LIMITS OF CATEGORIES, AND SHEAVES ON IND-SCHEMES

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1. Inverse limits of categories

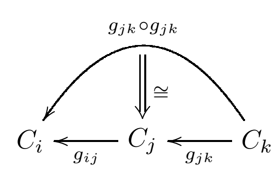
This notes aim to describe the categorical framework for discussing quasi coherent sheaves and D-modules on certain ind-schemes such as $GR_G, G(O)$ and $G(K)$. Our discussion is somewhat more general than in [1], where only ind-schemes of ind-finite type are discussed (hence $G(O)$ and $G(K)$ are excluded).

1.1. **The Data.** Throughout these notes, all categories are assumed to Abelian, and possess all direct limits (equivalently arbitrary direct sums). All functors are assumed to commute with direct filtered limits.

The datum we shall deal with is that of $\{C_i\}_{i\in I}$ a filtered, ordered system of categories, where for every $i < j$ we have a pair of adjoint functors (f_{ji}, g_{ij})

$$C_i \underset{g_{ij}}{\overset{f_{ji}}{\rightleftarrows}} C_j$$

together with a co-cycle of natural isomorphisms for the compositions of g's:



in turn, via the adjunctions, these give rise to uniquely determined natural transformations for the composition of the f's.

In light of the following examples we shall think of the f's as "going up", and the g's as "going down".

Example 1.1.

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Let $Y = \varinjlim Y_i$ a presentation of a reasonable strict ind-scheme over $\mathbb{C}$. I.e. Y_i's are schemes, $Y_i \xrightarrow{\iota_{ji}} Y_j$ are closed embeddings, and the ideal of Y_i in Y_j is finitely generated (this is needed to ensure that !-pullback exists).

(1) $C_i = \mathfrak{Q}co\,(Y_i)$ and

$$\mathfrak{Q}co\,(Y_i) \underset{g_{ij}=\iota_{ij}^!}{\overset{f_{ji}=\iota_{ji*}}{\rightleftarrows}} \mathfrak{Q}co\,(Y_j)$$

Actually even better, just think of an ind-affine scheme, i.e. let A be an abelian, complete, separated topological ring whose toplogy is generated by a filtered system of open ideals $\{I_i\}$, s.t. $I_i + I_j$ is finitely generated over $I_i \cap I_j$ (In the affine case this assumption can be relaxed); then $C_i = A/I_i - \text{mod}$.

(2) $C_i = D - \text{mod}(Y_i)$, same maps as above.

1.2. **Inverse limits.** To the data $\{C_i, g_{ji}, f_{ij}\}$ one might try to associate four limits; either inverse or inductive and with respect to either $f's$ or $g's$.

Our object of interest is $C := \varprojlim_{g_{ji}} C_i$, this beast is defined by the same universal property that always characterizes inverse limits, this time in the 2-category of categories. It is a category whose objects consist of sequences of "strictly[1] g_{ji}-compatible" objects, i.e. a sequence $x_i \in C_i$, and isomorphisms $x_i \xrightarrow{\cong} g_{ij}(x_j)$. Morphisms are sequences of morphisms $\{x_i \to x_i'\}_{i \in I}$ which are compatible in the sense that following squares commute

$$\begin{array}{ccc} x_i & \longrightarrow & g_{ji}(x_j) \\ \downarrow & & \downarrow \\ x_i' & \longrightarrow & g_{ji}(x_j') \end{array}$$

Alternatively, think of I as an index category, and think of $\cup C_i$ as a fibered category over I. Then $\varprojlim_{g_{ji}} C_i$ is it's category of g_{ji}-cartesian sections (cf. the definition of quasi coherent sheaves on algebraic stacks).

