Vanishing cycles for algebraic $\mathcal{D}$-modules

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1 Introduction

1.1 The gluing problem

Let X be a smooth variety over an algebraically closed field $\Bbbk$ of characteristic 0, and let $f : X \to \Bbbk$ be a regular function. Assume that f is smooth away from the locus $Y = f^{-1}(0)$. We have varieties and embeddings as depicted in the diagram

$$Y \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U = X - Y.$$

For any space Z we let $\mathrm{Hol}(\mathcal{D}_Z)$ denote the category of holonomic $\mathcal{D}_Z$-modules. The main focus of this (purely expository) thesis will be answering the following slightly vague question.

Question 1.1. Can one "glue together" the categories $\mathrm{Hol}(\mathcal{D}_Y)$ and $\mathrm{Hol}(\mathcal{D}_U)$ to recover the category $\mathrm{Hol}(\mathcal{D}_X)$?

Our approach to this problem will be to define functors of (unipotent) nearby and vanishing cycles along Y, $\Psi_f : \mathrm{Hol}(\mathcal{D}_U) \to \mathrm{Hol}(\mathcal{D}_Y)$ and $\Phi_f : \mathrm{Hol}(\mathcal{D}_X) \to \mathrm{Hol}(\mathcal{D}_Y)$ respectively. Using these functors and some linear algebra, we will build from $\mathrm{Hol}(\mathcal{D}_U)$ and $\mathrm{Hol}(\mathcal{D}_Y)$ a *gluing category* equivalent to $\mathrm{Hol}(\mathcal{D}_X)$. This strategy is due to Beilinson, who gives this construction (and in fact does so in greater generality) in the extraordinarily concise article [B]. Since that paper omits many essential details, there seems to be a place for a more leisurely exegesis (one might say "baby version") of the article, specialized to the setting of holonomic $\mathcal{D}$-modules.

A word of motivation: beyond the intrinsic interest of Question 1.1, the vanishing cycles and related functors for $\mathcal{D}$-modules which we shall construct (and corresponding constructions for perverse sheaves) have a wealth of applications. However, such applications – for instance to representation theory [BB] and algebraic geometry – are far beyond our scope.

1.2 A simple case

To fix ideas and make Question 1.1 more well-defined, let us consider the simplest relevant example. For definiteness, set $\Bbbk = \mathbb{C}$. Let $X = \mathbb{A}^1$, and let f be the coordinate t on X, so that $Y = \{0\}$. Write $O_X = \mathbb{C}[t]$, $O_U = \mathbb{C}[t, t^{-1}]$, $O_{X,0} = \mathbb{C}[t]_{(t)}$, and $O_Y = \mathbb{C}$. Differential operators on X and U are generated by the vector field $\partial = \frac{\mathrm{d}}{\mathrm{d}\,t}$.

Before considering $\mathcal{D}$-modules at all, we remark that one can formulate a much easier commutative analogue of Question 1.1, asking about categories of $\mathcal{O}$-modules rather than $\mathcal{D}$-modules. To obtain an Artinian category $\mathrm{Mod}(O_X)^{\mathrm{fd}}$ of O_X-modules analogous to $\mathrm{Hol}(\mathcal{D}_X)$, we restrict our attention to those which are finite-dimensional as $\mathbb{C}$-vector spaces. In analogy with holonomic $\mathcal{D}$-modules, these are the O_X-modules whose support is as small as possible in dimension: any finite-dimensional O_X-module M is supported on the 0-dimensional set of roots of the characteristic polynomial of $t \in O_X$ acting on M. This perspective makes it clear that there is an equivalence of categories

$$\mathrm{Mod}(O_X)^{\mathrm{fd}} \leftrightarrow \mathrm{Mod}(O_U)^{\mathrm{fd}} \times \mathrm{Mod}(O_{X,0})^{\mathrm{fd}}$$

given (from left to right) by localization and (from right to left) by taking direct sum. Furthermore, an object in $\mathrm{Mod}(O_{X,0})^{\mathrm{fd}}$ is precisely a finite-dimensional vector space – i.e., an object of $\mathrm{Mod}(O_Y)^{\mathrm{fd}}$ – equipped with a nilpotent endomorphism, multiplication by t. Thus the righthand side can be regarded as "glued" from $\mathrm{Mod}(O_U)^{\mathrm{fd}}$ and $\mathrm{Mod}(O_Y)^{\mathrm{fd}}$ using linear algebra.

In the case of $\mathcal{D}$-modules, keeping X, Y, U as we have defined them, it would be naïve to expect quite so simple an answer as was found in the commutative case. Nonetheless, since in this example Y is a point, holonomic $\mathcal{D}_Y$-modules are nothing more than finite-dimensional vector spaces, so it is still reasonable to expect a fairly simple answer. To obtain a really nice answer we shall be slightly more restrictive about the $\mathcal{D}_X$-modules we are considering. In particular, let us consider the subcategory $\mathrm{Hol}^0_{\mathrm{reg}}(\mathcal{D}_X)$ of regular holonomic $\mathcal{D}$-modules with no singularities away from the origin. These are precisely those holonomic $\mathcal{D}_X$-modules whose restrictions to U are not only $\mathcal{O}_U$-coherent, but *regular integrable connections.*[1] For example, let $P(t, \partial)u = 0$ be an algebraic differential equation on $X = \mathbb{C}$ of order n, where

$$P(t, \partial) = \sum_{i=0}^{n} a_i(t)\partial^i$$

and the a_i are polynomials. The corresponding holonomic $\mathcal{D}_X$-module $\mathcal{D}_X/\mathcal{D}_X P$ is in $\mathrm{Hol}^0_{\mathrm{reg}}(\mathcal{D}_X)$ if and only if $\mathrm{ord}_{t=0}\, a_i \geq \mathrm{ord}_{t=0}\, a_n + n - i$ for each i, and a similar condition holds at $t = \infty \in \mathbb{P}^1 \supset X$ (in terms of a suitable local coordinate). By a classical theorem of Fuchs, such a condition at a point p where the differential equation has a singularity, is equivalent to a "moderate growth" condition on the solutions u to the equation near p (cf. e.g. [HTT, Thm. 5.1.4]).

Let $\mathrm{Loc}_U \subset \mathrm{Hol}(\mathcal{D}_U)$ denote the full subcategory of regular integrable connections as above. Justifying the notation, the famous Riemann-Hilbert Correspondence entails that the category Loc_U is equivalent to the category of local systems $\mathcal{L}$ on the punctured affine line $\mathbb{C}^\times$, by taking sheaf of local solutions to the differential equation corresponding to the $\mathcal{D}$-module . Let $\mathrm{Hol}^0_{\mathrm{reg}}(\mathcal{D}_X) \subset \mathrm{Hol}(\mathcal{D}_X)$ denote the subcategory of those modules whose restriction to U is in Loc_U. A more refined version of Question 1.1 in our simple setup is the following.

Question 1.2. Is there a linear algebraic construction of a "gluing category" $\mathrm{Glue}(U, Y)$ from Loc_U and $\mathrm{Hol}(\mathcal{D}_Y)$ = finite dimensional vector spaces, such that $\mathrm{Hol}^0_{\mathrm{reg}}(\mathcal{D}_X)$ is equivalent to $\mathrm{Glue}(U, Y)$?

The category of local systems on $\mathbb{C}^\times$ is equivalent to the category of representations of $\pi_1(\mathbb{C}^\times, 1) = \mathbb{Z}$, an object of which is simply a finite dimensional vector space V (obtained as the stalk $\mathcal{L}_1$ of $\mathcal{L}$ at $1 \in \mathbb{C}^\times$) equipped with an invertible linear operator u (obtained as the monodromy action around the puncture). Consequently Loc_U is not much more complicated than $\mathrm{Hol}(\mathcal{D}_Y)$: it is itself constructed entirely in terms of linear algebra. So an equivalent version of the gluing problem in this setup is

Question 1.3. Is the category $\mathrm{Hol}^0_{\mathrm{reg}}(\mathcal{D}_X)$ equivalent to a category of collections of vector spaces and specified linear maps among them?

We will answer Question 1.3 affirmatively in §4.2

[1] See [Bor, III] or [HTT, Chs. 4&5] for the definition of this notion.

1.3 Outline of the rest of this thesis

A crucial tool for us will be the notion of the b-function of a holonomic $\mathcal{D}_U$-module. This is discussed in §2. In §3 the main results are proved: we construct nearby and vanishing cycle functors for $\mathcal{D}$-modules, and demonstrate their main properties. In §4 we use these functors to construct the required gluing category to answer Question 1.1.

1.4 Background and notation

I have tried to make this thesis more or less self-contained, modulo some category theoretic constructions recalled in Appendix A. An elementary overview of algebraic $\mathcal{D}$-modules, including (mostly without proof) all the facts about them we need below can be found in Appendix B.

There are competing notations for the various functors and categories used when working with $\mathcal{D}$-modules; we largely, but not entirely, follow [Ber] and [G]. Since this thesis is somewhat notationally heavy, a summary of our notation can be found in Appendix C, which also serves as a "Quick Reference" for the basic definitions and properties of the categories and functors we discuss.

A few loose notational conventions: capital letters usually denote varieties (X, Y, Z, U); capital script letters $(\mathcal{F}, \mathcal{G}, \mathcal{M}, \mathcal{N}, \cdots)$ denotes sheaves (all our sheaves are quasicoherent); capital Greek letters are mostly reserved for functorial operations on sheaves (Π, Ψ, Φ, Ξ); lowercase Greek letters generally denote either morphisms (α, β, γ) of sheaves or sections $(\mu, \varphi, \psi, \xi)$ of sheaves.

1.5 Acknowledgments

I owe an enormous debt to Dennis Gaitsgory. He guided me as I learned about $\mathcal{D}$-modules, and introduced me to the references [B] and [BG] as I was learning about b-functions; this thesis would not have been possible without his help. Thanks, too, to my family and to my girlfriend Connie, for putting up with me during the writing process. Finally, thanks to my roomates Rosen, Charlie, Peter, and Prabhas for invaluable moral support.

2 The lemma on b-functions

2.1 Statement

Consider the "XYU" setup of §1.1:

$$f^{-1}(0) = Y \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U = X - Y.$$

The ring $\mathcal{O}_U$ differs from $\mathcal{O}_X$ in that we can divide by f in the former, but not in the latter. In $\mathcal{D}_X$, however, a quasi-inverse to multiplication by f may already exist. In fact such a quasi-inverse does exist, in the following sense.

Theorem 2.1 (Lemma on b-functions)**.** *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_U$-module and $m \in \mathcal{M}$ a section. Then there exist $d_{ij}[s] \in \mathcal{D}_X[s]$ and $0 \neq b(s) \in \Bbbk[s]$ such that for all $n \in \mathbb{Z}$ we have the identities*

$$d[n](f^n m) = b(n) \cdot f^{n-1} m.$$

Theorem 2.1 is due independently to Bernstein [Ber2] and Sato.[2] The monic generator of the ideal in $\Bbbk[s]$ of polynomials b satisfying the theorem is known as the **b-function** or **Bernstein-Sato polynomial** of m. The point of the theorem is that away from the finitely many integer roots of the b-function, one can in fact "divide by f" in $\mathcal{D}_X$. By taking $\mathcal{M} = \mathcal{O}_U, m = 1$ we obtain the "classical" b-function lemma, which provides for just the sort of quasi-inverse described above. A well-known, but hardly representative, example of the classical b-function of a polynomial f is the following.

Example 2.2. Let $X = \mathbb{A}^r$ with coordinates $t_1, \ldots, t_r$, and $\partial_1 = \frac{\partial}{\partial t_1}, \ldots, \partial_r = \frac{\partial}{\partial t_r}$ the generators for vector fields on X, so that $D_X = \Bbbk\langle t_1, \ldots, t_r, \partial_1, \ldots, \partial_r\rangle$ with the relations $[\partial_i, t_j] = \delta_{ij}$ and $[\partial_i, \partial_j] = [t_i, t_j] = 0$. Let $f = \sum t_i^2$ and $\Delta = \sum \partial_i^2$ (the Laplacian). It is easy to compute explicitly

$$\Delta(f^{n+1}) = 4(n+1)(n + \frac{r}{2})f^n.$$

So $d = \frac{\Delta}{4} \in D_X \subset D_X[s]$ and $b(s) = (s+1)(s + \frac{r}{2})$ satisfy the conditions of the theorem; the latter is the b-function of f. It is worth pointing out that in general there need not be any resemblance between a polynomial and its b-function; the fact that in this case f and b are both quadratic is a coincidence. The fact that b has negative rational numbers as roots, however, is a general phenomenon, a deep theorem of Kashiwara [K].

Theorem 2.1 has a number of applications. A standard consequence is that it can be used to show [Ber, 3.8] that direct image along an open embedding preserves holonomicity.For another application we recall the fundamental classification result concerning irreducible holonomic $\mathcal{D}$-modules.

Theorem 2.3. *[Ber, 3.14] Let $\alpha : Z \to W$ be a locally closed embedding with Z irreducible, and let $\mathcal{E}$ be an irreducible holonomic $\mathcal{D}_Z$-module. Let*

$$\alpha_{!*}\mathcal{E} = \mathrm{im}(\mathcal{H}^0\alpha_!\mathcal{E} \to \mathcal{H}^0\alpha_*\mathcal{E})$$

denote the Goresky-MacPherson extension of $\mathcal{E}$.Then $\alpha_{!*}\mathcal{E}$ is a holonomic $\mathcal{D}_W$-module, and is the unique irreducible subquotient of $\alpha_*\mathcal{E}$ (or $\alpha_!\mathcal{E}$) with nonzero restriction to Z. Moreover, any irreducible holonomic $\mathcal{D}_W$-module $\mathcal{F}$ is of the form $\alpha_{!*}\mathcal{E}$ for some affine embedding $\alpha : Z \to W$ and some irreducible $\mathcal{O}$-coherent $\mathcal{D}_Z$-module $\mathcal{E}$. □

For a general holonomic $\mathcal{D}_U$-module $\mathcal{M}$ the Goresky-MacPherson extension $j_{!*}\mathcal{M}$ defined as in the theorem gives the smallest submodule of $j_*\mathcal{M}$ whose restriction to U is $\mathcal{M}$. We shall see below (Corollary 2.8) that the b-function lemma gives an algorithm for computing this module. Somewhat similarly, Gaitsgory and Beilinson [BG] use the b-function lemma to give an algorithm for computing $j_!\mathcal{M}$. The most important use of the b-function lemma in this thesis, Theorem 3.31, also involves the relationship between $j_!\mathcal{M}$ and $j_*\mathcal{M}$, in a special case.

[2]But I am not aware of a reference for Sato's work.

2.2 Proof

To prove the b-function lemma we first give a reformulation. For this we need a special module, which (following [BG]) we denote by "f^s". We begin by considering the sheaves $\mathcal{O}_U[s] = \mathcal{O}_U \underset{\Bbbk}{\otimes} \Bbbk[s]$ and $\mathcal{D}_U[s] = \mathcal{D}_U \underset{\Bbbk}{\otimes} \Bbbk[s]$ (where $\Bbbk$ and $\Bbbk[s]$ denote constant sheaves and s is a formal variable) of polynomials in s whose coefficients are regular functions and differential operators, respectively. The module "f^s" is defined as the free $\mathcal{O}_U[s]$-module of rank 1, generated by a formal symbol f^s. The $\mathcal{D}_U$-action is induced by the formula

$$\xi(f^s) = s\xi(f)f^{-1} \cdot f^s,$$

for any vector field ξ on U.

Theorem 2.4. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_U$-module and $m \in \mathcal{M}$ a section. There exists a polynomial $b(s) \in \Bbbk[s]$ and a differential operator $d(s) \in \mathcal{D}_U[s]$ such that the identity*

$$d(s) \cdot (fm \otimes f^s) = b(s)(m \otimes f^s)$$

holds in the $\mathcal{D}_U[s]$-module $\mathcal{M} \otimes$ "f^s" $= \mathcal{M} \underset{\mathcal{O}_U}{\otimes}$ "f^s".[3]

From this theorem we may readily deduce the b-function lemma in its standard form. By the theorem, we have an identity

$$b(m \otimes f^s) = d(m \otimes f \cdot f^s)$$

in $\mathcal{M} \otimes$ "f^s" for some $b \in \Bbbk[s]$ and $d \in \mathcal{D}_U[s]$. There is a homomorphism of $\mathcal{D}_U$-modules $\mathrm{ev}_{s=n} : \mathcal{M} \otimes$ "f^s" $\to \mathcal{M}$ for any $n \in \mathbb{Z}$, which sends $m \otimes g(s) \cdot f^s \mapsto g(n)f^n m$. Indeed, it suffices to check that this respects the action of "vector field coefficient polynomials in s", i.e. of $\xi(s) = \sum s^i \xi_i \in \Theta_U[s]$. This is true because the $\mathcal{D}_U[s]$-module $\mathcal{M} \otimes$ "f^s" was constructed in a manner respecting the Leibniz and chain rules: we have

$$\begin{aligned}\mathrm{ev}_{s=n}\left(\xi(s) \cdot (m \otimes f^s)\right) &= \mathrm{ev}_{s=n} \sum (\xi_i(m) \otimes s^i f^s + m \otimes s^{i+1}\xi_i(f)f^{-1}f^s) \\ &= \sum (n^i f^n \xi_i(m) + n^{i+1}\xi_i(f)f^{n-1}m) = \sum n^i \xi_i(f^n m) = \xi(n) \cdot \mathrm{ev}_{s=n}(m \otimes f^s).\end{aligned}$$

So the identity $b(m \otimes f^s) = d(m \otimes f f^s)$ entails $b(n)f^n m = d(n) \cdot (f^{n+1}m)$ in $\mathcal{M}$ as desired.

