Proof of the Hecke Property

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Last time, we introduced a lot of general stuff about group actions on categories, equivariant objects and convolution. We will apply all that to a very concrete setting: let G be a reductive group and X be a projective curve. Our main focus for this lecture will be the group $G((t))$ (and its incarnations $G(\widehat{\mathcal{K}}_x)$ for $x \in X$), and the many categories it acts on.

For example, $G((t))$ acts on the affine Grassmannian Gr_G via left multiplication, and thus induces an action of $G((t))$ on the category $D'\text{-mod}(\mathrm{Gr}_G)$ (the prime means "critically twisted"). The subcategory of $G[[t]]$ strongly equivariant objects, $D'\text{-mod}(\mathrm{Gr}_G)^{G[[t]]}$, is by definition the spherical category Sph_G introduced by Ryan on Feb 16. Then the general convolution functor we defined last time coincides with the one Ryan introduced back then:

$$\tilde{*} : \mathrm{Sph}_G \times \mathrm{Sph}_G \longrightarrow \mathrm{Sph}_G.$$

Of course we can consider fancier convolution functors. For any $x \in X$, $G(\widehat{\mathcal{K}}_x)$ acts on $\mathrm{Bun}_G^{\infty,x}$ by changing the level structure at x, and we have seen that $\mathrm{Bun}_G = \mathrm{Bun}_G^{\infty,x}/G(\widehat{\mathcal{O}}_x)$. Therefore, the general setup of the previous lecture produces a convolution functor:

$$*_x : \mathrm{Sph}_G \times D'\text{-mod}(\mathrm{Bun}_G) \longrightarrow D'\text{-mod}(\mathrm{Bun}_G).$$

The above can also be defined in a family over X, and we get the convolution functor:

$$* : \mathrm{Sph}_{G,X} \times D'\text{-mod}(\mathrm{Bun}_G) \longrightarrow D'\text{-mod}(X \times \mathrm{Bun}_G). \tag{1}$$

We can realize the latter geometrically by using the **global Hecke stack** $\mathcal{H}$. By definition, this stack represents the functor:

$$S \longrightarrow \mathcal{H}(S) = \{(x, T_1, T_2, \alpha),\ x : S \longrightarrow X,$$
$$T_1, T_2 \text{ are } G-\text{torsors on } X \times S,\ T_1|_{X\times S-\Gamma_x} \xrightarrow{\alpha} T_2|_{X\times S-\Gamma_x}\}.$$

The projections onto the T_1 and the (T_2, x) components, respectively, give rise to the following two morphisms:

$$\begin{array}{ccc} & \mathcal{H} & \\ {\scriptstyle \overleftarrow{h}} \swarrow & & \searrow {\scriptstyle \overrightarrow{h}} \\ \mathrm{Bun}_G & & X \times \mathrm{Bun}_G \end{array}$$

By patiently unwinding the definition, one sees that the convolution functor of (1) coincides with:

$$\mathcal{S} * \mathcal{M} = \overrightarrow{h}_!(\widetilde{\mathcal{S}} \otimes \overleftarrow{h}^*(\mathcal{M})).$$

Recall the Satake equivalence of categories:

$$\mathrm{Rep}_{G^L} \cong \mathrm{Sph}_G, \quad V \longrightarrow \mathcal{S}_V.$$

Any G^L-local system $\widetilde{\sigma}$ on X can be tensored up with any $V \in \mathrm{Rep}_{G^L}$, to give a $D-$module $V_{\widetilde{\sigma}}$ on X. In other words, $\widetilde{\sigma}$ is a tensor functor from Rep_{G^L} to D-mod(X).

Definition 1 *In this lecture, a Hecke **eigensheaf** with **eigenvalue** $\widetilde{\sigma}$ is a $D'-$module $\mathcal{M}$ on Bun_G, together with a compatible collection of isomorphisms:*

$$\mathcal{S}_V * \mathcal{M} \cong V_{\widetilde{\sigma}} \boxtimes \mathcal{M} \qquad \in D'\text{-mod}(X \times \mathrm{Bun}_G), \tag{2}$$

for all $V \in \mathrm{Rep}_{G^L}$, which are unital and multiplicative. Unital means that the above isomorphism is the identity when $V = \mathbb{C}$. Multiplicative means that the following diagram must commute for all $V, W \in \mathrm{Rep}_{G^L}$:

