrak{g}\text{-mod}, \qquad \Gamma(\mathcal{M}) = \Gamma(\mathcal{Y}, \mathcal{M}).$$

The globalization Γ is the right adjoint of the localization loc, and both functors respect the strong $G-$actions on the categories in question. This means that the action maps (3) are compatible with these functors, as is the identitication (7).

Example 13 *Let $K \subset G$ be a pair of group ind-schemes. We can take the category $\mathfrak{g}$-mod, and the restricted action of K to this category. Then:*

$$\mathfrak{g}\text{-mod}^K \cong (\mathfrak{g}, K)\text{-mod},$$

where the category on the right denotes Harish-Chandra modules.

In the next lecture, we will also need a *twisted* version of the above. Namely, suppose we have a central extension $0 \longrightarrow \mathbb{C} \longrightarrow \mathfrak{g}' \longrightarrow \mathfrak{g} \longrightarrow 0$, which induces an extension of the formal completions at the unit:

$$1 \longrightarrow \widehat{\mathbb{C}} \longrightarrow \widehat{G}_1' \longrightarrow \widehat{G}_1 \longrightarrow 1$$

The object on the left denotes the completion of $\mathbb{C}$ at 0. The weak notions of action and equivariance stay the same as before. Now suppose our weak action of G on $\mathcal{C}$ comes with a trivializion α on $\widehat{G}_1'$. If this trivialization satisfies:

$$\Psi : \text{triv}^*_{\widehat{\mathbb{C}}} \overset{\alpha|_{\widehat{\mathbb{C}}}}{\cong} \text{act}^*_{\widehat{\mathbb{C}}} \cong \text{triv}^*_{\widehat{\mathbb{C}}}, \qquad \Psi(1 \otimes \mathcal{C}) = e^t \otimes \mathcal{C}, \tag{11}$$

for all $f(t) \in \mathcal{O}_{\widehat{\mathbb{C}}} = \mathbb{C}[[t]]$, then we call the action **twisted strong** [2].

We would like to define twisted strongly equivariant objects now, but we can only do so for a subgroup $H \subset G$ over which the central extension of Lie algebras splits. For $X \in \mathcal{C}^{w,H}$, we have two trivializations of it to $\widehat{H}_1'$: one provided by (4) and one provided by the trivialization to $\widehat{G}_1'$ of the previous

[2]The second isomorphism above just reflects the fact that the map $\widehat{\mathbb{C}} \longrightarrow \widehat{G}_1$ is trivial

paragraph. If these two trivializations satisfy relation (11) (in particular, if their composition comes from $\widehat{\mathbb{C}}$), then we call X a **twisted strongly** $H-$**equivariant** object.

Throughout the above discussion, we have mostly dealt with a weak/strong action of G on an abelian category $\mathcal{C}$. However, this always induces a weak/strong action on the category of complexes $\mathbf{C}(\mathcal{C})$, and from there to a weak/strong action on the derived categories $\mathbf{D}(\mathcal{C}), \mathbf{D}^+(\mathcal{C}), \mathbf{D}^-(\mathcal{C}), \mathbf{D}^b(\mathcal{C})$. In fact, many important constructions (such as convolution) work best at the level of the derived categories.

To go on any further, we have to define $D-$modules on an ind-scheme $\mathcal{Y}$. Today we will only present the general idea, sweeping the categorical details under the rug until a future date. We always assume our ind-scheme is *good*, i.e. that we can write it as:

$$\mathcal{Y} = \varinjlim_i \mathcal{Y}_i,$$

where $\mathcal{Y}_i$ are schemes (possibly of infinite type) such that:

$$\mathcal{Y}_i = \varprojlim_j \mathcal{Y}_{ij}, \tag{12}$$

where $\mathcal{Y}_{ij}$ are schemes of finite type. The definition of "good" further requires that the injective limit is defined via closed embeddings $\psi_{ii'} : \mathcal{Y}_i \hookrightarrow \mathcal{Y}_{i'}$ (for $i < i'$), that the ideal of $\mathcal{Y}_i$ inside $\mathcal{Y}_{i'}$ is finitely generated, and that the projective limit is given with respect to smooth, surjective morphisms $\phi^i_{jj'} : \mathcal{Y}_{ij'} \twoheadrightarrow \mathcal{Y}_{ij}$ (for $j' > j$).