Our proof of Theorem 2.4 follows [Ber]. We begin by extendings scalars to the field of rational functions $K = \Bbbk(s)$. For a variety Z over $\Bbbk$ we denote the extended variety $Z \underset{\operatorname{Spec}\Bbbk}{\times} \operatorname{Spec} K$ by $\widehat{Z}$. Note that for a $\Bbbk$-algebra A we have $\mathrm{Der}_K(A \otimes_\Bbbk K) = \mathrm{Der}_\Bbbk(A) \otimes_\Bbbk K$, which entails $\mathcal{D}_{\widehat{Z}} = \mathcal{D}_Z \otimes_\Bbbk K = \mathcal{D}_Z(s)$. Similarly, for a $\mathcal{D}_Z$-module $\mathcal{F}$ we denote the extended $\mathcal{D}_{\widehat{Z}}$-module $K \otimes_\Bbbk \mathcal{F}$ by $\widehat{\mathcal{F}}$. With this notation, we have the following lemma.

Lemma 2.5. $\mathcal{N}_U = \widehat{\mathcal{M}} \underset{\mathcal{O}_{\widehat{U}}}{\otimes} \widehat{\text{“}f^s\text{”}}$ is a holonomic $\mathcal{D}_{\widehat{U}}$-module.

[3] For the definition of the internal tensor product $\mathcal{D}$-module structure, see Appendix B.3.3.

Proof. Since $\mathrm{S.S.}(\widehat{\mathcal{F}}) = \widehat{\mathrm{S.S.}(\mathcal{F})} \subset \widehat{\mathrm{T}^*Z} \cong \mathrm{T}^*\widehat{Z}$ for any $\mathcal{D}_Z$-coherent module $\mathcal{F}$, the extension of a holonomic module is holonomic. In particular, $\widehat{\mathcal{M}}$ is holonomic. Moreover the module $\widehat{\text{“}f^s\text{”}}$ is $\mathcal{O}_{\widehat{U}}$-coherent, hence holonomic. Internal tensor product $\bullet \overset{\mathrm{L}}{\underset{\mathcal{O}_{\widehat{U}}}{\otimes}} \bullet$ preserves holonomicity (cf. Appendix B.5.4). Since $\widehat{\text{“}f^s\text{”}}$ is $\mathcal{O}_{\widehat{U}}$-flat, the claim follows. $\square$

Proof of Theorem 2.4. Note that $\mathcal{N}_U$ from the lemma is the restriction to $\widehat{U}$ of the $\mathcal{D}_{\widehat{X}}$-module

$$\mathcal{N} = j_*\widehat{\mathcal{M}} \underset{\mathcal{O}_{\widehat{X}}}{\otimes} j_*\widehat{\text{“}f^s\text{”}}.$$

By the “extension lemma” (Appendix B.5.2) there exists a holonomic $\mathcal{D}_{\widehat{X}}$-submodule $\mathcal{N}' \subset \mathcal{N}$ such that $\mathcal{N}'|_{\widehat{U}} = \mathcal{N}_U$. Then the quotient $\mathcal{N}/\mathcal{N}'$ is supported on the hypersurface $\widehat{Y} = \widehat{X} - \widehat{U}$ cut out by f. So if we regard $m \otimes f^s$ as a section of $\mathcal{N}$, then by the Nullstellensatz there exists k_0 such that $m \otimes f^{k_0} f^s \in \mathcal{N}'$.

Since $\mathcal{N}'$ is holonomic so is the submodule $\mathcal{D}_{\widehat{X}}(m \otimes f^{k_0} f^s)$, and in particular the latter has finite length. So the descending chain of $\mathcal{D}_{\widehat{X}}$-submodules of $\mathcal{N}'$

$$\mathcal{D}_{\widehat{X}}(m \otimes f^{k_0} f^s) \supset \mathcal{D}_{\widehat{X}}(m \otimes f^{k_0+1} f^s) \supset \cdots$$

must eventually stabilize. In particular, for some $k \geq k_0$, we have

$$\mathcal{D}_{\widehat{X}}(m \otimes f^k f^s) = \mathcal{D}_{\widehat{X}}(m \otimes f^{k+1} f^s).$$

But note that there are compatible isomorphisms

$$\begin{array}{ccc} \mathcal{D}_{\widehat{X}}(m \otimes f^s) & \xrightarrow{\cong} & \mathcal{D}_{\widehat{X}}(m \otimes f^k f^s) \\ \uparrow & & \uparrow \\ \mathcal{D}_{\widehat{X}}(m \otimes f \cdot f^s) & \xrightarrow{\cong} & \mathcal{D}_{\widehat{X}}(m \otimes f^{k+1} f^s) \end{array}$$

given on the $\mathcal{D}_{\widehat{X}}$-generators by multiplication by f^k. It follows that

$$\mathcal{D}_{\widehat{X}}(m \otimes f^s) = \mathcal{D}_{\widehat{X}}(m \otimes f \cdot f^s)$$

just as well. So there is some $\widetilde{d}(s) \in \mathcal{D}_{\widehat{X}} = \mathcal{D}_X(s)$ such that

$$m \otimes f^s = \widetilde{d}(s)(m \otimes f \cdot f^s).$$

If we clear the denominators of $\widetilde{d}$, we find $b \in \Bbbk[s]$ and $d \in \mathcal{D}_X[s]$ satisfying

$$b_0(s)(m \otimes f^k f^s) = d_0(s)(m \otimes f^{k+1} f^s).$$

This concludes the proof of the b-function lemma. $\square$

Remark 2.6. In the special case $\mathcal{M} = \mathcal{O}_U, m = 1, X = \mathbb{A}^n$ the holonomicity Lemma 2.5 can be bypassed, removing the dependence upon any significant facts about $\mathcal{D}$-modules. (This was the original setting in which Bernstein proved the existence of b-functions.) In this case we can show directly that $\mathcal{D}_{\widehat{X}} f^s$ is holonomic. Indeed, it suffices to consider the global problem of showing that the module $A_n(K) \cdot f^s$ is holonomic over the Weyl algebra $A_n(K) = \Gamma(\widehat{X}, \mathcal{D}_{\widehat{X}})$ (Appendix B.1). By definition of the $\mathcal{D}$-action there is an inclusion of $A_n(K)$-modules $A_n(K) \cdot f^s \hookrightarrow O_{\widehat{U}} \cdot f^s = K[t_1, \dots, t_n][f^{-1}] \cdot f^\lambda$. Since a submodule of a holonomic $\mathcal{D}$-module is holonomic, it is therefore enough to check that $O_{\widehat{U}} \cdot f^s$ is $A_n(K)$-holonomic.

Now we use the fact that we are on affine space, for $A_n(K)$ admits in addition to the usual filtration by the order of differential operators another filtration known as the *Bernstein filtration*, defined by giving degree 1 to all the t_i and ∂_j. Bernstein showed [Ber2] that an A_n-module M if holonomic if it admits a good filtration M^k with respect to the Bernstein filtration on A_n, such that $\dim M^k \leq \frac{e}{n!} k^n + o(k^d)$ for some positive integer e (which is the *multiplicity* of the holonomic module M). It is straightforward to check that the module $O_{\widehat{U}} \cdot f^s$ admits such a filtration $\mathrm{Fil}^\bullet$ with $e = n(\deg(f) + 1)$, where Fil^k is generated by expressions $g(s) f^{-k} \cdot f^s$ for $g \in \Bbbk(s)[t_1, \dots, t_n]$ satisfying $\deg_{t_1,\dots,t_n} g \leq k(1 + \deg f)$.

Remark 2.7. The proof of the b-function lemma we have given is, of course, entirely nonconstructive. However, algorithms for computing the b-function of a holonomic $\mathcal{D}$-module along a hypersurface Y have recently been devised by T. Oaku [O].

2.3 A consequence

In the sequel we will need the following corollary of the b-function lemma. We remain in the XYU situation above.

Corollary 2.8. Let $\mathcal{M}_U$ be a holonomic $\mathcal{D}_U$-module generated by sections $m_1, \dots, m_n$. Then for $k \gg 0$ one can compute the Goresky-MacPherson extension as $j_{!*}\mathcal{M}_U = \sum_\ell \mathcal{D}_X f^k m_\ell$.

Proof. Let $\mathcal{M}_k = \sum_\ell \mathcal{D}_X f^k m_\ell$. Consider the quotient $j_*\mathcal{M}_U / j_{!*}\mathcal{M}_U$. Since $j_*\mathcal{M}_U$ and $j_{!*}\mathcal{M}_U$ both restrict to $\mathcal{M}_U$, the quotient is supported on Y. In particular, it follows that the sections $f^k m_\ell$ are all in $j_{!*}\mathcal{M}_U$ for sufficiently large k. So for $k \gg 0$ we have $\mathcal{M}_k \subset j_{!*}\mathcal{M}_U$. Now it is clear that $j^*\mathcal{M}_k = \mathcal{M}_U$ for any k (since f^k becomes invertible upon restriction). If $\mathcal{M}_k \subsetneq j_{!*}\mathcal{M}_U$, choose a nonzero section of the quotient $j_{!*}\mathcal{M}_U / \mathcal{M}_K$. It is again supported on Y, so it is annihilated by some power of f. Therefore the $\mathcal{M}_k$ (for sufficiently large k) form an increasing sequence of submodules of $j_{!*}\mathcal{M}_U$, which can only stabilize once $\mathcal{M}_k = j_{!*}\mathcal{M}_U$.

So it suffices to show that the sequence *does* stabilize, which will follow from the b-function lemma. Let b_ℓ be the b-function of m_ℓ. In particular, for k greater than all the integer roots of b_ℓ, one has $f^k m_\ell \in \mathcal{D}_X f^{k+1} m_\ell$. Hence for sufficiently large k (bigger than all the integer roots of all the b-functions b_ℓ) we obtain $\mathcal{M}_k \subset \mathcal{M}_{k+1}$, which means the sequence stabilizes as claimed. ☐

3 Nearby cycles, maximal extension, and vanishing cycles functors

This section is the heart of the thesis. The goal is to define functors $\Psi_f : \mathrm{Hol}(\mathcal{D}_U) \to \mathrm{Hol}(\mathcal{D}_Y)$ and $\Phi_f : \mathrm{Hol}(\mathcal{D}_X) \to \mathrm{Hol}(\mathcal{D}_Y)$ which will be used in §4 for the construction of the gluing category. The results of §2 play a small but crucial role in the definition of these functors (see the proof of Theorem 3.31).

3.1 Monodromy Jordan blocks

We return (temporarily) to the XYU setup

$$\{0\} = Y \overset{i}{\hookrightarrow} X = \mathbb{A}^1_{\mathbb{C}} \overset{j}{\hookleftarrow} U = \mathbb{C}^\times = X - Y$$

of Section 1.2. A natural class of indecomposable objects in Loc_U are those local systems with unipotent monodromy of a single Jordan block. The Fuchs conditions mentioned in Section 1.2 correspond, it can be shown, to the condition on a differential equation that it be equivalent (in the sense of giving rise to isomorphic $\mathcal{D}$-modules) to one of the form

$$t\partial \begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_k(t) \end{pmatrix} = \Gamma \begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_k(t) \end{pmatrix}$$

for a constant matrix Γ. The monodromy around the origin of the corresponding local system of solutions is $\exp(2\pi i \Gamma)$. So we may concern ourselves with those $\mathcal{D}_U$-modules on which the **logarithm of monodromy** operator $t\partial$ acts nilpotently.

Specifically, the $\mathcal{D}_U$-module corresponding to a nilpotent Jordan block of size n will be essential in what follows, so let us describe it. Since we are in an affine setting, we may freely pass between sheaves and ordinary modules by taking global sections (or in the other direction, localizing). Note that $D_U = \Gamma(U, \mathcal{D}_U)$ is $\mathbb{C}[t, t^{-1}, \partial]$ where $\partial = \frac{\partial}{\partial t}$ satisfies $[\partial, f] = \partial(f)$. Consider the D_U-module

$$J^{(n)} = D_U\langle e_1, \ldots, e_n\rangle / D_U\langle \theta e_1, \theta e_2 - e_1, \ldots, \theta e_n - e_{n-1}\rangle, \qquad \theta = t\partial. \tag{1}$$

Write $\mathcal{J}^{(n)}$ for the associated $\mathcal{D}_U$-module. The corresponding analytic $\mathcal{D}$-module $\mathcal{J}^{(n)}_{\mathrm{an}}$ has solutions $\mathcal{H}om_{\mathcal{D}_U^{\mathrm{an}}}(\mathcal{J}^{(n)}_{\mathrm{an}}, \mathcal{O}_U^{\mathrm{an}})$ given by the local system of solutions to the corresponding system of differential equations

$$t\frac{\mathrm{d}}{\mathrm{d}\,t}\vec{u}(t) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \vec{u}(t) = \Gamma \vec{u}(t).$$

Taking the local system of solutions to this equation, we see that $\mathcal{J}^{(n)}$ corresponds to the powers of the logarithm. An alternate presentation of $J^{(n)}$ is is

$$J^{(n)} = \sum_{k=0}^{n-1} O_U \cdot \log^k$$

where $\log^k$ is a formal symbol corresponding to $k! \cdot e_{k+1}$ in the presentation above, which ∂ acts upon by $t\partial \log^k = k \log^{k-1}$ (and of course $\partial \log^0 = 0$).

Now return to the general XYU setup

$$f^{-1}(0) = Y \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U = X - Y$$

for a regular function $f : X \to \Bbbk$ on a smooth variety X over an arbitrary algebraically closed field $\Bbbk$ of characteristic zero, such that $f|_U$ is smooth. We generalize the notion above.

Definition 3.1. The **monodromy Jordan block of size** n refers to the $\mathcal{D}_{\Bbbk^\times}$-module $\mathcal{J}^{(n)}$ defined by equation (1), or the corresponding $\mathcal{D}_U$-module $(f|_U)^! \mathcal{J}^{(n)}[\dim X - 1] = (f|_U)^* \mathcal{J}^{(n)}[1 - \dim X]$, which we denote by $\mathcal{J}_f^{(n)}$.

Note that since $f|_U$ is smooth, the inverse image $(f|_U)^![\dim X - 1]$ coincides with the ordinary sheaf-theoretic inverse image $(f|_U)^\Delta$ and is exact. So we may reduce many properties of the $\mathcal{J}_f^{(n)}$ to the corresponding properties of the $\mathcal{J}^{(n)}$. The $\mathcal{D}_{\Bbbk^\times}$-modules $\mathcal{J}^{(n)}$ are $\mathcal{O}_{\Bbbk^\times}$-coherent, hence holonomic; the same is true of the $\mathcal{J}_f^{(n)}$s, which are actual *modules* rather than simply complexes.

Plainly there exists a filtration

$$0 \subset \mathcal{J}^{(1)} \subset \mathcal{J}^{(2)} \subset \cdots \subset \mathcal{J}^{(n)}, \tag{2}$$

with $\mathcal{J}^{(k)}$ spanned by $e_1, \ldots, e_k \in \mathcal{J}^{(n)}$. The subquotients $\mathcal{J}^{(i+1)}/\mathcal{J}^{(i)}$ are each isomorphic to $\mathcal{J}^{(1)}$. Note that $D_{\Bbbk^\times}/D_{\Bbbk^\times} t\partial = D_{\Bbbk^\times}/D_{\Bbbk^\times}\partial$ and the commutation relations in $D_{\Bbbk^\times}$ can be used to bring any ∂'s to the righthand side, so this quotient is isomorphic to $O_{\Bbbk^\times}$. In other words, $\mathcal{J}^{(1)} \cong \mathcal{O}_{\Bbbk^\times}$. So we obtain a corresponding filtration of the $\mathcal{J}_f^{(n)}$s with subquotients isomorphic to $\mathcal{J}_f^{(1)}$. The pullback $\mathcal{J}_f^{(1)}$ is $\mathcal{O}_U$.

The filtrations above induce natural maps

$$\mathcal{J}^{(1)} \rightleftarrows \cdots \rightleftarrows \mathcal{J}^{(n-1)} \rightleftarrows \mathcal{J}^{(n)} \rightleftarrows \cdots$$

(since all the subquotients are the same), and similarly for the $\mathcal{J}_f^{(n)}$. Let us describe the modules

$$\varprojlim_n \mathcal{J}_f^{(n)} \qquad \text{and} \qquad \varinjlim_n \mathcal{J}_f^{(n)}$$

more explicitly.

Let s be a formal variable and recall the $\mathcal{D}_U[s]$-module "f^s" defined as in Section 2; it is $\mathcal{O}_U[s] \cdot f^s$, where f^s is regarded as a formal symbol, and the action of vector fields given by

$$\xi \cdot f^s = s f^{-1} \xi(f) \cdot f^s$$

defines the $\mathcal{D}_U[s]$-module structure. Write "t^s" for the corresponding $\mathcal{D}_{\Bbbk^\times}[s]$-module. The sheaf-theoretic inverse image $f^{-1}\mathcal{O}_{\Bbbk^\times}$ is the subsheaf of $\mathcal{O}_U$ obtained by taking a section $g(t, t^{-1})$ of $\mathcal{O}_{\Bbbk^\times}$ and making the formal substitution $t \mapsto f$, regarding the result as a function on U. Observe that there is an obvious identification of

$$(f|_U)^! \text{“}t^s\text{”}[\dim X - 1] = f^{-1}(\mathcal{O}_{\Bbbk^\times}[s] \cdot t^s) \underset{f^{-1}\mathcal{O}_{\Bbbk^\times}}{\otimes} \mathcal{O}_U$$

with $\mathcal{O}_U[s] \cdot t^s$. After making the additional formal substitution $t^s \mapsto f^s$, one sees that the inverse image $\mathcal{D}_U$-module structure on $(f|_U)^!$"t^s" (defined by $\xi \cdot (g(s) \cdot t^s \otimes h) = g(s) \cdot t^s \otimes \xi(h) + \partial(g(s) \cdot t^s) \otimes h\xi(f)$) coincides with that of the $\mathcal{D}_U[s]$-module "f^s".

Lemma 3.2. $\mathcal{J}_f^{(n)} \cong \frac{\text{“}f^s\text{”}}{s^n \text{“}f^s\text{”}} = \frac{\mathcal{O}_U[s] \cdot f^s}{s^n \mathcal{O}_U[s] \cdot f^s}$.