C admits component maps $C_i \xleftarrow{g_i} C$ (for $x = (x_i) \in C$, $g_i(x) = x_i$). These are appropriately compatible with g_{ji}'s, i.e. have natural isomorphisms

[1] In contrast with lax sequences, where the maps are not required to be isomorphisms.

$$\begin{array}{ccc} & & C \\ & {}^{g_i}\swarrow \;\; \Nwarrow & \downarrow {}^{g_j} \\ C_i & \underset{g_{ij}}{\longleftarrow} & C_j \end{array}$$

Example 1.2. In example 1 objects of this inverse limit are often called !-sheaves. When $Y = \mathrm{spf} A$, this inverse limit is no more than the category of continuous A-mod's, which are topologically discrete.

Remark 1.3. C is an abelian category, and contains all filtered direct limits. In particular, kernels and filtered direct limits are taken termwise, but co-kernels require some more tampering, this is done in 1.6.

1.3. **Mapping out of C[2].** The description above makes mapping out of C easy (as is always the case for inverse limits). Note that so far we have made no use of the f's, the role they play is to allow a description of functors out of C.

Proposition 1.4. *The functors g_i admit left adjoints f_i*

$$C_i \underset{g_i}{\overset{f_i}{\rightleftarrows}} C$$

Proof. Fix $i \in I$, in order to define f_i we must define a g_{ji} compatible family of functors $\left(C_i \xrightarrow{(f_i)_j} C_j \right)$ ($f_i's$ components). Indeed, given an object $x \in C_i$ let

$$(f_i(x))_j = \varinjlim_{k>i,j} g_{jk} \circ f_{ki}(x)$$

This is directed by the morphisms, for every $k' > k$,

$$g_{jk} \circ f_{ki}(x) \xrightarrow{\text{adjunction}} g_{jk} \circ g_{kk'} \circ f_{k'i}(x) = g_{jk'} \circ f_{k'i}(x)$$

It is routine to check that objects are g_{ji} compatible. Defining f_i on morphisms is straightforward as well.

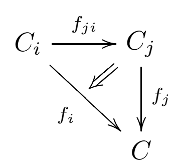
[2]This subsection is not used until deinition 2.8.

Let us show that (f_i, g_i) are adjoint.

$$
\begin{aligned}
\mathrm{Hom}_C\left(f_i(x_i), (y_j)\right) &= \\
&= \varprojlim_{j>i} \mathrm{Hom}_{C_j}\left(\varinjlim_{k>j} g_{jk} \circ f_{ki}(x_i), y_j\right) \\
&= \varprojlim_{j>i} \varprojlim_{k>j} \mathrm{Hom}_{C_j}\left(g_{jk} \circ f_{ki}(x_i), g_{jk}\left(y_k\right)\right) \\
&= \varprojlim_{k>i} \varprojlim_{k>j>i} \mathrm{Hom}_{C_j}\left(g_{jk} \circ f_{ki}(x_i), g_{jk}\left(y_k\right)\right) \\
&= \varprojlim_{k>i} \mathrm{Hom}_{C_j}\left(f_{ki}(x_i), y_k\right) \\
&= \mathrm{Hom}_{C_i}\left(x_i, y_i\right)
\end{aligned}
$$

The 4th equality follows from the fact that the Hom-sets in question form a double directed system with respect to j and k. □

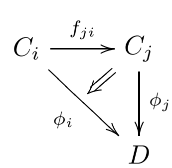
Remark 1.5. The f_i's are f_{ij} compatible, i.e. there exist natural isomorphisms



Construction 1.6. We may use 1.4 to construct objects of C as follows. Given the data of $c_i \in C_i$ and a compatible family of morphisms[3] $f_{ji}(c_i) \to c_j$ (equivalently, a compatible family $c_i \to g_{ij}(c_j)$), the (f_i, g_i)-adjunction makes $f_i(c_i)$ a directed system of objects in C, to which we associate the object $\varinjlim f_i(c_i)$. For instance, this is the method to construct co-kernel in C from the termwise co-kernels (which are a g-lax, but not strict, sequence).