We define the category of D-modules on the scheme (12) by:

$$D\text{-mod}(\mathcal{Y}_i) := \varprojlim_j D\text{-mod}(\mathcal{Y}_{ij}),$$

with respect to the maps $\phi^i_{jj'*}$. Then we define the category of D-modules on the good ind-scheme $\mathcal{Y}$ as:

$$D\text{-mod}(\mathcal{Y}) := \varprojlim_i D\text{-mod}(\mathcal{Y}_i), \tag{13}$$

with respect to the maps $\psi^{ii'!}$. The detail we are sweeping under the rug is the definition of these maps, but it's not hard to believe that they are somehow induced from the closed embeddings $\psi^{ii'}$.

Now suppose G is a group ind-scheme acting strongly on an abelian category $\mathcal{C}$. We will now define twisted products and convolution, but we need to make a compromise: though all the subsequent results hold for group ind-schemes G, we can so far only state them rigorously for a group scheme G. Therefore, we define the **twisted product** functor as:

$$D\text{-mod}(G) \times \mathcal{C} \longrightarrow D\text{-mod}(G) \otimes \mathcal{C},$$
$$(\mathcal{S}, X) \longrightarrow \mathcal{S} \widetilde{\boxtimes} X := \mathcal{S} \otimes_{\mathcal{O}_G} \operatorname{act}_G^*(X).$$

The fact that the action of G on $\mathcal{C}$ is strong implies the fact that the objects $\operatorname{act}^*(X)$ come equipped with a connection along G. In other words, they naturally sit inside $D\text{-mod}(G) \otimes \mathcal{C}$, and thus the above tensor product makes sense as a *tensor product of D-modules.*

The projection $\pi : G \longrightarrow \mathrm{pt}$ induces a push-forward map:

$$\pi_* = H_{\mathrm{DR}}(G, \cdot) : D\text{-mod}(G) \longrightarrow D\text{-mod}(\mathrm{pt}) = \mathrm{Vect},$$

and therefore naturally a functor we denote by the same letter:

$$H_{\mathrm{DR}}(G, \cdot) : D\text{-mod}(G) \otimes \mathcal{C} \longrightarrow \mathcal{C}.$$

This finally gives rise to the **convolution** functor:

$$D\text{-mod}(G) \times \mathcal{C} \longrightarrow \mathcal{C},$$
$$\mathcal{S} * C := H_{\mathrm{DR}}(G, \mathcal{S} \widetilde{\boxtimes} C).$$

Finally, let us note that the natural setting for this whole construction is at the level of the derived categories:

$$\mathbf{D}^+(D\text{-mod}(G)) \times \mathbf{D}^+(\mathcal{C}) \xrightarrow{*} \mathbf{D}^+(\mathcal{C}).$$

The above discussion also has equivariant versions: first of all, let us start with a subgroup $H \subset G$. Our first goal is to produce an equivariant convolution functor:

$$\mathbf{D}^+(D\text{-mod}(G))^H \times \mathbf{D}^+(\mathcal{C})^H \longrightarrow \mathbf{D}^+(\mathcal{C}). \tag{14}$$

In the left, we take equivariant modules with respect to the action of H on G by right multiplication. We have the following proposition:

Proposition 1 *For any reasonable ind-scheme $\mathcal{Y}$ acted on by a group ind-scheme H, there is a natural equivalence of categories:*

$$\mathbf{D}(D\text{-mod}(\mathcal{Y}))^H \cong \mathbf{D}(D\text{-mod}(\mathcal{Y}/H)).$$

The isomorphism is given by pull-back under the projection map $\mathcal{Y} \longrightarrow \mathcal{Y}/H$.