Proof. There is a morphism of $\mathcal{D}_{\Bbbk^\times}$-modules "$t^s$" $\to \mathcal{J}^{(n)}$, where the map $O_U[s] \cdot t^s \to J^{(n)}$ is defined by

$$s^k t^s \mapsto e_{n-k} \quad (0 \le k < n), \qquad s^k t^s \mapsto 0 \quad (k \ge n),$$

and extended O_U-linearly. (Recall that the e_ℓ are the generators of $J^{(n)}$.) This is $\mathcal{D}_U$-linear because

$$\partial f s^k t^s = s^k \partial(f) t^s + s^{k+1} f t^{-1} t^s \mapsto \partial(f) e_{n-k} + f t^{-1} e_{n-k-1} = \partial(f) e_{n-k} + f \partial(e_{n-k}) = \partial(f e_{n-k})$$

by the relations in $J^{(n)}$. The map induces an isomorphism $\mathcal{J}^{(n)} \cong$ "t^s"$/s^n$"t^s". So there is an exact sequence $0 \to s^n$"t^s" $\to$ "t^s" $\to \mathcal{J}^{(n)} \to 0$. Applying $(f|_U)^!$ and using the remark immediately preceding the lemma, the claim follows immediately. □

Remark 3.3. Another (somewhat cute) way of proving the lemma is to formally expand $t^s = \exp(s \log t) = \sum_{k=0}^{\infty} \frac{s^k \log(t)^k}{k!} = \sum_{k=0}^{\infty} s^k e_{k+1} \in \bigcup_{n=0}^{\infty} \mathcal{J}^{(n)}[[s]]$. Then map

$$\text{“}t^s\text{”} \ni g(s) \cdot t^s \mapsto \operatorname*{Res}_{s=0} \left(\frac{1}{s^n} g(s) t^s \right) \in \mathcal{J}^{(n)},$$

computing the residue formally with respect to this expansion. One can check that this amounts to the same map used above.

Definition 3.4. For integers $a \le b$ write $\mathcal{J}^{a,b} = s^a$"t^s"$/s^b$"t^s" (resp. $\mathcal{J}_f^{a,b} = s^a$"$f^s$"$/s^b$"$f^s$"). This is an $\mathcal{O}_{\Bbbk^\times}$ (resp. $\mathcal{O}_U$) -coherent, hence holonomic, $\mathcal{D}_{\Bbbk^\times}[s]$ (resp. $\mathcal{D}_U[s]$) -module.

By the lemma, we can identify $\mathcal{J}_{(f)}^{(n)} = \mathcal{J}_{(f)}^{0,n}$, and

$$\mathcal{J}_f^{0,\infty} = \varprojlim_b \mathcal{J}_f^{0,b} = \varprojlim_b \frac{\mathcal{O}_U[s] \cdot f^s}{s^b \mathcal{O}_U[s] \cdot f^s} = \mathcal{O}_U[[s]] \cdot f^s,$$

with the same $\mathcal{D}_U$-action as before. With respect to the identification in the lemma, the inclusion $\mathcal{J}_f^{0,n} \hookrightarrow \mathcal{J}_f^{0,n+k}$ is given by the map $\sigma_k =$ multiplication by s^k. In fact, for any a, b, k the map σ_k gives an isomorphism $\mathcal{J}_{(f)}^{a,b} \cong \mathcal{J}_{(f)}^{a+k,b+k}$. Now note that there is an isomorphism of injective systems of $\mathcal{D}_U[s]$-modules

$$\begin{array}{ccccccc}
\mathcal{J}_f^{0,1} & \xhookrightarrow{\sigma_1} & \mathcal{J}_f^{0,2} & \xhookrightarrow{\sigma_1} & \mathcal{J}_f^{0,3} & \xhookrightarrow{\sigma_1} & \cdots \\
\cong \downarrow \sigma_{-1} & & \cong \downarrow \sigma_{-2} & & \cong \downarrow \sigma_{-3} & & \\
\mathcal{J}_f^{-1,0} & \hookrightarrow & \mathcal{J}_f^{-2,0} & \hookrightarrow & \mathcal{J}_f^{-3,0} & \hookrightarrow & \cdots
\end{array}$$

From this we see that

$$\mathcal{J}^{-\infty,0} = \varinjlim_a \mathcal{J}_f^{a,0} = \varinjlim_a \frac{s^{-a} \mathcal{O}_U[s] \cdot f^s}{\mathcal{O}_U[s] \cdot f^s} = \frac{\mathcal{O}_U((s)) \cdot f^s}{\mathcal{O}_U[[s]] \cdot f^s}.$$

Definition 3.5. Denote $\mathcal{O}_U((s)) \cdot f^s$ by $\mathcal{J}_f^{-\infty,\infty}$ (with the same formula for the $\mathcal{D}_U$-action as before).

Observe that $\mathcal{J}_f^{-\infty,\infty}$ is *not* a coherent $\mathcal{D}$-module. Nonetheless, it has a number of extremely nice properties. For instance:

- The monodromy s acts invertibly upon it.
- In a certain ill-defined sense we will come to grips with below, the Goresky-MacPherson map for Laurent series $j_!\mathcal{J}_f^{-\infty,\infty} \to j_*\mathcal{J}_f^{-\infty,\infty}$ is an *isomorphism*. This despite the fact that this map fails to be an isomorphism on the level of the coherent modules $\mathcal{J}_f^{a,b}$. Passing to the limit and colimit effectively pushes both the kernel and cokernel "off the page" out to infinity. (See Example 3.38 for an elaboration of this point.)
- Consequently the Goresky-MacPherson map is *injective* for power series $\mathcal{J}_f^{0,\infty}$. So it is reasonable to study quotients such as $j_*\mathcal{J}_f^{0,\infty}/j_!\mathcal{J}_f^{0,\infty}$. We will find that this quotient is a $\mathcal{D}_X$-module supported on Y which captures some of the structure of $\mathcal{D}_U$-modules.

We shall use the modules $\mathcal{J}_f^{\cdot,\cdot}$ to define the nearby and vanishing functors we desire, in terms of the tensor products $\mathcal{M} \otimes_{\mathcal{O}_U} \mathcal{J}_f^{\cdot,\cdot}$ for a $\mathcal{D}_U$-module $\mathcal{M}$. Ultimately we would like to use notions such as

$$j_!(\mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{J}_f^{-\infty,\infty}) = j_!(\mathcal{M}((s)) \cdot f^s).$$

However, this expression is utter nonsense because, since $\mathcal{J}_f^{-\infty,\infty}$ is not coherent, the tensor product here is neither a holonomic $\mathcal{D}_U$-module nor a complex of such modules. So we cannot apply the direct image functor $j_!$ at all, and even the functor j_* would be intractable to compute.

Therefore our first task must be to produce an appropriate analogue of this object, for which we have sensible versions of these functors.

3.2 The category of pro-ind holonomic $\mathcal{D}$-modules

Before defining the correct analogue of $\mathcal{M} \otimes \mathcal{J}_f^{-\infty,\infty}$ in §3.3.1, we must define the proper category in which to regard "limits" of holonomic $\mathcal{D}_Z$-modules. Here we introduce this category and study some important technical properties.

Remark 3.6. Recall that holonomic $\mathcal{D}_Z$-modules form an abelian subcategory of $\mathrm{Mod}(\mathcal{D}_Z)$ closed under subquotients and extensions. All the constructions of this subsection will work for an arbitrary abelian category $\mathscr{A}$ in place of $\mathrm{Hol}(\mathcal{D}_Z)$. However, we have no need for any more generality than this.

3.2.1 The category $\lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_Z)$

Consider the poset

$$\Pi = \{(a,b) \in \mathbb{Z}^2 \mid a \leq b\} \text{ with the partial order } (a,b) \preceq (a',b') \Leftrightarrow a \geq a', b \geq b'.$$

We can regard Π as a category and consider the functor category $\mathrm{Hol}(\mathcal{D}_Z)^\Pi$ of Π-shaped **diagrams** in $\mathrm{Hol}(\mathcal{D}_Z)$. Explicitly, these are collections $\{\mathcal{F}^{a,b} \mid a \leq b\}$ of holonomic $\mathcal{D}_Z$-modules and morphisms $\{\mu_{a,b,c,d} : \mathcal{F}^{a,b} \to \mathcal{F}^{a',b'} \mid a \geq a', b \geq b'\}$ of $\mathcal{D}_Z$-modules, such that for $a \geq a' \geq a'', b \geq b' \geq b''$, the diagram

$$\mathcal{F}^{a,b} \xrightarrow{\mu_{a,b,a',b'}} \mathcal{F}^{a',b'} \xrightarrow{\mu_{a',b',a'',b''}} \mathcal{F}^{a'',b''}, \qquad \mu_{a,b,a'',b''} : \mathcal{F}^{a,b} \to \mathcal{F}^{a'',b''}$$

commutes. Morphisms α between such diagrams are defined in the obvious way as collections of morphisms $\alpha^{a,b}$ compatible with the transition maps μ for the source and the target. It is clear that $\mathrm{Hol}(\mathcal{D}_Z)^\Pi$ is an abelian category, with kernels and cokernels constructed on the level of individual objects $\mathcal{F}^{a,b}$ in $\mathrm{Hol}(\mathcal{D}_Z)$.

Example 3.7. The $\mathcal{J}_f^{a,b}$ give an object of $\mathrm{Hol}(\mathcal{D}_U)^\Pi$. This is precisely why we want to work with this category: we would like to generalize this example to the objects

$$\mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{J}_f^{a,b} = \mathcal{M} \underset{\mathcal{O}_U}{\otimes} \frac{s^a \text{``}f^s\text{''}}{s^b \text{``}f^s\text{''}} = \frac{s^a \mathcal{M}[[s]] \cdot f^s}{s^b \mathcal{M}[[s]] \cdot f^s}$$

for $\mathcal{M} \in \mathrm{Hol}(\mathcal{D}_U)$. Ultimately this will allow us to study the limits $\mathcal{M}[[s]] \cdot f^s$ and $\mathcal{M}((s)) \cdot f^s$ (analogous to $\mathcal{J}_f^{0,\infty}, \mathcal{J}_f^{-\infty,\infty}$) and, more importantly, the $\mathcal{D}_X$-modules obtained therefrom via $j_!$ and j_*.

Next we distinguish a full subcategory $\mathrm{Hol}(\mathcal{D}_Z)^\Pi_\leftrightarrow \subset \mathrm{Hol}(\mathcal{D}_Z)^\Pi$: those diagrams $\{\mathcal{F}^{a,b}\}$ such that for any $a \leq b \leq c$ the sequence of $\mathcal{D}_Z$-modules

$$0 \to \mathcal{F}^{b,c} \to \mathcal{F}^{a,c} \to \mathcal{F}^{a,b} \to 0$$

is exact. We call such diagrams **admissible**.

Lemma 3.8. $\mathrm{Hol}(\mathcal{D}_Z)^\Pi_\leftrightarrow$ is an exact category[4] with respect to the class $\mathcal{E}$ of short exact sequences of admissible diagrams.

Proof. We appeal to a general criterion for the exactness of a subcategory an abelian category, Proposition A.3. It suffices to check that admissible diagrams are closed under extensions. Let

$$0 \to \mathcal{F}^{\cdot,\cdot} \to \mathcal{G}^{\cdot,\cdot} \to \mathcal{H}^{\cdot,\cdot} \to 0$$

be a short exact sequence of diagrams in $\mathrm{Hol}(\mathcal{D}_Z)^\Pi$ such that the first and third terms are admissible. Let $a \leq b \leq c$, and consider the diagram below. By hypothesis all the columns and the first and third rows are exact. So by the 9 Lemma, the middle row is exact as well. □

[4]See Appendix A.2.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}^{b,c} & \longrightarrow & \mathcal{F}^{a,c} & \longrightarrow & \mathcal{F}^{a,b} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{G}^{b,c} & \longrightarrow & \mathcal{G}^{a,c} & \longrightarrow & \mathcal{G}^{a,b} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{H}^{b,c} & \longrightarrow & \mathcal{H}^{a,c} & \longrightarrow & \mathcal{H}^{a,b} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Diagram for proof of Lemma 3.8.

We next distinguish a special class Σ of morphisms in $\mathrm{Hol}(\mathcal{D}_Z)^{\Pi}_{\leftrightarrow}$.

Definition 3.9. We call a function $\varphi : \mathbb{Z} \to \mathbb{Z}$ **well-behaved** if (i) φ is order preserving: $\varphi(a) > \varphi(b)$ for $a > b$; and (ii) φ is of bounded distance from the identity: there exists N such that $|\varphi(i) - i| < N$ for all i.[5]

Given a well-behaved map φ there is an additive functor $\widetilde{\varphi}$ from $\mathrm{Hol}(\mathcal{D}_Z)^{\Pi}_{\leftrightarrow}$ to itself defined on a diagram $\mathcal{F}^{\cdot,\cdot}$ by $\widetilde{\varphi}(\mathcal{F})^{a,b} = \mathcal{F}^{\varphi(a),\varphi(b)}$. The arrows in the diagram $\widetilde{\varphi}(\mathcal{F})$ are given by the morphisms

$$\mu_{\varphi(a),\varphi(b),\varphi(a'),\varphi(b')} : \widetilde{\varphi}(\mathcal{F})^{a,b} \to \widetilde{\varphi}(\mathcal{F})^{a',b'}.$$

On morphisms $\alpha : \mathcal{F} \to \mathcal{G}$, this functor acts by $\widetilde{\varphi}(\alpha)^{a,b} = \alpha^{\varphi(a),\varphi(b)}$. If $\mathcal{F}$ is an admissible diagram then so is $\widetilde{\varphi}(\mathcal{F})$: if $a \leq b \leq c$ then $\varphi(a) \leq \varphi(b) \leq \varphi(c)$ so

$$\mathcal{F}^{\varphi(b),\varphi(c)} \hookrightarrow \mathcal{F}^{\varphi(a),\varphi(c)} \twoheadrightarrow \mathcal{F}^{\varphi(a),\varphi(b)}$$

is short exact, which is precisely the admissibility criterion for $\widetilde{\varphi}(\mathcal{F})$. Note that $\widetilde{\mathbf{1}_{\mathbb{Z}}}$ is the identity functor.

Now suppose we are given two well-behaved functions φ, ψ such that $\varphi(i) \leq \psi(i)$ for all $i \in \mathbb{Z}$. (We abbreviate this by $\psi \geq \varphi$.) Then one can check that $\mu_{\psi(a),\psi(b),\varphi(a),\varphi(b)}$: $\widetilde{\psi}(\mathcal{F})^{a,b} \to \widetilde{\varphi}(\mathcal{F})^{a,b}$ gives a natural transformation $\widetilde{\psi} \to \widetilde{\varphi}$. Let Σ be the collection of all morphisms $\widetilde{\psi}(\mathcal{F}) \to \widetilde{\varphi}(\mathcal{F})$ produced in this way for various $\mathcal{F}$ and φ, ψ.

Example 3.10. Let φ be a well-behaved map. For $Z = U$, one can consider in addition to the diagram $\mathcal{J}_f^{a,b}$ the diagram $\widetilde{\varphi}\mathcal{J}_f^{a,b} = s^{\varphi(a)}\text{“}f^s\text{”}/s^{\varphi(b)}\text{“}f^s\text{”}$. Since φ is order-preserving, we have

$$\lim_{b\to\infty} \varphi(b) = \infty, \qquad \lim_{a\to-\infty} \varphi(a) = -\infty.$$

[5]Beilinson [B] omits the constraint that the distance of φ from $\mathbf{1}_{\mathbb{Z}}$ be bounded, but I was unable to prove that the construction works without this extra hypothesis. Specifically, we use property (ii) below to prove that a particular class of morphisms is well-suited for localizing the category of admissible diagrams, in the sense that the resulting localization is still an exact category. It may be possible to make this conclusion without property (ii).

So when we pass to the limit $\varprojlim_b \widetilde{\varphi}\mathcal{J}_f^{a,b}$ we obtain $s^{\varphi(a)}\mathcal{O}_U[[s]]\cdot f^s$, and passing to the limit $\varinjlim_a \varprojlim_b \widetilde{\varphi}\mathcal{J}_f^{a,b}$ we still obtain $\mathcal{O}_U((s))\cdot f^s$, the same $\mathcal{D}_U$-module obtained by taking the limits of the $\mathcal{J}_f^{a,b}$s. It follows from the hypothesis that φ is of bounded distance from the identity, that the canonical maps

$$s^{\varphi(a)}\text{“}f^s\text{”}/s^{\varphi(b)}\text{“}f^s\text{”} \to s^a\text{“}f^s\text{”}/s^b\text{“}f^s\text{”}$$

induce the identity map on $\mathcal{O}_U((s))\cdot f^s$ after passing to the $\varinjlim_a \varprojlim_b$. Therefore, if our goal is to approximate the properties of non-coherent $\mathcal{D}$-modules obtained as “Laurent series” by working in the category of admissible diagrams, we would like to insist that $\widetilde{\varphi}\mathcal{J}_f^{\cdot,\cdot} \to \mathcal{J}_f^{\cdot,\cdot}$ be invertible. Unfortunately, it is *not* invertible in $\mathrm{Hol}(\mathcal{D}_U)^{\Pi}_{\leftrightarrow}$!

As this example illustrates, this is a situation which calls for *localizing* the category $\mathrm{Hol}(\mathcal{D}_Z)^{\Pi}_{\leftrightarrow}$ with respect to the collection of morphisms Σ. (See Appendix A.2 for basic facts about localization of categories.) For technical reasons we first must add all isomorphisms to Σ, and then take the closures of the resulting collection of morphisms under composition. The next lemma guarantees, by abstract nonsense, that the localization will have nice properties.

Lemma 3.11. Σ is multiplicative in the sense of Definition A.1.

Proof. Properties (1) and (2) of Definition A.1 were ensured by construction.