Remark 1.7. If it happens that $f_{ij} \circ g_{ji} \cong 1_{C_i}$ (equivalently f_{ij} is fully faithfull), as is the case in both the examples in 1.1, then $f_j : C_i \to C$ simply amounts to taking an object of C_j $f^*\#@$-ing it up along all $i > j$ and g'ing down along all $i < j$ (cf. exercise 1.2).

Proposition 1.8. *For any category D (satisfying our assumptions) the datum of a functor $\Phi : C \to D$ is equivalent to that of a collection of functors and a co-cycle of natural isomorphisms*



[3] i.e. a lax f-sequence

That is $\varprojlim_{g_{ji}} C_i = C = \varinjlim_{f_{ij}} C_i$ [4].

Proof. Given Φ, define $\phi_i = \Phi \circ f_i$, the adjunction is used to shown f_{ij}-compatibility.

Conversely, assume we are given the datum of the ϕ_i's and natural isomorphisms. Note that for every $x = (x_i) \in C$ we get a directed system of objects in D by

$$\phi_i(x_i) \cong \phi_j \circ f_{ji}(x_i) \to \phi_j(x_j)$$

(the last map is given by the adjunction of (f_{ij}, g_{ji})). Define $\Phi(x) = \varinjlim \phi_j(x_j)$ on objects, and termwise on morphisms.

To complete the claim it remains to present an isomorphisms of functors

$$\Phi \cong \varinjlim_i \Phi \circ f_i \circ g_i \quad \text{and} \quad \phi_i \cong \varinjlim_j \phi_j \circ g_j \circ f_i$$

Unraveling (and remembering all functors commute with direct limits) Both of these reduce to the existence of

$$1_C \cong \varinjlim_i f_i \circ g_i$$

We leave this as an exercise in unraveling the definitions and commuting limits. □

Remark 1.9. If one is only interested the construction of a functor $\Phi : C \to D$, one can relax the requirement in 1.8, that the transformations be isomorphisms, and allow any co-cycle of natural transformations. The same construction is used. Of course, distinct data may now give rise to equivalent functors.

1.4. **Compactly generated categories.** Next we introduce the notions of compact objects and categories, which simplify the story above. Recall that maps in to inverse limits are easily understood, but mapping out is generally trickier.

Definition 1.10. An object $x \in C$ is called *compact* if for any directed system of objects the natural map below is an isomorphism

$$\mathrm{Hom}(x, \varinjlim x_i) \to \varinjlim \mathrm{Hom}(x, x_i)$$

For a category C, we denote by C^c it's full subcategory of compact objects.

Definition 1.11. A category C is called *compactly generated* if every object in C is the direct limit of compact objects.

Example 1.12. For any quasi-projective scheme X consider $\mathfrak{Q}co\,(X)$; it's compact objects are finitely presented sheaves, and it is compactly generated.

[4]In the sense that is satisfies the appropriate universal property in the 2-category of inductively complete categories and functors which preserve inductive limits.

Observation 1.13. (1) A compactly generated category is completely determind by it's compact objects in the sense that for any two objects, choosing a limit presentation we may describe their morphisms as well.

$$\mathrm{Hom}(c,d) = \mathrm{Hom}(\varinjlim c_i, \varinjlim d_j) = \varprojlim_i \varinjlim_j (c_i, d_j)$$

(2) If a functor $F : C \to D$ admits a right adjoint, G, which commutes with direct limits, then F sends compact objects to compact objects. If $c \in C$ is compact

$$\mathrm{Hom}_D\left(F(c), \varinjlim d_i\right) = \mathrm{Hom}_C\left(c, \varinjlim G(d_i)\right) = \varinjlim \mathrm{Hom}_D(c, G(d_i)) = \varinjlim \mathrm{Hom}_D(F(c), d_i)$$

Now let use return to our system of categories $\{C_i\}$ and it's inverse limit C.

Proposition 1.14. *If each C_i is compactly generated, then so is C. An object in C is compact iff it is isomorphic to $f_i(c_i)$ for some $i \in I$ and $c_i \in C_i^c$.*

Proof. By the observation above, f_i carries the compact objects of C_i to compact objects in C. Since f_i also commutes with direct limits, the image of each f_i consists of compactly generated objects. As noted in the proof of 1.8, for every $c \in C$ we have $c \cong \varinjlim f_i g_i(c)$. This proves the first assertion.