With this, defining a functor (14) becomes equivalent to defining:

$$\mathbf{D}^+(D\text{-mod}(G/H)) \times \mathbf{D}^+(\mathcal{C})^H \longrightarrow \mathbf{D}^+(\mathcal{C}). \tag{15}$$

Given $\mathcal{S} \in \mathbf{D}^+(D\text{-mod}(G/H))$ and $X \in \mathbf{D}^+(\mathcal{C})^H$, we can form their twisted product over H:

$$\mathcal{S}\widetilde{\boxtimes}_H X := \mathcal{S} \otimes_{\mathcal{O}_{G/H}} \text{act}_G^*(X) \in \mathbf{D}^+(D\text{-mod}(G/H)) \otimes \mathcal{C}. \tag{16}$$

The above makes sense because the object $\text{act}_G^*(X)$ is strongly $H-$equivarient, and thus can be descended to a D-module on G/H (tensor $\mathcal{C}$). The desired convolution $\mathcal{S} * X$ of (15) is then just the push-forward under $G/H \longrightarrow \text{pt}$ of the object (16).

Let us now take another subgroup $H' \subset G$, acting on the group via left multiplications. If $\mathcal{S} \in \mathbf{D}^+(D\text{-mod}(G/H))$ of (16) is strongly $H'-$equivariant, then so will $\mathcal{S}\widetilde{\boxtimes}_H X$. It therefore descends to an object:

$$\mathcal{S}\widetilde{\boxtimes}_H^{H'} X \in \mathbf{D}(D\text{-mod}(H'\backslash G/H)) \otimes \mathcal{C}.$$

If we push this object forward via $H'\backslash G/H \longrightarrow \text{pt}$, then we obtain the doubly equivariant convolution product:

$$\mathbf{D}^+(D\text{-mod}(G))^{H',H} \times \mathbf{D}^+(\mathcal{C})^H \longrightarrow \mathbf{D}^+(\mathcal{C})^{H'}, \tag{17}$$

or, which is equivalent by Proposition 1,

$$\mathbf{D}^+(D\text{-mod}(H'\backslash G/H)) \times \mathbf{D}^+(\mathcal{C})^H \longrightarrow \mathbf{D}^+(\mathcal{C})^{H'}.$$

Example 14 *The first example of convolution comes along for* $G = \mathbb{C}((t))$*,* $H = H' = \mathbb{C}[[t]]$ *and* $\mathcal{C} = D\text{-mod}(\text{Gr}_G)$*. Then*

$$D\text{-mod}(G/H)^{H'} = \mathcal{C}^H = \mathcal{C}^{H'} = \text{Sph},$$

and the convolution $\text{Sph} \times \text{Sph} \longrightarrow \text{Sph}$ *defined by* (17) *is just the standard multiplication on* Sph*.*

Example 15 *More generally, suppose we have a group ind-scheme* G *acting on an ind-scheme* $\mathcal{Y}$*, and take any subgroup* $H \subset G$*. Consider:*

$$\begin{array}{ccc} G \times_H \mathcal{Y} & \xrightarrow{a} & \mathcal{Y} \\ {\scriptstyle p}\big\downarrow & & \\ G/H & & \end{array}$$

In his Feb 16 talk, Ryan used this diagram to introduce a convolution functor:

$$D\text{-mod}(G/H) \times D\text{-mod}(\mathcal{Y})^H \longrightarrow D\text{-mod}(\mathcal{Y}),$$
$$(\mathcal{S}, X) \longrightarrow a_*(p^*\mathcal{S} \otimes \widetilde{X}).$$

This is just the particular case of our construction in the case $\mathcal{C} = D\text{-mod}(\mathcal{Y})$*.*

Example 16 *In the setting of Examples 10 and 12, recall that the categories* $\mathfrak{g}\text{-mod}$ *and* $D\text{-mod}(\mathcal{Y})$ *both carry strong* G *actions, and the functors* loc *and* Γ *between them respect this action. Because these functors respect the strong* G *action, they also respect the convolution product:*

$$\text{loc}(\mathcal{S} * V) = \mathcal{S} * \text{loc}(V),$$
$$\Gamma(\mathcal{S} * \mathcal{M}) = \mathcal{S} * \Gamma(M).$$