To establish property (3), let $\mathcal{F} \xleftarrow{\alpha} \widetilde{\varphi}\mathcal{G} \xrightarrow{s} \widetilde{\psi}\mathcal{G}$ be a diagram in $\mathrm{Hol}(\mathcal{D}_Z)^{\Pi}_{\leftrightarrow}$ with the righthand morphism in Σ and $\varphi \geq \psi$. Since φ and ψ are both of bounded distance from $\mathbf{1}_{\mathbb{Z}}$, they are of bounded distance from one another. Suppose $|\varphi(i) - \psi(i)| < N$. Define $\rho(i) = i - N$. Then ρ is a constrained strictly monotonic function and is $\leq \mathbf{1}_{\mathbb{Z}}$. One has $\psi(i) - \rho\varphi(i) = K - (\varphi(i) - \psi(i)) \geq 0$. So $\psi \geq \rho\varphi$. Hence there is a morphism $\beta : \widetilde{\psi}\mathcal{G} \to \widetilde{\rho}\mathcal{F}$ defined as the composition

$$\widetilde{\psi}\mathcal{G} \to \widetilde{\rho\varphi}\mathcal{G} = \widetilde{\rho}\widetilde{\varphi}\mathcal{G} \xrightarrow{\widetilde{\rho}\alpha} \widetilde{\rho}\mathcal{F}.$$

There is also a morphism $t : \mathcal{F} = \widetilde{\mathbf{1}}_{\mathbb{Z}}\mathcal{F} \to \widetilde{\rho}\mathcal{F}$ in Σ. It is straightforward to check that $\beta s = t\alpha$. The dual assertion is proved analogously.

For property (4), let $\psi \geq \varphi$ and suppose

$$\widetilde{\psi}\mathcal{F} \xrightarrow{s\in\Sigma} \widetilde{\varphi}\mathcal{F} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathcal{G}$$

is a diagram in $\mathrm{Hol}(\mathcal{D}_Z)^{\Pi}_{\leftrightarrow}$, and $(\alpha - \beta)s = 0$. By the reasoning above there exists $\rho \leq \mathbf{1}_{\mathbb{Z}}$ such that $\varphi \geq \rho\psi$. Then setting $t : \mathcal{G} \to \widetilde{\rho}\mathcal{G}$, the composition $t(\alpha - \beta)$ factors as

$$\widetilde{\varphi}\mathcal{F} \to \widetilde{\rho\psi}\mathcal{F} \xrightarrow{\widetilde{\rho}((\alpha-\beta)s)} \widetilde{\rho}\mathcal{G}.$$

Since $(\alpha - \beta)s = 0$, this composition is zero. So $t\alpha = t\beta$ as desired. Again, the dual case is proved similarly. □

Definition 3.12. The category of **pro-ind holonomic** $\mathcal{D}_Z$-modules is the localization $\Sigma^{-1}\mathrm{Hol}(\mathcal{D}_Z)^{\Pi}_{\leftrightarrow}$, which we denote by $\lim_{\leftrightarrow}\mathrm{Hol}(\mathcal{D}_Z)$. The localization functor will be denoted by $\lim_{\leftrightarrow} : \mathrm{Hol}(\mathcal{D}_Z)^{\Pi}_{\leftrightarrow} \to \lim_{\leftrightarrow}\mathrm{Hol}(\mathcal{D}_Z)$.

By Proposition A.2 we have the following corollary of Lemma 3.11.

Corollary 3.13. The category $\lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_Z)$ is exact with respect to the class $\mathcal{E}$ of localizations of exact sequences in $\mathrm{Hol}(\mathcal{D}_Z)^{\Pi}_{\leftrightarrow}$. □

Clearly the localization functor $\lim\limits_{\leftrightarrow}$ is exact.

Corollary 3.14. $(\lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_Z))^{op} = \lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_Z)^{op}$. Moreover, an exact (covariant resp. contravariant) functor $F : \mathrm{Hol}(\mathcal{D}_{Z_1}) \to \mathrm{Hol}(\mathcal{D}_{Z_2})$ induces an exact (covariant resp. contravariant) functor $F : \lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_{Z_1}) \to \lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_{Z_2})$ defined by $F\lim\limits_{\leftrightarrow}\mathcal{F}^{a,b} = \lim\limits_{\leftrightarrow} F\mathcal{F}^{a,b}$. □

Using the standard description of morphisms in localized categories (see Appendix A.2) the reader can check that a morphism between the objects $\lim\limits_{\leftrightarrow}\mathcal{F}^{\cdot,\cdot}$ and $\lim\limits_{\leftrightarrow}\mathcal{G}^{\cdot,\cdot}$ in $\lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_Z)$ is represented by a pair (φ, α) where φ is any constrained strictly monotonic function on the integers and $\alpha : \mathcal{F}^{\cdot,\cdot} \to \varphi\mathcal{G}^{\cdot,\cdot}$ is a morphism in the diagram category. Two such pairs (φ, α) and (ψ, β) represent the same morphism if and only if the maps

$$\mu_{\varphi(a),\varphi(b),\min(\varphi(a),\psi(a)),\min(\varphi(a),\psi(a))} \circ \alpha^{a,b} \text{ and } \mu_{\psi(a),\psi(b),\min(\varphi(a),\psi(a)),\min(\varphi(a),\psi(a))} \circ \beta^{a,b}$$

agree. More significantly for us, the criterion for a morphism $\mathcal{F}^{\cdot,\cdot} \to \mathcal{G}^{\cdot,\cdot}$ of admissible diagrams in the diagram category to induce an isomorphism on the level of their $\lim\limits_{\leftrightarrow}$s, is that this morphism becomes invertible after applying one of the natural transformations $\widetilde{\varphi}$, for some well-behaved $\varphi \geq \mathbf{1}_{\mathbb{Z}}$.

3.2.2 The embedding $\mathrm{Hol}(\mathcal{D}_Z) \hookrightarrow \lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_Z)$

Any $\mathcal{M} \in \mathrm{Hol}(\mathcal{D}_Z)$ has a natural trivial decreasing filtation $\mathcal{M}^i = \mathcal{M}$ for $i < 0$, $\mathcal{M}^i = 0$ for $i \geq 0$. Correspondingly we may define for any $a \leq b$ a Π-diagram

$$\mathcal{M}^{a,b} = \mathcal{M}^a/\mathcal{M}^b = \begin{cases} \mathcal{M}, & a < 0, b \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let us call this the *trivial diagram corresponding to* $\mathcal{M}$. The maps in this diagram are the obvious ones, and it is clear that $\{\mathcal{M}^{a,b}\}$ is admissible. It is not difficult to check that $\mathcal{M} \rightsquigarrow \mathcal{M}^{\cdot,\cdot}$ embeds $\mathrm{Hol}(\mathcal{D}_Z)$ as a full subcategory of $\mathrm{Hol}(\mathcal{D}_Z)^{\Pi}_{\leftrightarrow}$. Moreover, this embedding is exact with respect to the exact structure $\mathcal{E}$ on the category of admissible diagrams. The composition of the localization functor with this embedding gives an exact functor $\mathrm{Hol}(\mathcal{D}_Z) \to \lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_Z)$.

Lemma 3.15. This functor is an exact embedding (but is not necessarily full).

Proof. It is injective on objects because $\lim\limits_{\leftrightarrow}$ is. It is faithful because no morphism between trivial diagrams corresponding to a morphism of objects of $\mathrm{Hol}(\mathcal{D}_Z)$ is in the class Σ. □

3.2.3 $\Bbbk[[s]]$ objects and $\Bbbk((s))$ objects

We have yet to capture all the structure of the $\mathcal{D}_U$-module $\mathcal{J}_f^{-\infty,\infty}$ of Laurent series. We are missing two components, the action of the logarithm of monodromy $\partial t = s$ on this module, and a filtration structure. Here we concern ourselves with the first of there.

The s-action on Laurent series makes $\mathcal{J}_f^{-\infty,\infty}$ a "$\Bbbk[[s]]$-object" in $\mathrm{Hol}(\mathcal{D}_U)$. This means that it is equipped with an action of $\Bbbk[[s]]$ by $\mathcal{D}_U$-endomorphisms, or in other words, that it is a $\mathcal{D}_U[[s]]$-module.

Futhermore, each $\mathcal{J}_f^{a,b}$ is a (s^b-torsion) $\Bbbk[[s]]$-module. Multiplication by s is a $\mathcal{D}_U$-linear endomorphism of $\mathcal{J}_f^{a,b}$. Moreover, this map is compatible with the natural maps among the $\mathcal{J}_f^{a,b}$. So it induces an endomorphism $\underset{\leftrightarrow}{\lim} s$ of $\underset{\leftrightarrow}{\lim}\mathcal{J}_f^{a,b}$, which we abusively denote by s. This gives $\underset{\leftrightarrow}{\lim}\mathcal{J}_f^{a,b}$ the structure of a $\Bbbk[[s]]$-object in the pro-ind category.

There is a commutative diagram

$$\begin{array}{ccc} \mathcal{J}_f^{a,b} & \xrightarrow{\sigma} & \mathcal{J}_f^{a+1,b+1} \\ & {\scriptstyle s}\searrow & \downarrow{\scriptstyle \mathrm{can}} \\ & & \mathcal{J}_f^{a,b} \end{array}$$

where σ is also given by multiplication by s. The maps σ induce an isomorphism $\underset{\leftrightarrow}{\lim}\mathcal{J}_f^{a,b} \to \underset{\leftrightarrow}{\lim}\mathcal{J}_f^{a+1,b+1}$. The vertical maps induce a morphism of diagrams in the localizing class Σ, and hence an isomorphism $\underset{\leftrightarrow}{\lim}\mathcal{J}_f^{a+1,b+1} \to \underset{\leftrightarrow}{\lim}\mathcal{J}_f^{a,b}$. Therefore s acts as an *automorphism* of $\underset{\leftrightarrow}{\lim}\mathcal{J}_f^{a,b}$. We might say that this makes this *particular* pro-ind module a $\Bbbk((s))$-object.

By the same construction, each $\underset{\leftrightarrow}{\lim}\mathcal{J}_{f,k}^{a,b}$ is a $\Bbbk[[s]]$-object. One can easily check that the action of s shifts by 1 the indexing on the filtration of $\underset{\leftrightarrow}{\lim}\mathcal{J}_f^{a,b}$ by the $\underset{\leftrightarrow}{\lim}\mathcal{J}_{f,k}^{a,b}$.

Generalizing these properties, we make the following definition.

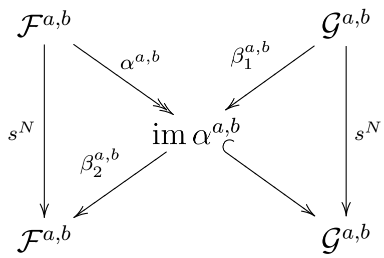
Definition 3.16. We say $\underset{\leftrightarrow}{\lim}\mathcal{F}^{a,b}$ is a $\Bbbk[[s]]$**-object** if the $\mathcal{F}^{a,b}$ are s^b-torsion $\mathcal{D}_Z[s]$-modules, equipped with shift isomorphisms $\sigma_k : \mathcal{F}^{a,b} \to \mathcal{F}^{a+k,b+k}$, so that the pro-ind limit obtains a $\Bbbk[[s]]$-module structure by the same construction above. We call $\underset{\leftrightarrow}{\lim}\mathcal{F}^{a,b}$ a $\Bbbk((s))$**-object** if it is a $\Bbbk[[s]]$-object on which s acts invertibly.

We will need the following criterion for the invertibility of a morphism of $\Bbbk((s))$-objects in $\underset{\leftrightarrow}{\lim}\,\mathrm{Hol}(\mathcal{D}_Z)$.

Proposition 3.17. Let $\alpha = \underset{\leftrightarrow}{\lim}\alpha^{a,b} : \underset{\leftrightarrow}{\lim}\mathcal{F}^{a,b} \to \underset{\leftrightarrow}{\lim}\mathcal{G}^{a,b}$ be a morphism of of $\Bbbk((s))$-objects.[6] Suppose there exists a fixed integer $N \geq 0$ such that for all $a \leq b$ both the kernel and cokernel of $\alpha^{a,b} : \mathcal{F}^{a,b} \to \mathcal{G}^{a,b}$ are annihilated by s^N. Then α is an isomorphism.

[6]Meaning α commutes with s.

Proof. The hypothesis entails that $s^N : \mathcal{G}^{a,b} \to \mathcal{G}^{a,b}$ factors through $\operatorname{im} \alpha^{a,b}$, and that $s^N : \mathcal{F}^{a,b} \to \mathcal{F}^{a,b}$ factors through $\mathcal{F}^{a,b} / \ker \alpha^{a,b} \cong \operatorname{im} \alpha^{a,b}$ as well. In other words, there exist maps $\beta_1^{a,b} : \mathcal{G}^{a,b} \to \operatorname{im} \alpha^{a,b}$ and $\beta_2^{a,b} : \operatorname{im} \alpha^{a,b} \to \mathcal{F}^{a,b}$ such that the diagram



commutes. Define $\beta^{a,b} = \beta_2^{a,b} \beta_1^{a,b}$. These are compatible: β_1 is simply multiplication by s^N and β_2 is multiplying by s^N and taking the (now unique) lift to $\mathcal{F}^{a,b}$; both procedures are compatible with the maps in the diagrams $\mathcal{F}$ and $\mathcal{G}$. So the $\beta^{a,b}$ induce a morphism $\beta = \underset{\leftrightarrow}{\lim} \beta^{a,b} : \underset{\leftrightarrow}{\lim} \mathcal{G}^{a,b} \to \underset{\leftrightarrow}{\lim} \mathcal{F}^{a,b}$ in $\underset{\leftrightarrow}{\lim} \operatorname{Hol}(\mathcal{D}_Z)$. Note that we have $\beta^{a,b} \alpha^{a,b}(m) = (\alpha^{a,b})^{-1} \alpha^{a,b}(s^{2N} m) = s^{2N} m$ and similarly $\alpha^{a,b} \beta^{a,b} = s^{2N}$. So both compositions $\alpha\beta$ and $\beta\alpha$ are multiplication by s^{2N}. Since α is a map of $\Bbbk((s))$ objects, multiplication by s^{2N} is invertible on either one. Consequently $s^{-2N}\beta$ inverts α. $\square$

3.2.4 Admissible filtrations

Another structure on $\mathcal{J}_f^{-\infty,\infty}$ is that it is exhaustively filtered by $\mathcal{D}_U[[s]]$-submodules $\mathcal{J}_f^{a,\infty}$. We can define truncated versions of the diagram $\{\mathcal{J}_f^{a,b}\}$ by

$$\mathcal{J}_{f,k}^{a,b} = s^{\max(a,k)} \text{“}f^s\text{”} / s^{\max(b,k)} \text{“}f^s\text{”}.$$

Observe that $\varinjlim_a \varprojlim_b \mathcal{J}_{f,k}^{a,b} = s^k \mathcal{O}_U[[s]] \cdot f^s = \mathcal{J}_f^{k,\infty}$. It is a simple exercise to verify that there are natural embeddings $\mathcal{J}_{f,k}^{\cdot,\cdot} \hookrightarrow \mathcal{J}_{f,\ell}^{\cdot,\cdot}$ for $k \geq \ell$, and that the $\mathcal{J}_{f,k}^{\cdot,\cdot}$ are all admissible. So $\mathcal{J}_f^{\cdot,\cdot}$ is *admissibly $\mathbb{Z}$-filtered* by the $\mathcal{J}_{f,k}^{\cdot,\cdot}$ in the diagram category:

$$\mathcal{J}_f^{\cdot,\cdot} \supset \cdots \supset \mathcal{J}_{f,-2}^{\cdot,\cdot} \supset \mathcal{J}_{f,-1}^{\cdot,\cdot} \supset \cdots .$$

Consequently there is a filtration by admissible monomorphisms in the exact category $\underset{\leftrightarrow}{\lim} \operatorname{Hol}(\mathcal{D}_U)$

$$\underset{\leftrightarrow}{\lim} \mathcal{J}_f^{a,b} \supset \cdots \supset \underset{\leftrightarrow}{\lim} \mathcal{J}_{f,-2}^{a,b} \supset \underset{\leftrightarrow}{\lim} \mathcal{J}_{f,-1}^{a,b} \supset \underset{\leftrightarrow}{\lim} \mathcal{J}_{f,0}^{a,b} \supset \cdots$$

which is analogous to the filtration of $\mathcal{J}_f^{-\infty,\infty}$ by the $\mathcal{J}_f^{a,\infty}$.

Definition 3.18. We say $\underset{\leftrightarrow}{\lim} \mathcal{F}^{a,b}$ is **admissibly filtered** if it has a $\mathbb{Z}$-indexed decreasing filtration by monomorphisms $\cdots \supset \underset{\leftrightarrow}{\lim} \mathcal{F}_k^{a,b} \supset \underset{\leftrightarrow}{\lim} \mathcal{F}_{k+1}^{a,b} \supset \cdots$ which are admissible in the exact category $\underset{\leftrightarrow}{\lim} \operatorname{Hol}(\mathcal{D}_Z)$, such that $\mathcal{F}_k^{a,b} = \mathcal{F}^{\max(a,k),\max(b,k)}$.

Lemma 3.19. Let $k \leq \ell$, and assume $\lim\limits_{\leftrightarrow}\mathcal{F}^{a,b}$ is admissibly filtered. Then we may form the cokernel

$$\lim\limits_{\leftrightarrow}\mathcal{F}_k^{a,b}/\lim\limits_{\leftrightarrow}\mathcal{F}_\ell^{a,b} = \operatorname{coker}(\lim\limits_{\leftrightarrow}\mathcal{F}_\ell^{a,b} \hookrightarrow \lim\limits_{\leftrightarrow}\mathcal{F}_k^{a,b}) = \lim\limits_{\leftrightarrow}\mathcal{G}^{a,b}, \qquad \mathcal{G}^{a,b} = \mathcal{F}_k^{a,b}/\mathcal{F}_\ell^{a,b}.$$

This cokernel canonically isomorphic to $\mathcal{F}^{k,\ell} \in \mathrm{Hol}(\mathcal{D}_Z) \subset \lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_Z)$.