For the second assertion let $c \in C$ be a compact object. This is the first place we use that C_i and C are all abelian categories. Present $c = \varinjlim f_i(c_i)$ for $c_i \in C_i^c$, as c is compact the identity on c must factor $c \to f_i(c_i) \to c$ for some i . It follows $f_i(c_i) = k \oplus c$ for some object $k \in C$. k must be compact as well (direct summand of a compact) so the same argument shows there exists a surjection $f_j(k_j) \twoheadrightarrow k$, for some $k_j \in C_j^c$. With out loss of generality (replacing i and j by some $k > i, j$), we assume $i = j$ and so obtain a presentation $f_i(k_i) \xrightarrow{\alpha} f_i(c_i) \to c \to 0$.

To conclude, note

$$\begin{aligned}
\mathrm{Hom}_C(f_i(k_i), f_j(c_j)) &= \varinjlim_k \mathrm{Hom}_{C_k}(f_{ki}(k_i), f_{kj}(c_j)) \\
&= \mathrm{Hom}_{C_i}(k_i, g_i f_j(c_j)) \\
&= \varinjlim_k \mathrm{Hom}_{C_i}(k_i, g_{ik} f_{kj}(c_j)) \\
&= \varinjlim_k \mathrm{Hom}_{C_k}(f_{ki}(k_i), f_{kj}(c_j))
\end{aligned}$$

thus α comes from some map $f_{ki}(k_i) \xrightarrow{\alpha_k} f_{kj}(c_j)$. It follows that

$$c = h_k\left(\mathrm{coker}\left(f_{ki}(k_i) \xrightarrow{\alpha_k} f_{kj}(c_j)\right)\right)$$

hence comes from C_k. □

Observation 1.15. The point of all this is that $f_i(c_i)$, for all $c_i \in C_i^c$, generate all the objects in C. Hom's from a compact object considerably, for $c_i \in C_i^c$

$$\mathrm{Hom}_C\left(f_i(c_i), \left(c_j^{'}\right)\right)\Big) = \varinjlim_k \mathrm{Hom}_{C_k}\left(f_{ki}(c_i), c_k^{'}\right)$$

so every morphism comes from some "finite stage". Moreover, in order to define a functor $C \to D$, it suffices to define appropriately compatible functors $C_i^c \to D$[5].

2. D-modules on infinite type schemes and on ind-schemes

We wish to construct the objects in the title out of D-mod's on finite type schemes. The "ind" part in the title has essentially been taken care of, e.g. on a finite type ind-scheme, $Y = \varinjlim Y_i$ (i.e. Y_i are of finite type over $\mathbb{C}$. e.g. Gr_G), define

$$\text{D-mod}(Y) = \varprojlim_{\iota_{ji}^!} \text{D-mod}(Y_i)$$

One could do the same for an arbitrary ind-scheme - if only he could make sense of D-mod(Y_i), and $\iota_{ji}^!$ when the Y_i's are not of finite type. That is the purpose of this section.

2.1. D-mod's on schemes of infinite type.

Definition 2.1. A scheme Y is called *good* if it admits a presentation

$$Y = \varprojlim Y_j$$

where the Y_j's are schemes of finite type and $Y_j \xleftarrow{\pi_{jk}} Y_k$ is a smooth surjection.

Example 2.2.

(1) Any scheme of finite type is good.
(2) $\mathrm{spec}\mathbb{C}[x_1, x_2, \cdots] = \varprojlim \mathbb{A}^n$ is good. More generally, for any smooth finite type scheme$X$, the scheme $X[[t]] = \varprojlim X[t]/t^n$ is good , e.g. $G(O)$ is good.
(3) A closed subscheme of a good scheme, whose ideal is finitely generated, is good.
(4) $\mathrm{spec}\mathbb{C}[[t]]$ is not good (but $\mathrm{spf}\mathbb{C}[[t]]$ will be reasonable - the corresponding notion for ind-schemes).