$$\begin{array}{ccccc} \mathcal{S}_{V\otimes W} * \mathcal{M} & \xrightarrow{\cong} & (\mathcal{S}_V * \mathcal{S}_W) * \mathcal{M} & \xrightarrow{\cong} & \mathcal{S}_V * (\mathcal{S}_W * \mathcal{M}) \\ {\scriptstyle\cong}\downarrow & & & & \downarrow{\scriptstyle\cong} \\ (V \otimes W)_{\widetilde{\sigma}} \boxtimes \mathcal{M} & \xrightarrow[\cong]{} & V_{\widetilde{\sigma}} \otimes (W_{\widetilde{\sigma}} \boxtimes \mathcal{M}) & \xrightarrow[\cong]{} & \mathcal{S}_V * (W_{\widetilde{\sigma}} \boxtimes \mathcal{M}) \end{array}$$

Remark 1 *The actual definition of Hecke eigensheaf is a bit stronger: it requires compatible families of isomorphisms as in* (2) *for each $n \geq 1$, where* Sph *and $\mathcal{H}$ are replaced by Ryan's* Sph_n *and $\mathcal{H}_n$. We require this data to be "factorizable", i.e. possess appropriate compatibilities as n varies.*

Opers give us some very nice candidates for eigenvalues: recall the space of global opers $\mathrm{Op}_{G^L}(X) = \mathrm{Spec}\ \mathfrak{z}(X)$. In the first semester, we struggled to produce a map $\mathfrak{z}(X) \longrightarrow \Gamma(\mathrm{Bun}_G, D')$ called the *quantum Hitchin integrable system*. Under this map, any $D'-$module over Bun_G acquires a $\mathfrak{z}(X)-$module structure. Therefore, for any point $\sigma \in \mathrm{Op}_{G^L}(X)$, we can look at:

$$D'_\sigma := D' \otimes_{\mathfrak{z}(X)} k_\sigma \quad \in D'\text{-mod}(\mathrm{Bun}_G),$$

where $k_\sigma = \mathfrak{z}(X)/\mathfrak{m}_\sigma$. Our job for today is to prove the following:

Theorem 1 *For any oper σ, D'_σ is a Hecke eigensheaf with eigenvalue σ.*

Before we jump into the proof of the Theorem, let's take a minute to understand its scope. This theorem produces a particular Hecke eigensheaf, with eigenvalue any given local system σ which admits an oper structure. We cannot and do not yet say anything about general local systems.

The proof relies heavily on our earlier study of opers, and also the notion of localization-globalization. We need the latter in the following setting:

$$(\hat{\mathfrak{g}}'\text{-mod})^{G[[t]]} \xrightarrow{\mathrm{loc}} D'\text{-mod}(\mathrm{Bun}_G), \qquad \mathrm{loc}(V) = D' \otimes_{\hat{\mathfrak{g}}'} V,$$

$$D'\text{-mod}(\mathrm{Gr}_G) \xrightarrow{\Gamma} (\hat{\mathfrak{g}}'\text{-mod})^{G[[t]]}, \qquad \Gamma(\mathcal{M}) = \Gamma(\mathrm{Gr}_G, \mathcal{M}).$$

These functors commute with the convolution action $\mathrm{Sph}_G \times \cdot \longrightarrow \cdot$.

Lemma 1 *Recall that the vacuum module is* $\mathbb{V}' = \mathrm{Ind}_{\mathfrak{g}[[t]]\oplus\mathbb{C}}^{\hat{\mathfrak{g}}'}(\mathbb{C})$*, and let δ_1 be the $\delta-$function at the unit $1 \in \mathrm{Gr}_G$. Then we have:*

$$\mathrm{loc}(\mathbb{V}') = D', \tag{3}$$

$$\mathbb{V}' = \Gamma(\delta_1), \tag{4}$$

$$\mathcal{S} * \mathbb{V}' = \Gamma(\mathcal{S}), \tag{5}$$

for any $\mathcal{S} \in \mathrm{Sph}_G$.