Proof. By admissibility, one can prove that for $a \leq b$ one has

$$\mathcal{G}^{a,b} = \begin{cases} \mathcal{F}^{k,\ell} & b \geq \ell \geq k \geq a \\ \mathcal{F}^{a,\ell} & b \geq \ell \geq a \geq k \\ \mathcal{F}^{a,b} & \ell \geq b \geq a \geq k \\ \mathcal{F}^{k,b} & \ell \geq b \geq k \geq a \\ 0 & b \geq a \geq \ell \geq k \text{ or } \ell \geq k \geq b \geq a \end{cases}$$

Now consider the map $\varphi : \mathbb{Z} \to \mathbb{Z}$ defined by

$$\varphi(i) = \begin{cases} i + \ell & i \geq 0 \\ i + k & i < 0 \end{cases}$$

It is easy to verify that this is well-behaved in the sense of Definition 3.9, and that $\mathbf{1}_\mathbb{Z} \leq \varphi$. Moreover it has the properties

$$\varphi(b) \geq \ell \geq k \geq \varphi(a) \qquad b \geq 0 > a$$

$$\varphi(b) \geq \varphi(a) \geq \ell \geq k \qquad b \geq a \geq 0$$

$$\ell \geq k \geq \varphi(a) \geq \varphi(b) \qquad 0 > b \geq a$$

It follows that $\widetilde{\varphi}\mathcal{G}^{\cdot,\cdot} = \mathcal{F}^{k,\ell}$ as a diagram in $\mathrm{Hol}(\mathcal{D}_Z)^{\Pi}_{\leftrightarrow}$. So there is a map of diagrams $\mathcal{F}^{k,\ell} \to \mathcal{G}^{\cdot,\cdot}$ in the localizing class Σ, and therefore an isomorphism $\mathcal{F}^{k,\ell} \cong \lim\limits_{\leftrightarrow}\mathcal{G}^{a,b}$ in $\lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_Z)$. ☐

We will use this lemma to prove an important technical proposition.

Suppose we are given an isomorphism of admissibly filtered objects

$$\alpha = \lim\limits_{\leftrightarrow}\alpha^{a,b} : \lim\limits_{\leftrightarrow}\mathcal{F}_!^{a,b} \xrightarrow{\sim} \lim\limits_{\leftrightarrow}\mathcal{F}_*^{a,b}$$

in $\lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_Z)$. Note that we obtain maps $\alpha_k^{a,b} = \alpha^{\max(a,k),\max(b,k)} : \mathcal{F}_{!,k}^{a,b} \to \mathcal{F}_{*,k}^{a,b}$ for any k, and hence also a map $\alpha_k = \lim\limits_{\leftrightarrow}\alpha_k^{a,b} : \lim\limits_{\leftrightarrow}\mathcal{F}_{!,k}^{a,b} \to \lim\limits_{\leftrightarrow}\mathcal{F}_{*,k}^{a,b}$.

Key Technical Point 3.20. Observe that while not necessarily an isomorphism, α_k *is an admissible monomorphism!* For there is an embedding in the diagram category $\ker \alpha_k^{a,b} \to \ker \alpha^{a,b}$. By a diagram chase (cf. the proof of Lemma 3.8 above) one can verify that the kernel of a morphism of admissible diagrams is in fact an admissible diagam. We have $\lim\limits_{\leftrightarrow} \ker \alpha^{a,b} = 0$ in the pro-ind category. As an exercise, the reader may check that this implies $\lim\limits_{\leftrightarrow} \ker \alpha_k^{a,b} = 0$, and that this entails α_k is an admissible monomorphism.

This is an important point! It allows us to form objects such as the cokernel $\operatorname{coker}(\alpha_k)$ in the pro-ind category. It is *not* true that the image of an arbitrary morphism of admissible diagrams is admissible (try doing the diagram chase). In particular, this fails for just the example we need: we will ultimately want to study this cokernel when the αs are the Goresky-MacPherson map $j_! \to j_*$ for holonomic $\mathcal{D}$-modules. The Goresky-MacPherson functor $j_{!*} = \operatorname{im}(j_! \to j_*)$ preserves injectivity and surjectivity but is *not* exact in the middle. Consequently the diagram $\operatorname{im} \alpha^{a,b}$ need not be admissible in this case!

Proposition 3.21. Consider the cokernel $\operatorname{coker}_{k,\ell}$ of the admissible monomorphism $\varinjlim\!\!\!\!\varprojlim \mathcal{F}_!^{a,b} \overset{\alpha_\ell}{\hookrightarrow} \mathcal{F}_{*,\ell}^{a,b} \hookrightarrow \underset{\leftrightarrow}{\lim} \mathcal{F}_{*,k}^{a,b}$, for any $k \leq \ell$. This cokernel is isomorphic to an object of the subcategory $\operatorname{Hol}(\mathcal{D}_Z) \subset \underset{\leftrightarrow}{\lim} \operatorname{Hol}(\mathcal{D}_Z)$.

Proof. By the definition of isomorphisms in the pro-ind category, the map of diagrams $\alpha^{a,b}$ becomes invertible after applying one of the natural transformations $\widetilde{\varphi}$. In other words, there exists a well-behaved $\varphi \geq \mathbf{1}_{\mathbb{Z}}$ and an isomorphism of *diagrams* $\widetilde{\varphi}\mathcal{F}_! \to \widetilde{\varphi}\mathcal{F}_*$ such that the square

$$\begin{array}{ccc} \mathcal{F}_!^{a,b} & \xrightarrow[\sim]{\alpha^{a,b}} & \mathcal{F}_*^{a,b} \\ \sim\uparrow & & \uparrow\sim \\ \widetilde{\varphi}\mathcal{F}_!^{a,b} & \xrightarrow[\cong]{} & \widetilde{\varphi}\mathcal{F}_*^{a,b} \end{array}$$

commutes. Here and throughout this proof we use the convention that $\sim$ denotes a map which becomes an isomorphism after taking $\underset{\leftrightarrow}{\lim}$s, while $\cong$ denotes a genuine isomorphism of diagrams. We negate these symbols to indicate that the relevant property does *not* hold.

It follows immediately that there is also a commutative square of truncated diagrams

$$\begin{array}{ccc} (\widetilde{\varphi}\mathcal{F}_!)_\ell^{a,b} & \xrightarrow{\cong} & (\widetilde{\varphi}\mathcal{F}_*)_\ell^{a,b} \\ \not\sim\downarrow & & \downarrow\not\sim \\ \mathcal{F}_{!,\ell}^{a,b} & \xrightarrow[\alpha_\ell^{a,b}]{\not\sim} & \mathcal{F}_{*,\ell}^{a,b} \end{array}$$

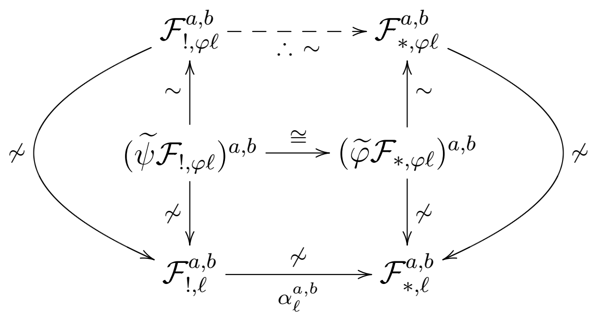
where the truncations of the vertical maps no longer become isomorphisms in the pro-ind category.

Temporarily denote by $\sharp$ either $!$ or $*$. Note that $(\widetilde{\varphi}\mathcal{F}_\sharp)_\ell^{a,b} = (\widetilde{\varphi}\mathcal{F}_\sharp)^{\max(\ell,a),\max(\ell,b)} = \mathcal{F}_\sharp^{\varphi(\max(\ell,a)),\varphi(\max(\ell,b))}$. Since φ is order preserving, this is the same thing as $\mathcal{F}_\sharp^{\max(\varphi\ell,\varphi a),\max(\varphi\ell,\varphi b)} = \mathcal{F}_{\sharp,\varphi\ell}^{\varphi a,\varphi b} = (\widetilde{\varphi}\mathcal{F}_{\sharp,\varphi\ell})^{a,b}$. We have canonical maps $(\widetilde{\varphi}\mathcal{F}_{\sharp,\varphi\ell})^{a,b} \xrightarrow{\sim} \mathcal{F}_{\sharp,\varphi\ell}^{a,b}$. The squares

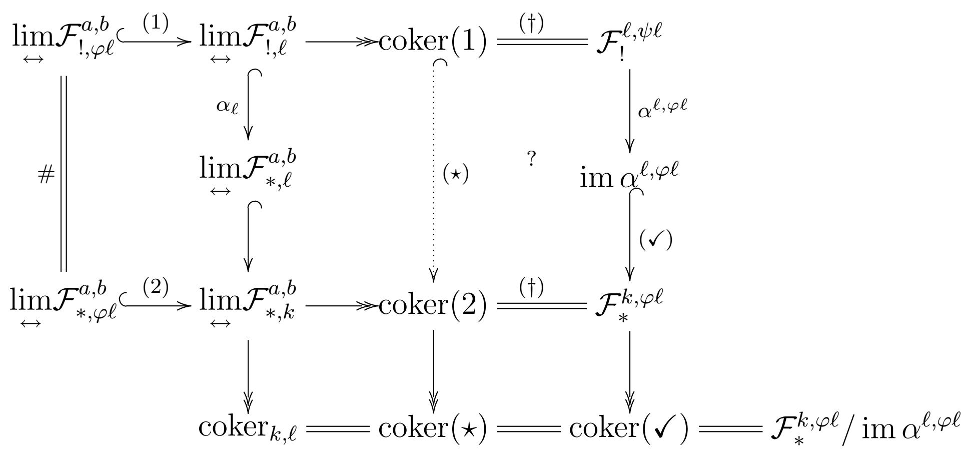
$$\begin{array}{ccc} (\widetilde{\varphi}\mathcal{F}_\sharp)_\ell^{a,b} & \xrightarrow{\not\sim} & \mathcal{F}_{\sharp,\ell}^{a,b} \\ \| & & \uparrow\not\sim \\ (\widetilde{\varphi}\mathcal{F}_{\sharp,\varphi\ell})^{a,b} & \xrightarrow[\sim]{} & \mathcal{F}_{\sharp,\varphi\ell}^{a,b} \end{array}$$

commute.

Hence we obtain a commutative diagram in $\mathrm{Hol}(\mathcal{D}_Z)^{\Pi}_{\leftrightarrow}$:



Passing to the pro-ind category, we therefore obtain the isomorphism # in the following commutative diagram in the pro-ind category, with exact sequences as indicated by $\hookrightarrow$ for admissible monomorphisms and $\twoheadrightarrow$ for admissible epimorphisms.



Here the isomorphisms labeled (†) are by Lemma 3.19, and the isomorphisms in the bottom row are by general abstract nonsense. We leave the commutativity of the rectangle marked ? as an exercise. □

An straightforward corollary of the proof is the following.

Corollary 3.22. If $k = \ell$ and the modules $\mathrm{coker}(\alpha^{a,b})$ are supported on a subvariety $W \subset Z$ for any a, b, then the holonomic module $\mathrm{coker}_{k,k}$ is also supported on W. □

3.3 Duality and direct image functors in $\lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_Z)$

By Corollary 3.14, exact functors between the categories of holonomic $\mathcal{D}$-modules we are interested in, induce analogues in the $\lim\limits_{\leftrightarrow}$s of these categories. We will apply these functors to the following collection of pro-ind holonomic $\mathcal{D}$-modules.

3.3.1 Definition of $\mathcal{M}^{-\infty,\infty}$ and $\mathcal{M}_k^{-\infty,\infty}$

For any holonomic $\mathcal{D}_U$-module $\mathcal{M}$ we define

$$\mathcal{M}^{-\infty,\infty} = \lim_{\leftrightarrow}(\mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{J}_f^{a,b}).$$

Note that $\mathcal{M} \otimes \mathcal{J}_f^{a,b}$ is a holonomic $\mathcal{D}_U$-module (see Appendix B.5.3), so these together form a diagram of the required sort. One can check directly that this diagram is admissible, using the fact that as $\Bbbk$-vector spaces $\mathcal{M} \otimes \mathcal{J}_f^{a,b} \cong \mathcal{M} \otimes_{\Bbbk} s^a\Bbbk[s]/s^b\Bbbk[s]$ and the maps among these are the obvious ones. Moreover since each $\mathcal{J}_f^{a,b}$ is a free $\mathcal{O}$-module, tensoring by it is an exact functor. Consequently $\mathcal{M} \mapsto \mathcal{M}^{-\infty,\infty}$ is an exact functor from $\mathrm{Hol}(\mathcal{D}_U)$ to $\lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_U)$. In addition, $\mathcal{M}^{-\infty,\infty}$ is an admissibly filtered $\Bbbk((s))$-object in the pro-ind category, filtered by

$$\mathcal{M}_k^{-\infty,\infty} = \lim_{\leftrightarrow}(\mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{J}_{f,k}^{a,b}).$$

The action of s is induced from the action of s on the $\mathcal{J}_f^{a,b}$s.

3.3.2 Duality

To analyze the duality functor on the pro-ind category, we will first need to understand it for the $\mathcal{J}^{a,b}$s.

Lemma 3.23. We have $\mathbb{D}\mathcal{J}^{a,b} \cong \mathcal{J}^{-b,-a}$, as $\mathcal{D}_{\Bbbk^\times}$-modules, and with respect to this isomorphism, $\mathbb{D}(\mathcal{J}^{a,b} \xrightarrow{s} \mathcal{J}^{a,b}) = \mathcal{J}^{-b,-a} \xleftarrow{s} \mathcal{J}^{-b,-a}$.

Proof. Define a pairing

$$\mathcal{J}^{a,b} \times \mathcal{J}^{-b,-a} \to \mathcal{J}^{0,1} = \mathcal{O}_{\Bbbk^\times}$$

by

$$\begin{aligned} \langle f(s), g(s)\rangle &= \langle (s^a x_0 + s^{a+1}x_1 + \cdots + s^{b-1}x_{b-a-1})t^s, (s^{-b}y_0 + s^{-b+1}y_1 + \cdots + s^{-a-1}y_{b-a-1})t^s\rangle \\ &= \operatorname*{Res}_{s=0} f(s)g(-s)\,\mathrm{d}\,s = -(x_0y_{b-a-1} + \cdots + x_{b-a-1}y_0). \end{aligned}$$

Here we have computed the residue formally after making the substitution $s \to -s$ (including $t^s \mapsto t^{-s}$ so that the two cancel).

For $\mathcal{O}$-coherent $\mathcal{D}$-modules, the duality functor $\mathbb{D}$ coincides with the usual duality for $\mathcal{O}$-modules, $\mathcal{H}om_{\mathcal{O}}(\bullet, \mathcal{O})$, with a canonically defined $\mathcal{D}$-action

$$(\xi\varphi)(m) = \xi\varphi(m) - \varphi(\xi m).$$

It is straightforward to check that the pairing above induces an isomorphism of $\mathcal{O}$-modules

$$\mathbb{D}\mathcal{J}^{a,b} = \mathcal{H}om_{\mathcal{O}_{\Bbbk^\times}}(\mathcal{J}^{a,b}, \mathcal{O}_{\Bbbk^\times}) \cong \mathcal{J}^{-b,-a}.$$

In fact this is an isomorphism of $\mathcal{D}$-modules as well. To see that it respects the action of ∂, note that it sends $s^{-b+k}t^s \in \mathcal{J}^{-b,-a}$ to $\varphi_k \in \mathcal{H}om_{\mathcal{O}_{\Bbbk^\times}}(\mathcal{J}^{a,b}, \mathcal{O}_{\Bbbk^\times})$ defined by $\varphi_k(s^{a+j}t^s) = -\delta_{j+k,b-a-1}$. In particular, $\partial(s^{-b+k}t^s) = s^{-b+k+1}t^{-1}t^s$ is sent to $t^{-1}\varphi_{k+1}$. We have

$$\begin{aligned} (\partial\varphi_k)(s^{a+j}t^s) = \partial\varphi_k(s^{a+j}t^s) - \varphi_k\partial(s^{a+j}t^s) \\ = -\partial\delta_{j+k,b-a-1} + \varphi_k(s^{a+j+1}t^{-1}t^s) \\ = -t^{-1}\delta_{k+j+1,b-a-1} = t^{-1}\varphi_{k+1}(s^{a+j}t^s), \end{aligned}$$

as desired. We leave the reader to check the last assertion (about the dual of the s-action). $\square$

Lemma 3.24. $\mathbb{D}\mathcal{J}_f^{a,b} \cong \mathcal{J}_f^{-b,-a}$ as $\mathcal{D}_U$-modules, in a manner respecting the s action.

Proof. This follows from the last lemma and the computation

$$\mathbb{D}_U\mathcal{J}_f^{a,b} = \mathbb{D}_U((f|_U)^!\mathcal{J}^{a,b}[\dim X - 1] = (\mathbb{D}_U(f|_U)^!\mathcal{J}^{a,b})[1 - \dim X] = \\ ((f|_U)^*\mathbb{D}_{\Bbbk^\times}\mathcal{J}^{a,b})[1 - \dim X] = ((f|_U)^*\mathcal{J}^{-b,-a})[1 - \dim X] = \mathcal{J}_f^{-b,-a}.$$

The s action on the $\mathcal{J}_f$s is induced from that on the $\mathcal{J}$s by functoriality, so the last assertion comes for free. □

From the considerations discussed in Appendix B.5.3 we obtain the following corollary.

Corollary 3.25. For any holonomic $\mathcal{D}_U$-module $\mathcal{M}$, we have an isomorphism

$$\mathbb{D}(\mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{J}_f^{a,b}) \cong \mathbb{D}\mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{J}_f^{-b,-a}$$

which respects the s-action. □

Taking duals sends the canonical maps $\mu_{a,b,a',b'}$ to $\mu_{-b',-a',-b,-a}$. Since $(a,b) \mapsto (-b,-a)$ is an automorphism of the poset Π, we obtain the following.

Proposition 3.26. There is a canonical isomorphism of $\Bbbk((s))$-objects in $\underset{\leftrightarrow}{\lim}\,\mathrm{Hol}(\mathcal{D}_U)$

$$\mathbb{D}(\mathcal{M}^{-\infty,\infty}) \cong (\mathbb{D}\mathcal{M})^{-\infty,\infty}. \quad \square$$

We will need to know how this behaves with respect to the admissible filtrations described above.

Proposition 3.27. $\mathbb{D}(\mathcal{M}_k^{-\infty,\infty}) \cong (\mathbb{D}\mathcal{M})^{-\infty,\infty}/(\mathbb{D}\mathcal{M})_{-k}^{-\infty,\infty}$.