Let $C_i = \text{D-mod}(Y_i)$. Since π_{ij} is assumed to be smooth π_{ij}^* exists, and we get an inverse system of categories, as in section 1, using the maps

[5]We stress all functors are assumed to commute with direct limits.

$$\begin{array}{ccccc}
\text{D-mod}(Y_i) & \underset{g_{ij}=\pi_{ji*}}{\overset{f_{ji}=\pi^*_{ji}}{\rightleftarrows}} & \text{D-mod}(Y_j) & \cdots \underset{\pi_{i*}}{\overset{\pi_i^*}{\rightleftarrows}} & \text{D-mod}(Y) \\
Y_i & \xleftarrow[\pi_{ji}]{} & Y_j & \cdots \longleftarrow & Y
\end{array}$$

set

$$\text{D-mod}(Y) := \varprojlim_{\pi_*} \text{D-mod}(Y_i) \cong \varinjlim_{\pi^*} \text{D-mod}(Y_i)$$

and denote the projections by $\text{D-mod}(Y) \xrightarrow{\pi_{i*}} \text{D-mod}(Y_j)$.

The discussion so far has been only in the context of the abelian category, however π_* on this level is unsatisfactory and we shall want to make sense of all this in a derived setting. In fact, all the notions of section 1 make sense for triangulated categories, equipped with DG-models. The issue is choosing which derived categories to use (left or right D-mod's). There is a caveat here and we postpone this point.

2.2. **D-mod operations.** Let $Y \xrightarrow{h} Z$ be a map of good schemes. We show that $h_* : \text{D-mod}(Y) \to \text{D-mod}(Z)$ always exists, and discuss when $h^!$ exists.

Construction 2.3. $h_* : \text{D-mod}(Y) \to \text{D-mod}(Z)$.

We must construct compatible functors $\text{D-mod}(Y) \to \text{D-mod}(Z_i)$. As Z_i is of finite type, h factors for j large enough

$$\begin{array}{ccc}
Y & \xrightarrow{h_i} & Z_i \\
{\scriptstyle \pi_j}\downarrow & \nearrow_{h_{ji}} & \\
Y_j & &
\end{array}$$

For such j, we define h_{i*} as the composition $\text{D-mod}(Y) \xrightarrow{\pi_{i*}} \text{D-mod}(Y_j) \xrightarrow{h_{ji*}} \text{D-mod}(Z_i)$ (note this does not depend on j, as long as it large enough so the map factors). There exists a co-cycle of natural isomorphisms

$$\begin{array}{ccc}
\text{D-mod}(Y) & \xrightarrow{h_i} & \text{D-mod}(Z_i) \\
& {\scriptstyle h'_{i*}}\searrow \;\; \swarrow\!\!\!\!\swarrow & \downarrow{\scriptstyle \pi_{ii'*}} \\
& & \text{D-mod}(Z_{i'})
\end{array}$$

thus we get our functor $\text{D-mod}(Y) \xrightarrow{h_*} \text{D-mod}(Z)$. Note that if $g : Z \to W$ then $g_* \circ h_* \cong (g \circ h)_*$.

Also note, on the abelian level h_{i*} is a composition of left exact functors, hence is left exact. h_* is an inverse limit of left exact functors hence also left exact.

In general, $h^!$ will not exist, however we do have the following special case.