The proof of the above lemma is immediate from the definitions. For our next ingredient, let us fix a point $x \in X$, and let its formal neighborhood be $\mathcal{D}_x \hookrightarrow X$. Consider the following closed embeddings:

$$\begin{array}{ccccc} \sigma & \hookrightarrow & \mathrm{Op}_{G^L}(X) & \hookrightarrow & \mathrm{Op}_{G^L}(\mathcal{D}_x) \\ & & \| & & \| \\ & & \mathrm{Spec}\ \mathfrak{z}(X) & \hookrightarrow & \mathrm{Spec}\ \mathfrak{z}_x \end{array}$$

Now we can take the universal opers on the above spaces (twisted by $V \in \mathrm{Rep}_{G^L}$), each of which is the restriction of the one on its right:

$$\begin{array}{ccccc} V_\sigma|_{\mathcal{D}_x} & & \mathcal{V}^{\mathrm{g}}|_{\mathcal{D}_x} & & \mathcal{V}^{\mathrm{l}} \\ \downarrow & & \downarrow & & \downarrow \\ \sigma \times \mathcal{D}_x & \hookrightarrow & \mathrm{Op}_{G^L}(X) \times \mathcal{D}_x & \hookrightarrow & \mathrm{Op}_{G^L}(\mathcal{D}_x) \times \mathcal{D}_x \end{array}$$

The letters g and l stand for *global* and *local*. Meanwhile, V_σ is the D'-module on X that appears in (2). Restrict all these bundles to the closed point $x \in \mathcal{D}_x$:

$$\begin{array}{ccccc} V_\sigma|_x & & \mathcal{V}^{\mathrm{g}}_x & & \mathcal{V}^{\mathrm{l}}_x \\ \downarrow & & \downarrow & & \downarrow \\ \sigma & \hookrightarrow & \mathrm{Op}_{G^L}(X) & \hookrightarrow & \mathrm{Op}_{G^L}(\mathcal{D}_x) \end{array}$$

In particular, we see that $\mathcal{V}^{\mathrm{l}}_x$ is a $\mathfrak{z}_x$-module. The following result was proved in Sam's talk.

Theorem 2 *For any point $x \in X$ and any local coordinate at x, we have an identification $\mathfrak{z}_x \cong \mathrm{End}(\mathbb{V}')$. With this in mind, we have:*

$$\Gamma(\mathcal{S}_V) \cong \mathcal{V}^{\mathrm{l}}_x \otimes_{\mathfrak{z}_x} \mathbb{V}'.$$

Moreover, these isomorphisms are unital and multiplicative in V.

Proof of Theorem 1: Start by using (3), (4) and (5):

$$\mathcal{S}_V * D'|_x = \mathrm{loc}(\mathcal{S}_V * \mathbb{V}') \cong \mathrm{loc}(\Gamma(\mathcal{S}_V)).$$

Then we can apply Theorem 2 and again (3):

$$\mathcal{S}_V * D'|_x \cong \mathrm{loc}(\mathcal{V}_x^{\mathrm{l}} \otimes_{\mathfrak{z}_x} \mathbb{V}') = \mathcal{V}_x^{\mathrm{l}} \otimes_{\mathfrak{z}_x} D' = \mathcal{V}_x^{\mathrm{g}} \otimes_{\mathfrak{z}(X)} D'.$$

The last equality takes place because $\mathfrak{z}_x$ acts on D' through its quotient $\mathfrak{z}(X)$. Now let's twist the above by k_σ, i.e. restrict to the fiber above σ:

$$\mathcal{S}_V * D'_\sigma|_x = (\mathcal{S}_V * D')|_x \otimes_{\mathfrak{z}(X)} k_\sigma = (\mathcal{V}_x^{\mathrm{g}} \otimes_{\mathfrak{z}(X)} D') \otimes_{\mathfrak{z}(X)} k_\sigma =$$

$$= (\mathcal{V}_x^{\mathrm{g}} \otimes_{\mathfrak{z}(X)} k_\sigma) \otimes_{k_\sigma} (D' \otimes_{\mathfrak{z}(X)} k_\sigma) = V_\sigma|_x \otimes D'_\sigma.$$

All the above identifications respect the crystal structure of D-modules on X, and therefore give an isomorphism $\mathcal{S}_V * D'_\sigma \cong V_\sigma \boxtimes D'_\sigma$. These isomorphisms are unital and multiplicative because so are the isomorphisms of Theorem 2.