(The "quotient" really means cokernel of the corresponding admissible inclusion.) *Proof.* A straightforward corollary of Lemma 3.19 is that $\mathrm{coker}(\underset{\leftrightarrow}{\lim}\mathcal{F}_k^{a,b} \hookrightarrow \underset{\leftrightarrow}{\lim}\mathcal{F}^{a,b}) = \underset{\leftrightarrow}{\lim}_k\mathcal{F}^{a,b}$, where ${}_k\mathcal{F}^{a,b} = \mathcal{F}^{\min(a,k),\min(b,k)}$. It is clear that we have $\mathbb{D}\mathcal{J}_{f,k}^{a,b} = {}_{-k}\mathcal{J}_f^{-b,-a}$, so

$$\mathbb{D}(\mathcal{M}_k^{-\infty,\infty}) = \underset{\leftrightarrow}{\lim}(\mathbb{D}\mathcal{M} \otimes {}_{-k}\mathcal{J}_f^{-b,-a}).$$

Now we recognize the righthand side as

$$\mathrm{coker}\{(\mathbb{D}\mathcal{M})_{-k}^{-\infty,\infty} \hookrightarrow \mathbb{D}(\mathcal{M}^{-\infty,\infty})\} = \mathrm{coker}\{\underset{\leftrightarrow}{\lim}(\mathbb{D}\mathcal{M} \otimes \mathcal{J}_{f,-k}^{a,b}) \hookrightarrow \underset{\leftrightarrow}{\lim}(\mathbb{D}\mathcal{M} \otimes \mathcal{J}_f^{a,b})\}. \quad \square$$

The object in the category of $\mathcal{D}_U((s))$ modules analogous to $\mathbb{D}(\mathcal{M}_k^{-\infty,\infty})$ is the "truncated Laurent series" $\mathcal{M} \otimes (\mathcal{O}_U((s)) \cdot f^s / s^{-k}\mathcal{O}_U[[s]] \cdot f^s)$.

3.3.3 Direct images

Next we discuss functors of direct image along the open embedding j.

Key Technical Point 3.28. *Note that j is an open affine embedding (it is obtained by base change from $\Bbbk^\times \hookrightarrow \Bbbk$) so $j_* = j.$ is exact, and hence $j_! = \mathbb{D}j_*\mathbb{D}$ is also exact.* In particular, they take holonomic modules to holonomic **modules**, not complexes! Consequently there are induced exact functors j_* and $j_!$ from $\lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_U)$ to $\lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_X)$.

Definition 3.29. For $\sharp = *$ or $!$ we denote $j_\sharp \mathcal{M}^{-\infty,\infty}$ by $\Pi^{f\sharp}\mathcal{M}$ or simply $\Pi^\sharp\mathcal{M}$.

The functor $\Pi^{f,\sharp} : \mathrm{Hol}(\mathcal{D}_U) \to \lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_X)$ is exact. If we write s for $j_\sharp(s) : \Pi^\sharp\mathcal{M} \to \Pi^\sharp\mathcal{M}$, we see that the structure of admissibly filtered $\Bbbk((s))$-object on $\mathcal{M}^{-\infty,\infty} \in \lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_U)$ induces the same structure on $\Pi^\sharp\mathcal{M} \in \lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_X)$, if we write $\Pi^\sharp_k\mathcal{M}$ for the admissible $\Bbbk[[s]]$-subobject $j_\sharp\mathcal{M}_k^{-\infty,\infty} \subset \Pi^\sharp\mathcal{M}$. The functors $\Pi^\sharp_k$ are also exact.

By Proposition 3.26 we obtain $\mathbb{D}\Pi^!\mathcal{M} = \Pi^*\mathbb{D}\mathcal{M}$ and vice versa. By Proposition 3.27 we have $\mathbb{D}\Pi^!_k\mathcal{M} = (\Pi^*\mathbb{D}\mathcal{M})/(\Pi^*_{-k}\mathbb{D}\mathcal{M})$ and vice versa.

Next we wish to produce a canonical morphism $\Pi^{f!} \to \Pi^{f*}$. Recall that for a holonomic $\mathcal{D}_U$-module $\mathcal{M}$ there is a morphism $j_!\mathcal{M} \to j_*\mathcal{M}$. Let $\gamma : j_!j^!j_*\mathcal{M} \to j_*\mathcal{M}$ be the (co?)unit morphism of the adjunction $j_! \leftrightarrow j^!$ at the object $j_*\mathcal{M}$. Since j is an open embedding, $j^* = j^! = j^\Delta =$ restriction, and $j^!j_*\mathcal{M} = \mathcal{M}$. So there is a canonical map (the identity) $\widetilde{\alpha} : \mathcal{M} \to j^!j_*\mathcal{M}$. The canonical morphism $\alpha : j_!\mathcal{M} \to j_*\mathcal{M}$ is given by

$$\gamma \circ (j_!\widetilde{\alpha}) : j_!\mathcal{M} \to j_!j^!j_*\mathcal{M} \to j_*\mathcal{M}.$$

This is entirely functorial, and hence compatible with the maps $\mathcal{M} \otimes \mathcal{J}_f^{a,b} \to \mathcal{M} \otimes \mathcal{J}_f^{a',b'}$, by naturality. Hence the morphisms

$$\alpha^{a,b} : j_!(\mathcal{M} \otimes \mathcal{J}_f^{a,b}) \to j_*(\mathcal{M} \otimes \mathcal{J}_f^{a,b})$$

give rise to a map $\alpha = \lim\limits_{\leftrightarrow} \alpha^{a,b} : \Pi^!\mathcal{M} \to \Pi^*\mathcal{M}$ in $\lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_X)$. This respects the s-action. Similarly, we obtain $\alpha_k = \lim\limits_{\leftrightarrow} \alpha_k^{a,b} = \lim\limits_{\leftrightarrow} \alpha^{\max(a,k),\max(b,k)} : \Pi^!_k\mathcal{M} \to \Pi^*_k\mathcal{M}$.

Remark 3.30. Despite the raised symbols, the $\Pi^\sharp$ and $\Pi^\sharp_k$ are actually *co*variant functors, as the construction makes clear. Justifying this breach of convention, it leaves room for the truncation index k below. Furthermore, we will see in the next section that the $\Pi^\sharp$s (without truncation) are merely temporary notation anyway.

3.4 Main isomorphism

All of the machinery of the pro-ind category was set up so that we may prove the following key theorem and its corollary.

Theorem 3.31. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_U$-module. In the category $\lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_X)$, the map*

$$\Pi^!\mathcal{M} \to \Pi^*\mathcal{M}$$

is an isomorphism.

Proof. Write $\alpha^{a,b} : j_!(\mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{J}_f^{a,b}) \to j_*(\mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{J}_f^{a,b})$ for the natural map, and $\alpha = \underset{\leftrightarrow}{\lim}\alpha^{a,b}$: $\Pi^!\mathcal{M} \to \Pi^*\mathcal{M}$. This is a morphism of $\Bbbk((s))$-objects in the category of pro-ind holonomic $\mathcal{D}_X$-modules. By Proposition 3.17 it suffices to show that there exists $N \geq 0$ such that s^N annihilates both $\ker \alpha^{a,b}$ and $\operatorname{coker} \alpha^{a,b}$ for all a, b. Since $\mathbb{D}\alpha^{a,b}$ is the canonical map for $\mathbb{D}\mathcal{M}$, it is enough to check this for cokernels. That is, it suffices to prove that

$$s^N j_*(\mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{J}_f^{a,b}) \subset \operatorname{im}(j_!(\mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{J}_f^{a,b}) = j_{!*}(\mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{J}_f^{a,b})$$

when $N \gg 0$, regardless of a and b.

For this we will use the results of §2. First, a consequence of Corollary 2.8 is that

$$j_{!*}(\mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{J}_f^{a,b}) = \sum_\ell \mathcal{D}_X[s](f^k m_\ell \otimes f^s) \subset \sum_\ell \sum_{i\in\mathbb{Z}} \mathcal{D}_X[s](f^i m_\ell \otimes f^s) = j_*(\mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{J}_f^{a,b}),$$

for $k \gg 0$, where m_ℓ runs through a collection of sections which generates the $\mathcal{D}_U$-module $\mathcal{M}$. To prove the theorem, it is therefore enough to check that for some $N \geq 0$, the module $\sum \mathcal{D}_X[s](f^k m_\ell \otimes f^s)$ contains $s^N f^i m_\ell \otimes f^s$ for all exponents $i \in \mathbb{Z}$, since these sections generate the $\mathcal{D}_X$-module $s^N j_*(\mathcal{M} \otimes \mathcal{J}_f^{a,b})$. In fact, however, $j_*(\mathcal{M} \otimes \mathcal{J}_f^{a,b})$ is a holonomic $\mathcal{D}_X$-module, so it is generated over $\mathcal{D}_X[s]$ by the $s^N f^i m_\ell \otimes f^s$ for $i \geq k'$ for some fixed $k' \ll 0$. So it suffices to find N so that $s^N f^{k'} m_\ell \otimes f^s, \dots, s^N f^{k-1} m_\ell \otimes f^s$ are contained in $\mathcal{D}_X[s](f^k m_\ell \otimes f^s)$, or equivalently, such that $s^N f^{k'} m_\ell \otimes f^s \in \mathcal{D}_X[s](f^k m_\ell \otimes f^s)$, for each ℓ.

Let b_ℓ be the b-function of m_ℓ. By Theorem 2.1 we have

$$b_\ell(s) \cdot m_\ell \otimes f^s = d_\ell(s) f m_\ell \otimes f^s \in \mathcal{D}_X[s](f m_\ell \otimes f^s),$$

where d_ℓ is a polynomial with coefficients in $\mathcal{D}_X$. It follows formally that

$$b_\ell(s+k-1) b_\ell(s+k-2) \dots b_\ell(s+k') f^{k'} m_\ell \otimes f^s = D_\ell(s) f^k m_\ell \otimes f^s \in \mathcal{D}_X[s](f^k m_\ell \otimes f^s),$$

where $D_\ell(s)$ is a (much bigger) polynomial with coefficients in $\mathcal{D}_X$. Denote the product of translates of b_ℓ which occurs in this equation by $B_\ell(s) \in \Bbbk[s]$. Let l_ℓ denote the number of integer roots of b_ℓ. Then certainly $s^{l_\ell} B_\ell(s)$ has positive order at $s = 0$. In particular it can be expanded as a power series $C_\ell(s) \in \Bbbk[[s]]$. Since $j_*(\mathcal{M} \otimes \mathcal{J}_f^{a,b})$ is s^b-torsion, it follows that the identity $\widetilde{C}_\ell B_\ell f^{k'} m_\ell \otimes f^s = s^{l_\ell} f^{k'} m_\ell \otimes f^s$ holds in $\mathcal{D}_X[s](f^{k'} m_\ell \otimes f^s)$, where $\widetilde{C}_\ell$ is the truncation of C_ℓ modulo s^b. Consequently

$$s^{l_\ell} f^{k'} m_\ell \otimes f^s = \widetilde{C}_\ell D_\ell f^k m_\ell \otimes f^s \in \mathcal{D}_X[s](f^k m_\ell \otimes f^s) \subset j_{!*}(\mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{J}_f^{a,b}).$$

Now if we take $N = \sum_\ell l_\ell$ and replace s^{l_ℓ} by s^N then the equation above holds for each ℓ, which proves the theorem. □

Definition 3.32. By the theorem $\Pi^!\mathcal{M} = \Pi^*\mathcal{M}$, so from now on we will denote this object in $\underset{\leftrightarrow}{\lim} \operatorname{Hol}(\mathcal{D}_Z)$ simply by $\Pi\mathcal{M}$.

The following crucial corollary of Theorem 3.31 is a consequence of Proposition 3.21 and Corollary 3.22, in light of the fact that each quotient $j_*(\mathcal{M} \otimes \mathcal{J}_f^{a,b}) / j_{!*}(\mathcal{M} \otimes \mathcal{J}_f^{a,b})$ is supported on Y. (Recall that by Kashiwara's theorem, cf. Appendix B.3.2, holonomic $\mathcal{D}_X$-modules supported on Y are equivalent to holonomic $\mathcal{D}_Y$-modules, via the functor $i^!$.)

Corollary 3.33. For any $k \leq \ell$, the correspondence

$$\mathcal{M} \mapsto \Pi_{!*}^{k,\ell}\mathcal{M} = \frac{\Pi_k^*\mathcal{M}}{\Pi_\ell^!\mathcal{M}}$$

gives an exact functor from $\mathrm{Hol}(\mathcal{D}_U)$ to $\mathrm{Hol}(\mathcal{D}_X)$. When $k = \ell$, $\Pi_{!*}^{k,k}\mathcal{M}$ is supported on Y, so $i^!\Pi_{!*}^{k,k}$ is an exact functor $\mathrm{Hol}(\mathcal{D}_U) \to \mathrm{Hol}(\mathcal{D}_Y)$. □

Here the "quotient" $\Pi_k^*\mathcal{M}/\Pi_\ell^!\mathcal{M}$ really means to take the cokernel of the admissible monomorphism $\Pi_\ell^!\mathcal{M} \overset{\alpha_\ell}{\hookrightarrow} \Pi_\ell^*\mathcal{M} \hookrightarrow \Pi_k^*\mathcal{M}$; cf. Key Technical Point 3.20. We should remark that exactness here follows from a version of the 9 Lemma for exact categories: $\Pi_\ell^!$ is an exact subfunctor of the exact functor Π_ℓ^*, and is moreover admissible, so the cokernel is exact as well.

We will need to know below that the $\Pi_{!*}^{k,\ell}$ are well-behaved with respect to duality. Propositions 3.26 and 3.27 imply

Lemma 3.34. $\mathbb{D}\Pi_k^!\mathcal{M} = \Pi\mathbb{D}\mathcal{M}/\Pi_{-k}^*\mathbb{D}\mathcal{M}$ and $\mathbb{D}\Pi_k^*\mathcal{M} = \Pi\mathbb{D}\mathcal{M}/\Pi_{-k}^!\mathbb{D}\mathcal{M}$. □

Proposition 3.35. $\mathbb{D}\Pi_{!*}^{k,\ell}\mathcal{M} = \Pi_{!*}^{-\ell,-k}\mathbb{D}\mathcal{M}$.

Proof. This is a formal consequence of Lemma 3.34: $\mathbb{D}\Pi_{!*}^{k,\ell}\mathcal{M} = \mathbb{D}\,\mathrm{coker}(\Pi_\ell^!\mathcal{M} \hookrightarrow \Pi_k^*\mathcal{M}) = \ker(\mathbb{D}\Pi_\ell^!\mathcal{M} \twoheadrightarrow \mathbb{D}\Pi_k^*\mathcal{M}) = \ker(\frac{\Pi\mathbb{D}\mathcal{M}}{\Pi_{-k}^*\mathbb{D}\mathcal{M}} \twoheadrightarrow \frac{\Pi\mathbb{D}\mathcal{M}}{\Pi_{-\ell}^!\mathbb{D}\mathcal{M}}) = \Pi_{!*}^{-\ell,-k}\mathbb{D}\mathcal{M}$. □

3.5 Nearby cycles functor

The results of the last subsection allow us (finally!) to define the nearby cycles, maximal extension, and vanishing cycles functors we need. We begin with nearby cycles.

Definition 3.36. The **unipotent nearby cycles functor** $\Psi_f : \mathrm{Hol}(\mathcal{D}_U) \to \mathrm{Hol}(\mathcal{D}_Y)$ is defined by

$$\Psi_f(\mathcal{M}) = \Pi_{!*}^{0,0}\mathcal{M}.$$

Theorem 3.37. *The functor Ψ_f has the following properties.*

(i) $\Psi_f(\mathcal{M})$ is a holonomic $\mathcal{D}_X$-module supported on Y.

(ii) Ψ_f is exact.

(iii) Ψ_f commutes with duality.

Proof. Immediate from Corollary 3.33 and Proposition 3.35. □

Example 3.38. Consider the case $\Psi_t(\mathcal{O}_{\Bbbk^\times})$. The entire point of the construction of the pro-ind category was that $\mathcal{O}_{\Bbbk^\times}^{-\infty,\infty}$ behaves exactly the like $\mathcal{D}_{\Bbbk^\times}$-module $\mathcal{J}^{-\infty,\infty} = \mathcal{O}_{\Bbbk^\times}((s)) \cdot t^s$, and the filtration $\mathcal{O}_{\Bbbk^\times,k}^{-\infty,\infty}$ like the $\mathcal{J}_k^{-\infty,\infty} = s^k\mathcal{O}_{\Bbbk^\times}[[s]] \cdot t^s$. In particular, $\Psi_t\mathcal{O}_{\Bbbk^\times}$ may be identified with "$j_*(\mathcal{O}_{\Bbbk^\times}[[s]] \cdot t^s)/j_!(\mathcal{O}_{\Bbbk^\times}[[s]] \cdot t^s)$". Here is an interpretation of this expression due to Ginzburg [G].

Recall that $\mathcal{J}^{(n)}$ has a filtration of length n with simple subquotients isomorphic to $\mathcal{J}^{(1)} \cong \mathcal{O}_{\Bbbk^\times}$. One can check that $j_*\mathcal{O}_{\Bbbk^\times}$ has a submodule isomorphic to $\mathcal{O}_\Bbbk$ and the quotient is the simple $\mathcal{D}_{\Bbbk}$-module δ_0 (the δ-function); see Appendix B.6, B.3. Consequently $j_*\mathcal{J}^{(n)}$ has a filtration of length $2n$ with simple subquotients (from "bottom" to "top")

$$\mathcal{O}, \delta_0, \mathcal{O}, \delta_0, \ldots, \mathcal{O}, \delta_0.$$

Both $\mathcal{O}$ and δ_0 are self-dual, and as we saw above so is $\mathcal{J}^{(n)}$ (disregarding the s-action). So $j_!\mathcal{O}_{\Bbbk^\times}$ has a filtration

$$\delta_0, \mathcal{O}, \delta_0, \mathcal{O}, \ldots, \delta_0, \mathcal{O}.$$

The canonical map takes the last $2n-1$ subquotients of $j_!$ to the first $2n-1$ subquotients of j_*. Passing to the projective limit $\mathcal{J}^{0,\infty}$ corresponds to continuing the *bottom* of these filtration indefinitely. We find (heuristically) that the modules $j_*(\mathcal{O}_{\Bbbk^\times}[[s]] \cdot t^s)$ and $j_!(\mathcal{O}_{\Bbbk^\times}[[s]] \cdot t^s)$ have respective filtrations

$$\cdots, \mathcal{O}, \delta_0, \mathcal{O}, \delta_0$$

$$\cdots, \delta_0, \mathcal{O}, \delta_0, \mathcal{O}.$$

Thus the map from the latter to the former is injective. (As per Theorem 3.31, it is the truncation of a map which becomes an isomorphism in the pro-ind limit, i.e. $j_!(\mathcal{O}_{\Bbbk^\times}((s))\cdot t^s) \cong j_*(\mathcal{O}_{\Bbbk^\times}((s)) \cdot t^s)$.) The δ_0 at the "bottom" of $j_!\mathcal{J}^{(n)}$ killed by the canonical map $j_! \to j_*$ has been relegated to irrelevance in the projective limit. Moreover, we see that the quotient module is the δ_0 at the "top" of $j_*\mathcal{J}^{0,\infty}$.