Proposition 2.4. *If $h : Y \hookrightarrow Z$ is a closed embedding, and the ideal of Y in Z is finitely generated; then h_* admits a right adjoint $h^! : D\text{-}mod(Z) \to D\text{-}mod(Y)$.*

Proof. As this is a local question, assume $Y = \mathrm{spec}A$ and $Z = \mathrm{spec}B$ are affine. Let $Z = \varprojlim_{j \in J} Z_j$ be a presentation of Z. In general, there is no reason for Y to have a presentation whose underlying poset is related to that of Z. However, in our case we can arrange for both schemes to be presented on the same poset and for the map to "factor through" the presentation as follows. Let $I \subset B$ be the ideal of Y in Z, it is finitely generated by assumption; whence we may assume that for all j $I \cap B_j$ generates I , where $B_j := \mathcal{O}_{Z_j}$. Certainly $A = B/I = \varprojlim_j B_i/B_i \cap I$, moreover the following square is cartesian by construction

$$\begin{array}{ccc} B_i/B_i \cap I & \longleftarrow & B_i \\ \uparrow & & \uparrow \\ B_j/B_i \cap I & \longleftarrow & B_j \end{array}$$

implying this is a good presentation of Y. Evidently $h = \varprojlim h_i$, where

$$h_i : Y_i := \mathrm{spec}B_i/B_i \cap I \to \mathrm{spec}B_i = Z_i$$

Since for every i we have the adjoint pair

$$\text{D-mod}(Y_i) \underset{h_i^!}{\overset{h_{i*}}{\rightleftarrows}} \text{D-mod}(Z_i)$$

we are led to define $h^! = \varprojlim_{\iota^!_{ji}} h^!_i$. This is indeed seen to be a right adjoint to h_*. □

Remark 2.5. It is obvious from the proof that $h^!$ exists in somewhat greater generality than closed embeddings, i.e. whenever Y and Z may be presented on the same poset, I, and the map h factors through this presentation, with cartesian squares as above. For instance this is the case when $Y \xrightarrow{h} Z$ is *finitely presented*, i.e. factors locally

$$Y \to Z \times \mathbb{A}^n \to Z$$

where the first map has a finitely generated ideal.

2.3. **D-mod's on reasonable ind-schemes.** A "reasonable" ind-scheme will be any scheme where the constructions above go through, namely

Definition 2.6. An ind-scheme Y is called *reasonable* if it admits a presentation

$$Y = \varinjlim_i \varprojlim_j Y_{ij}$$

where Y_{ij} are schemes of finite type, $Y_{ij} \to Y_{ik}$ is smooth and surjective, and $Y_i \to Y_{i'}$ is a closed embedding (of schemes) with locally finitely generated ideal.

For a reasonable ind-scheme Y, we define $\text{D-mod}(Y) = \varprojlim_{\iota^!_{i'i}} \text{D-mod}(Y_i)$. Note we could have defined this category using the universal presentation (using all good closed sub-schemes of Y). As any other presentation is co-final in the universal one, the definition does not depend on the presentation.

Example 2.7.

(1) Any ind-scheme of ind-finite type is reasonable.
(2) $G(K)$ is a reasonable ind-scheme[6]. To see this recall that $Gr_G = G(K)/G(O) = \varinjlim_i Z_i$, for some finite type schemes Z_i, let $Y_i = Z_i \times_{Gr_G} G(K)$ it is a closed sub-scheme of $G(K)$. We proceed to show that $G(K) = \varinjlim_i Y_i$ is a good presentation of $G(K)$. Indeed, $G(O) = \varprojlim_j G[t]/t^j$, let $G_j = \ker\left(G(O) \to G[t]/t^j\right)$ then

$$Y_i = \varprojlim_j Y_i/G_j$$

If $j > j'$ then the map $Y_i/G_j \to (Y_i/G_j)/G_{j'} = Y_i/G_{j'}$ is smooth and surjective, hence the presentation of Y_i above is good.

Definition 2.8 (De-Rham cohomology)**.** Define the functor $H_{DR}(Y, -) := \varinjlim_i H_{DR}(Y_i, -)$ using 1.8, by noting that the latter functors are π_{i*}-compatible.

References

[1] Beilinson; Drinfeld. Hitchin's integrable system. *http : //www.math.harvard.edu/* ∼ *gaitsgde/grad*_2009.

[6] However, even when X is smooth, $X((t))$ is not always reasonable