Heuristically, then, we expect $\Psi_t(\mathcal{O}_{\Bbbk^\times}) = \delta_0 \in \mathrm{Hol}(\mathcal{D}_X)$, which is indeed supported on $\{0\} = \Bbbk - \Bbbk^\times$ as per part (i) of the theorem above. Under the Kashiwara equivalence, this module is taken to $i^!\delta_0 = \mathbb{C}$, the unique simple holonomic $\mathcal{D}$-module on a point. Indeed, this computation is justified by the proof of Proposition 3.21, which implies $\Psi_t\mathcal{O}_{\Bbbk^\times} = j_*\mathcal{J}^{0,N}/j_{!*}\mathcal{J}^{0,N}$ for $N = \varphi(0)$ with $\varphi \geq \mathbf{1}_{\mathbb{Z}}$ the well-behaved function $\mathbb{Z} \to \mathbb{Z}$ occuring in that proof. We saw above that any such quotient is δ_0.

3.6 Shifts and the maximal extension functor

3.6.1 Shifts and duality for $\Pi^{k,\ell}_{!*}$, $k \neq \ell$

The nearby cycles were supported on Y, but we will also need a $\Pi^{k,\ell}_{!*}$ with $k \neq \ell$ and thus not supported on Y. Unfortunately, these do not *quite* commute with duality, according to Proposition 3.35. Fortunately, there is an easy fix. Recall that the shift isomorphisms $\sigma_k : \mathcal{F}^{a,b} \to \mathcal{F}^{a+k,b+k}$ induce the operator of multiplication by s^k on an admissibly filtered $\Bbbk((s))$-boject $\lim\limits_{\leftrightarrow}\mathcal{F}^{a,b}$, and given an isomorphism of $\Bbbk[[s]]$-subobjects $\lim\limits_{\leftrightarrow}\mathcal{F}^{a,b}_\ell \xrightarrow{\sigma_k} \lim\limits_{\leftrightarrow}\mathcal{F}^{a,b}_{\ell+k}$. (*A priori* this only makes sense for $k \geq 0$, but since these maps are isomorphisms we may formally denote σ_k^{-1} by σ_{-k}.) In particular, we obtain isomorphisms

$$\Pi^\sharp_\ell\mathcal{M} \xrightarrow{\sigma_k} \Pi^\sharp_{\ell+k}\mathcal{M}$$

in $\lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_X)$. These quite obviously induce an isomorphism

$$\Pi^{k,\ell}_{!*}\mathcal{M} \xrightarrow{\sigma_j} \Pi^{k+j,\ell+j}_{!*}\mathcal{M}.$$

Hence we have a canonical isomorphism

$$\mathbb{D}\Pi^{k,\ell}_{!*}\mathcal{M} \cong \Pi^{-\ell,-k}_{!*}\mathbb{D}\mathcal{M} \xrightarrow{\sigma_{k+\ell}} \Pi^{k,\ell}_{!*}\mathbb{D}\mathcal{M}.$$

Definition 3.39. The **shifted nearby cycles** functor is

$$\Psi_f^{(i)}\mathcal{M} = \Pi_{!*}^{i,i}\mathcal{M} \overset{\sigma_i}{\leftarrow} \Psi_f\mathcal{M}.$$

By definition $\mathbb{D}\Psi_f^{(i)}\mathcal{M} = \Psi_f^{(-i)}\mathbb{D}\mathcal{M}$.

3.6.2 Maximal extension functor

Definition 3.40. The **maximal extension functor** $\Xi_f^{(i)} : \mathrm{Hol}(\mathcal{D}_U) \to \mathrm{Hol}(\mathcal{D}_X)$ is defined by

$$\Xi_f^{(i)}\mathcal{M} = \Pi_{!*}^{i,i+1}\mathcal{M} \overset{\sigma_i}{\leftarrow} \Pi_{!*}^{0,1}\mathcal{M} = \Xi_f\mathcal{M}.$$

Parts (iv)-(vi) of the next theorem are why the Ξ_f are significant.

Theorem 3.41. *The functors Ξ_f have the following properties.*

(i) $\Xi_f^{(i)}(\mathcal{M})$ *is a holonomic $\mathcal{D}_X$-module.*

(ii) $\Xi_f^{(i)}$ *is exact.*

(iii) $\mathbb{D}\Xi_f^{(i)}\mathcal{M} = \Xi_f^{(-i-1)}\mathbb{D}\mathcal{M}$.

(iv) There are canonical exact sequences

$$0 \to j_!(\mathcal{M}) \overset{\alpha_-}{\to} \Xi_f^{(i)}(\mathcal{M}) \overset{\beta_-}{\to} \Psi_f^{(i)}(\mathcal{M}) \to 0$$

$$0 \to \Psi_f(\mathcal{M})^{(-i-1)} \overset{\beta_+}{\to} \Xi_f^{(-i-1)}(\mathcal{M}) \overset{\alpha_+}{\to} j_*(\mathcal{M}) \to 0$$

which are interchanged by duality.

(v) $\alpha_+\alpha_- : j_!\mathcal{M} \to j_*\mathcal{M}$ *is the canonical map* α.

(vi) $\beta_-\beta_+ : \Psi_f(\mathcal{M}) \to \Psi_f(\mathcal{M})$ *is multiplication by s (a.k.a., the monodromy operator).*

Proof. Parts (i)-(iii) are clear.

By Lemma 3.19, $j_!\mathcal{M} \cong j_!(\mathcal{M} \otimes \mathcal{J}_f^{i,i+1}) \cong \Pi_i^!\mathcal{M}/\Pi_{i+1}^!\mathcal{M}$ in $\lim\limits_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_X)$, for any i. The exact sequence

$$\Pi_i^!\mathcal{M} \hookrightarrow \Pi_i^*\mathcal{M} \twoheadrightarrow \Psi_f^{(i)}\mathcal{M}$$

in the pro-ind category induces an exact sequence

$$\frac{\Pi_i^!\mathcal{M}}{\Pi_{i+1}^!\mathcal{M}} \hookrightarrow \frac{\Pi_i^*\mathcal{M}}{\Pi_{i+1}^!\mathcal{M}} \twoheadrightarrow \Psi_f^{(i)}\mathcal{M}.$$

The first object is $j_!\mathcal{M}$ by the remark just above and the second object is $\Xi_f^{(i)}\mathcal{M}$ by definition; so this is an exact sequence of holonomic $\mathcal{D}_X$-modules. Part (iv) follows from this by duality, observing that the dual of the trivial diagram corresponding to a holonomic module is the trivial diagram corresponding to its dual.

Part (v) is more or less immediate from the fact that α_- is induced from $\alpha_i : \Pi_i^! \hookrightarrow \Pi_i^*$, while α_+ is just a projection. Likewise for part (vi), β_- is just a projection and we leave it to the reader to confirm that its dual is multiplication by s. (Check it first for projections $\mathcal{J}^{a,b} \to \mathcal{J}^{a,b-1}$ using the explicit description of the duality on these modules given in §3.3.2.) $\square$

Example 3.42. Continuing Example 3.38, let us compute $\Xi_t(\mathcal{O}_{\Bbbk^\times})$. The proof of Proposition 3.21 implies in this case that $\Xi_t(\mathcal{O}_{\Bbbk^\times}) = j_*\mathcal{J}^{0,N}/j_{!*}\mathcal{J}^{1,N}$. In this case the Goresky-MacPherson map $j_!\mathcal{J}^{1,N} \to \mathcal{J}_*^{1,N}$ (simply a version of the map for $\mathcal{J}^{0,N-1}$ shifted one degree in filtration by the operator σ) hits the portion of the filtration of $j_*\mathcal{J}^{0,N}$ indicated below:

$$\overbrace{\mathcal{O}, \delta_0, \cdots, \mathcal{O}}, \delta_0, \mathcal{O}, \delta_0.$$

The leftover quotient giving the maximal extension is filtered by $\delta_0, \mathcal{O}, \delta_0$. It is *not* supported at the origin. It has $j_*\mathcal{O}_{\Bbbk^\times}$ as a quotient. Moreover it is the *largest* extension of $j_*\mathcal{O}_{\Bbbk^\times}$ whose restriction to $\Bbbk^\times$ is $\mathcal{O}_{\Bbbk^\times}$, which explains the terminology; see [G, 4.6.20] for more on this point.

3.7 Vanishing cycles functor

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module, and write $\mathcal{M}_U = j^!\mathcal{M} = j^*\mathcal{M}$ for the restriction of $\mathcal{M}$ to U. Let $\gamma_- : j_!\mathcal{M}_U = j_!j^!\mathcal{M} \to \mathcal{M}$ and $\gamma_+ : \mathcal{M} \to j_*j^*\mathcal{M} = j_*\mathcal{M}_U$ be the adjunction morphisms. We can write down a diagram of $\mathcal{D}_X$-modules

$$j_!\mathcal{M}_U \overset{(\alpha_-,\gamma_-)}{\to} \Xi_f(\mathcal{M}_U) \oplus \mathcal{M} \overset{(\alpha_+,-\gamma_+)}{\to} j_*\mathcal{M}_U. \tag{3}$$

The following is clear.

Lemma 3.43. (α_-, γ_-) is injective, and $(\alpha_+, -\gamma_+)$ is surjective. $\square$

In fact (3) is a complex: by Theorem 3.41(v) the composition $\alpha_+\alpha_- = \alpha : j_!\mathcal{M}_U \to j_*\mathcal{M}_U$; by applying $j_!j^!$ to the adjunction morphism $\gamma_+ : \mathcal{M} \to j_*\mathcal{M}_U$ we obtain a commutative square

$$\begin{array}{ccc} j_!\mathcal{M}_U & \xrightarrow{\gamma_-} & \mathcal{M} \\ \Big\| {\scriptstyle j_!\widetilde{\alpha}} & & \Big\downarrow {\scriptstyle \gamma_+} \\ j_!j^!j_*\mathcal{M}_U & \xrightarrow[\gamma]{} & j_*\mathcal{M}_U \end{array}$$

and consequently $\gamma_+\gamma_- = \gamma \circ j_!\widetilde{\alpha} = \alpha$ as discussed in §3.3.3.

Definition 3.44. The functor of **vanishing cycles along** Y, $\Phi_f : \mathrm{Hol}(\mathcal{D}_X) \to \mathrm{Hol}(\mathcal{D}_Y)$, is defined as

$$\Phi_f(\mathcal{M}) = \frac{\ker(\alpha_+, -\gamma+)}{\mathrm{im}(\alpha_-, \gamma_-)}.$$

This time the basic facts about the functor are not quite as obvious.

Theorem 3.45. *The functor Φ_f has the following properties.*

(i) $\Phi_f(\mathcal{M})$ *is a holonomic* $\mathcal{D}_X$*-module supported on* Y*.*

(ii) Φ_f *is exact.*

(iii) Φ_f *commutes with duality.*

(iv) There exist canonical exact sequences

$$0 \to i^!\mathcal{M} \to \Phi_f(\mathcal{M}) \xrightarrow{v} \Psi_f(\mathcal{M}_U) \to \mathcal{H}^1 i^!\mathcal{M} \to 0$$

$$0 \to i^*\mathcal{M} \to \Psi_f(\mathcal{M}_U) \xrightarrow{u} \Phi_f(\mathcal{M}) \to \mathcal{H}^1 i^*\mathcal{M} \to 0$$

which are interchanged by duality.

(v) The compositions vu *and* uv *are nilpotent.*

Proof. (i) That $\Phi_f\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, at least, comes for free, as it is a subquotient of the holonomic module in the middle of (3). To analyze its support, we first produce the exact functor $j^!$ of restriction from $\varinjlim\!\!\!\!\!\varprojlim \mathrm{Hol}(\mathcal{D}_X)$ to $\lim_{\leftrightarrow} \mathrm{Hol}(\mathcal{D}_U)$ in the usual way. Since $j^!\Pi^!_k\mathcal{M}_U = j^!\Pi^*_k\mathcal{M}_U = \lim_{\leftrightarrow}\mathcal{M}_U \otimes \mathcal{J}^{a,b}_{f,k}$, the cokernel $j^!\Pi^{k,\ell}_{!*}\mathcal{M} = \mathcal{M}_U \otimes \mathcal{J}^{k,\ell}_f$ by Lemma 3.19. In particular $j^!\Xi_f\mathcal{M}_U \cong \mathcal{M}_U$. Therefore, applying $j^!$ to (3) we obtain

$$\mathcal{M}_U \to \mathcal{M}_U \oplus \mathcal{M}_U \to \mathcal{M}_U. \tag{4}$$

Both $j^!\gamma_-$ and $j^!\gamma_+$ are the identity on $\mathcal{M}_U$. Applying $j^!$ to the exact sequences of Theorem 3.41(iv) and observing that $j^!\mathcal{M}_f\mathcal{M}_U = 0$, we see $j^!\alpha_\pm$ is an isomorphism, which one can check is actually the identity. So the morphisms in (4) are $(\mathbf{1},\mathbf{1})$ and $(\mathbf{1},-\mathbf{1})$. The homology in the middle is trivial. So $j^!\Phi_f\mathcal{M} = 0$.

(ii) The functor which sends $\mathcal{M}$ to the diagram (3) for $\mathcal{M}$ is exact, as each term in the diagram is an exact functor of $\mathcal{M}$. So a short exact sequenc eof $\mathcal{D}_X$-modules $\mathcal{M}$ gives rise to a short exact sequence of diagrams (3). We can regard these as chain complexes over $\mathrm{Hol}(\mathcal{D}_X)$. It is well-known that a short exact sequence of chain complexes gives rise to a long exact sequence on their homology. But in this case, each diagram (3) has homology concentrated in a single degree (the middle term), so the "long exact sequence" is actually short exact.

(iii) The diagram (3) is self-dual.

(iv) Note that in these exact sequences we abusively write $\Psi_f\mathcal{M}_U$ (resp. $\Phi_f\mathcal{M}$) for $i^!\Psi_f\mathcal{M}_U$ (resp. $i^!\Phi_f\mathcal{M}$). The map u is defined by $u(\psi) = (\beta_+\psi, 0)$. This lands in $\ker(\alpha_+, -\gamma_+)$ because $\alpha_+\beta_+ = 0$ by Theorem 3.41(iv). The map v is defined by $v(\varphi) = v(\xi,\mu) = \beta_-\xi$. This is well-defined because $\beta_-\alpha_- = 0$ by the same theorem. It is not hard to check that u and v are interchanged by duality. We will not need the other assertions of (iv), so we omit their proofs.

(v) The composition $vu\psi = \beta_-\beta_+\psi = s\psi$ by Theorem 3.37(vi). The operator s on any $j_\sharp(\mathcal{M}_U \otimes \mathcal{J}^{a,b}_f)$ is nilpotent. By the proof of Proposition 3.21, s therefore acts nilpotently on each $\Pi^{k,\ell}_{!*}\mathcal{M}_U$, and in particular on $\Psi_f\mathcal{M}_U$. By duality it follows that uv is nilpotent as well. □

4 The gluing category

With all this machinery we can answer Question 1.1.

4.1 Definition of the gluing category

We remain in the same XYU setup as always.

Definition 4.1. The **gluing category** $\mathrm{Glue}(U, Y)$ has as objects collection of data $(\mathcal{M}_U, \mathcal{M}_Y, v, u)$ consisting of

- A holonomic $\mathcal{D}_U$-module $\mathcal{M}_U$,
- A holonomic $\mathcal{D}_Y$-module $\mathcal{M}_Y$,[7] and
- Morphisms $\Psi_f\mathcal{M}_U \xrightarrow{u} \mathcal{M}_Y \xrightarrow{v} \Psi_f\mathcal{M}_U$ such that $vu = \beta_-\beta_+$, where the $\beta_\pm$ denote the maps of Theorem 3.41(iv) for $\mathcal{M}_U$.

Definition 4.2. The **ungluing functor** $\mathrm{unglue} : \mathrm{Hol}(\mathcal{D}_X) \to \mathrm{Glue}(U, Y)$ is defined by

$$\mathrm{unglue}(\mathcal{M}) = (\mathcal{M}_U, \Phi_f\mathcal{M}, u, v)$$

using the notation of Theorem 3.45. The **gluing functor** $\mathrm{glue} : \mathrm{Glue}(U, Y) \to \mathrm{Hol}(\mathcal{D}_X)$ is defined by

$$\mathrm{glue}(\mathcal{M}_U, \mathcal{M}_Y, u, v) = \frac{\ker(\beta_-, -v)}{\mathrm{im}(\beta_+, u)}$$

with respect to the complex

$$\Psi_f(\mathcal{M}_U) \overset{(\beta_+,u)}{\hookrightarrow} \Xi_f(\mathcal{M}_U) \oplus \mathcal{M}_Y \overset{(\beta_-,-v)}{\twoheadrightarrow} \Psi_f(\mathcal{M}_U). \tag{5}$$

It is clear that gluing, much like ungluing, is exact – cf. the proof of Theorem 3.45(ii). The main theorem is

Theorem 4.3. *The functors* glue *and* unglue *are mutually quasi-inverse equivalences of categories.* □

The proof is discussed in §4.3.

This theorem gives an affirmative answer to Question 1.1: all the data about a $\mathcal{D}_X$-module are captured by its restriction to U and its vanishing cycles along Y, and conversely almost any pair $\mathcal{M}_U, \mathcal{M}_Y$ can be obtained in this way, subject merely to the condition that the morphisms u and v exist.

[7] For the purposes of this section, "$\mathcal{D}_Y$-module" means "$\mathcal{D}_X$-module supported on Y", which by Kashiwara's theorem is the same thing, and means we need not bother writing the $i^!$s.

4.2 Example

Here we return to the simple example from §1.2 ($X = \mathbb{A}^1, f = t$) and analyze what the Theorem 4.3 says in this case.

Corollary 4.4. The category $\mathrm{Hol}^0_{\mathrm{reg}}(\mathcal{D}_X)$ is equivalent to the category of diagrams

$$M \underset{u}{\overset{v}{\rightleftarrows}} N$$

of vector spaces, with $\mathbf{1}_M - uv$ and $\mathbf{1}_N - vu$ invertible.

Proof. As remarked in §1.3, Loc_U is equivalent to the category of vector spaces V equipped with a monodromy automorphism μ, while $\mathrm{Hol}(\mathcal{D}_Y)$ is just vector spaces. Let us analyze the nearby cycles functor from this perspective. Recall that for us Loc_U consists of local systems. Any such local system contains a maximal incomposable sub-object on which the monodromy acts unipotently, or equivalently, on which $\mathbf{1}_V - \mu$ acts nilpotently. Given (V, μ) representing an object of Loc_U, write V^0 for the corresponding maximal subspace of which $\mathbf{1} - \mu$ is nilpotent, and ν for the restriction of this operator of V^0.

The unique irreducible local system with unipotent monodromy is taken by the Riemann-Hilbert correspondence to the $\mathcal{D}_U$-module $\mathcal{O}_U = \mathcal{J}^{0,1}$. (Indeed, there is only one such object up to isomorphism, as can be seen by thinking about it as a representation of $\pi_1(U)$, and therefore it must be $\mathcal{O}_U$.) So any $\mathcal{M}_U \in \mathrm{Loc}_U$ has a submodule $\mathcal{M}^0_U$ (corresponding to V^0) with a filtration with irreducible subquotients isomorphic to $\mathcal{O}_U$. On the other hand, the irreducible local system on the punctured line with *non*unipotent monodromy are, in their manifestation as $\mathcal{D}$-modules, the ones commonly denoted "t^λ" for $\lambda \in \mathbb{C} - \mathbb{Z}$. Formally this is the quotient "t^s"$/(s - \lambda)$"t^s" $\in \mathrm{Hol}(\mathcal{D}_U)$. It is isomorphic to $\mathcal{O}_U \cdot t^\lambda$ where ∂ acts by $\partial(f t^\lambda) = (\partial f + t^{-1} f \lambda) t^\lambda$. The logarithm of monodromy $t\partial$ acts on the generator by λ, so the eigenvalue of the monodromy of the corresponding irreducible local system is $\exp(2\pi i \lambda)$. Consider the tensor product "t^λ" $\otimes \mathcal{J}^{0,n}$. By definition the b-function of this tensor product is $s + \frac{1}{\lambda}$. This has no integer roots by assumption. By the proofs of Theorem 3.31 and Corollary 2.8, it follows that $j_!(\text{“}t^\lambda\text{”} \otimes \mathcal{J}^{0,n}) \to j_*(\text{“}t^\lambda\text{”} \otimes \mathcal{J}^{0,n})$ is an isomorphism, and that $\Psi_t(\text{“}t^\lambda\text{”}) = 0$. So Ψ_t only captures the part of a $\mathcal{D}_U$-module with unipotent monodromy (which is why we call it the "unipotent part" of the nearby cycles functor).

Therefore $\Psi_t(\mathcal{M}_U) = \Psi_t(\mathcal{M}^0_U)$. For each irreducible subquotient $\mathcal{O}_U$ of $\mathcal{M}^0_U$ we get by Example 3.38 a copy of the $\mathcal{D}_X$-module δ_0. Applying the functor $i^!$ to get a vector space ($\mathcal{D}$-module on the origin), we find $\Psi_t(\mathcal{M}_U)$ is the vector space V^0. The nilpotent operator $\mathbf{1} - \mu$ is the logarithm of monodromy $t\partial = s = \beta_- \beta_+$ acting on the nearby cycles.

Hence the gluing category is equivalent to the category of linear algebra data

$$\{(V, \mu), W, V^0 \xrightarrow{u} W \xrightarrow{v} V^0\}$$

where V is a vector space, μ is an automorphism of V, W is another vector space, V^0 is the maximal subspace of V on which $\mathbf{1} - \mu$ is nilpotent, and u and v are linear maps such that $vu = \mathbf{1} - \mu$. By Theorem 4.3 (or rather the obvious version of it proved by replacing $\mathrm{Hol}(\mathcal{D}_U)$ by Loc_U and $\mathrm{Hol}(\mathcal{D}_X)$ by $\mathrm{Hol}^0_{\mathrm{reg}}(\mathcal{D}_X)$ everywhere[8]) this category is equivalent to $\mathrm{Hol}^0_{\mathrm{reg}}(\mathcal{D}_X)$. Note that this already gives a positive answer to Question 1.3.

[8] One can show (although it is not entirely obvious) that $\mathrm{Hol}^0_{\mathrm{reg}}(\mathcal{D}_X)$ is a full abelian subcategory of $\mathrm{Hol}(\mathcal{D}_X)$, so the proof does indeed go through. See [Ber, 4].

In fact we can further simplify the description of this linear algebraic category to yield the corollary. Indeed, the category above is equivalent to the category of diagrams of vector spaces $M \overset{v}{\to} N \overset{u}{\to} M$ (with $\mathbf{1} - uv$ and $\mathbf{1} - vu$ invertible) as in the statement of the corollary. The functor giving the equivalence takes the collection (M, N, u, v) to $\{(N, \mathbf{1} - vu), M^0, u, v\}$ where M^0 is the maximal subspace of M on which uv is nilpotent. Since this equivalence is purely an exercise in diagram chasing, we leave it to the reader. □

4.3 Proof of Theorem 4.3

Beilinson [B] proves this by reducing the gluing and ungluing functors, through a series of linear algeraic manipulations, to a mutually inverse pair of "reflection" functors on a category of *diads* (a certain type of diagram in any exact category). Here we take a more direct approach, constructing (one of) the natural isomorphisms which realize the equivalence explicitly. In contrast, Beilinson's method merely provides a neat *recipe* for producing such natural isomorphisms.[9]

Let us show that $\widetilde{\mathcal{M}} = \text{glue} \circ \text{unglue}(\mathcal{M})$ is naturally isomorphic to $\mathcal{M}$. This is a diagram chase.

The $\mathcal{D}_X$-module $\widetilde{\mathcal{M}}$ is the cohomology of the complex

$$\Psi_f \mathcal{M}_U \overset{(\beta_+, u)}{\hookrightarrow} \Xi_f \mathcal{M}_U \oplus \Phi_f \mathcal{M} \overset{(\beta_-, -v)}{\twoheadrightarrow} \Psi_f \mathcal{M}_U$$

in the notation of Theorem 3.45.

Recall that the first map sends

$$\Psi_f \mathcal{M}_U \ni \psi \mapsto \beta_+ \psi \oplus [\beta_+ \psi \oplus 0] \in \Xi_f \mathcal{M}_U \oplus \Phi_f \mathcal{M},$$

where the brackets denote the cohomology class of $\beta_+ \psi \oplus 0 \in \ker(\alpha_+, -\gamma_+)$; the second sends

$$\Xi_f \mathcal{M}_U \oplus \Phi_f \mathcal{M} \ni \xi \oplus \varphi \mapsto \beta_-(\xi - \xi')$$

where $\widetilde{\varphi} = \xi' + \mu'$ denotes a lift of φ to $\ker(\alpha_+, -\gamma_+) \subset \Xi_f \mathcal{M}_U \oplus \mathcal{M}$.

(By definition $\Phi_f \mathcal{M}$ is the cohomology of the complex (3) from §3.7, which we rewrite here for convenience:

$$j_! \mathcal{M}_U \overset{(\alpha_-, \gamma_-)}{\hookrightarrow} \Xi_f \mathcal{M}_U \oplus \mathcal{M} \overset{(\alpha_+, -\gamma_+)}{\twoheadrightarrow} j_* \mathcal{M}_U.)$$

Let $\widetilde{\mu} \in \widetilde{\mathcal{M}}$.

Lift $\widetilde{\mu}$ to $\xi \oplus \varphi \in \ker(\beta_-, -v) \subset \Xi_f \mathcal{M}_U \oplus \Phi_f \mathcal{M}$.

Lift φ to $\widetilde{\varphi} = \xi' \oplus \mu' \in \ker(\alpha_+, -\gamma_+) \subset \Xi_f \mathcal{M}_U \oplus \mathcal{M}$.

Define $\widetilde{\omega}(\widetilde{\mu}) = \mu'$. Note that $\xi \oplus \varphi$ is well-defined up to the addition of $\beta_+ \psi \oplus [\beta_+ \psi \oplus 0]$. In particular φ is well-defined modulo cohomology classes of the form $[\beta_+ \psi \oplus 0]$. In particular $\widetilde{\varphi}$ is well-defined modulo $\text{im}\, \beta_+ \oplus 0$. In particular $\widetilde{\omega}(\widetilde{\mu})$ is well-defined.

[9] Far from the simplest ones, though!.

There are commutative diagrams

$$\begin{array}{ccc} j_!\mathcal{M}_U & \xhookrightarrow{\alpha_-} & \Xi_f\mathcal{M}_U \\ {\scriptstyle -\gamma_-}\downarrow & & \downarrow{\scriptstyle \alpha_+} \\ \mathcal{M} & \xrightarrow[-\gamma_+]{} & j_*\mathcal{M}_U \end{array} \qquad \begin{array}{ccc} \Psi_f\mathcal{M}_U & \xhookrightarrow{\beta_+} & \Xi_f\mathcal{M}_U \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \beta_-} \\ \Phi_f\mathcal{M}_U & \xrightarrow[v]{} & \Psi_f\mathcal{M}_U \end{array}$$

Given $\mu \in \mathcal{M}$, choose $\xi \in \Xi_f\mathcal{M}_U$ such that $\alpha_+\xi = \gamma_+\mu$.
Then $\varphi = [\xi \oplus \mu]$ gives an element of $\Phi_f\mathcal{M}$.
Choose ξ' such that $\beta_-\xi' = v\varphi$.
Then $\widetilde{\mu} = [\xi' \oplus \varphi]$ gives an element of $\widetilde{\mathcal{M}}$.
Define $\omega(\mu) = \widetilde{\mu}$.

We leave it to the reader to verify for herself that ω and $\widetilde{\omega}$ realize well-defined, mutually-inverse, canonical isomorphisms between $\mathcal{M}$ and $\widetilde{\mathcal{M}}$.

We also leave it to the reader to carry out the analogous diagram chase to show that $\widetilde{G} = \text{unglue} \circ \text{glue}(G)$ is canonically isomorphic to G, for gluing data $G \in \text{Glue}(U, Y)$. (The difficult part of the latter is to identify $\Phi_f(\text{glue}(G))$ with the part $\mathcal{M}_Y$ of G.) □

5 Epilogue

We have proved everything we're going to prove. However, there are a number of topics related to the content of this thesis which would be good places to begin further studies in the subject.[10] In this concluding section we list a few of these topics as "Problems" – which is not to say that they are unsolved, although the solutions do not seem to be in the literature.

5.1 Remarks on perverse sheaves

There is another notion of vanishing cycles functors in the setting of sheaves for the analytic topology on a variety, say over $\mathbb{C}$. They have their origins in Goresky-MacPherson intersection cohomology, and the definition we discuss in this subsection is due to Deligne [SGA7]. In this subsection I will try to say a few words about these functors (following [Di]) and their relationship to the analogues defined above for $\mathcal{D}$-modules in the algebraic category.

We begin with the topological setup for the definition.

5.1.1 The local Milnor fiber

Let X be a smooth affine complex variety and let $f : X \to \mathbb{C}$ be a non-constant analytic function on X. To fix some notation, denote the fibre $f^{-1}(t)$ by X_t, and choose a point $x \in X_0$ and a ball B of radius δ about x. Let D_ϵ be a small disk around the origin in $\mathbb{C}$ (with $\epsilon \ll \delta$), $D_\epsilon^\times = D_\epsilon - \{0\}$, $\mathbb{T} = f^{-1}(D_\epsilon) \cap B$ (a small "tube" around $B \cap X_0$), and $\mathbb{T}^\times = \mathbb{T} - X_0$. We allow the hypersurface X_0 to be singular, but choose ϵ small enough so that for $t \in D_\epsilon^\times$,

[10]Indeed, given another couple of months to work on this thesis I would have endeavored to treat some or all of the following material more completely.

the fiber $X_t \cap B$ is smooth, and moreover $f : \mathbb{T}^{\times} \to D_{\epsilon}^{\times}$ is a locally trivial fibration. (Indeed, it turns out that this is always possible, whether in the algebraic or the analytic setting; for a nice explanation of this result, see the expository article [Se].) In particular, all the nearby fibers $B \cap X_t$ "look the same"; any one is known as the *local Milnor fiber* F_x of f at x. A picture of the situation can be seen in the figure below.

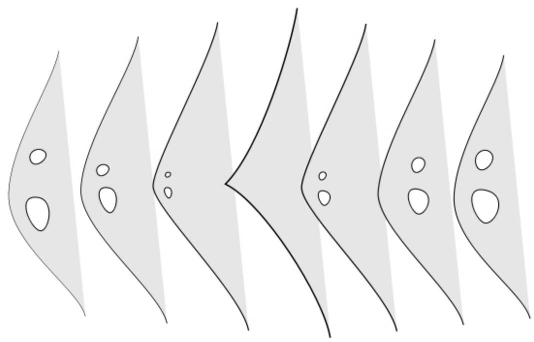


Figure 1: The Milnor fibration. (Picture from [Se].)

5.1.2 Nearby and vanishing cycle functors

A natural question to ask is how one can study the Milnor fiber F_x . The nearby and vanishing cycles functors associated to the function f on X provide tools for this purpose. We will briefly sketch the definition of functors ψ_f, φ_f on the bounded derived category $D_c^b(X)$ of constructible complexes of $\mathbb{C}_X$-modules. (We will gives the precise definition of this category here; see [Di]. Note, however, that for a morphism $\pi : Y \to X$ we obtain the usual inverse and direct image functors $\pi_!, \mathrm{R}\pi_* : \mathscr{C}_Y \to \mathscr{C}_X$ and $\pi^!, \pi^* : \mathscr{C}_X \to \mathscr{C}_Y$.)

We need names for some maps of spaces, which are indicated in the following diagram.

$$\begin{array}{ccccccc} F & \xrightarrow{\alpha} & \mathbb{T}^{\times} & \xhookrightarrow{j} & X & \xhookleftarrow{i} & X_0 \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle f} & & & & \\ \widetilde{D}_{\epsilon}^{\times} & \xrightarrow[\pi]{} & D_{\epsilon}^{\times} & & & & \end{array}$$

Here i and j are the natural inclusions, $\widetilde{D}_{\epsilon}^{\times} \to D_{\epsilon}^{\times}$ is the universal covering space, and the space F and the maps α and β are defined as the fiber product $\widetilde{D}_{\epsilon}^{\times} \underset{D_{\epsilon}^{\times}}{\times} \mathbb{T}^{\times}$. One can show that $F = F_x$ is the Milnor fiber.

With this notation, we can define the functor $\psi_f : D_c^b(X) \to D_c^b(X_0)$ of **nearby cycles**. Given a complex $\mathcal{F}^{\cdot} \in D_c^b(X)$ its nearby cycles are given by

$$\psi_f \mathcal{F}^{\cdot} = i^* \mathrm{R}(j\alpha)_*(j\alpha)^* \mathcal{F}^{\cdot} \in D_c^b(X_0).$$

The relationship between the functor ψ_f and the topology of the Milnor fiber can already be seen from the following fact, a direct computation from the definition.

Theorem 5.1. *[Di, Prop. 4.2.2] Consider the case $\mathcal{F}^\cdot = \mathbb{C}_X$. Then there is a canonical isomorphism*

$$\mathcal{H}^k(\psi_f \mathbb{C}_X)_x \cong \mathrm{H}^k(F_x; \mathbb{C})$$

between the stalk at x of the kth cohomology sheaf of the complex $\psi_f \mathbb{C}_X$, and the kth reduced singular cohomology group of the Milnor fiber F_x. [11] □

The functor $\mathrm{R}(j\alpha)_*$ is right adjoint to $(j\alpha)^*$, so the unit of the adjunction induces a **comparison morphism** $\mathrm{comp}_{\mathcal{F}^\cdot} : i^*\mathcal{F}^\cdot \to \psi_f \mathcal{F}^\cdot$. We define the functor of **vanishing cycles** $\varphi_f : D^b_c(X) \to D^b_c(X_0)$ by setting $\varphi_f \mathcal{F}^\cdot$ to be the mapping cone $\mathrm{Cone}(\mathrm{comp}_{\mathcal{F}^\cdot})$. The functors we have given naturally sit in a distinguished triangle in $D^b_c(X_0)$

$$i^*\mathcal{F}^\cdot \overset{\mathrm{comp}}{\to} \psi_f \mathcal{F}^\cdot \to \varphi_f \mathcal{F}^\cdot \overset{[1]}{\to} .$$

Corresponding to this triangle is a long exact sequence of cohomology sheaves. Setting $\mathcal{F}^\cdot = \mathbb{C}_X$ and using the isomorphism given in Theorem 5.1, the stalks at x of this long exact sequence are

$$\cdots \to \mathrm{H}^i(B \cap X_0; \mathbb{C}) \to \mathrm{H}^i(F_x; \mathbb{C}) \to \mathcal{H}(\varphi_f \mathbb{C}_X)_x \to \mathrm{H}^{i+1}(B \cap X_0; \mathbb{C}) \to \cdots$$

(reduced cohomology groups). The geometric significance of this sequence can be seen by considering its dual; we obtain

$$\cdots \to \mathrm{H}_{i+1}(B \cap X_0; \mathbb{C}) \to \mathcal{H}(\varphi_f \mathbb{C}_X)_x^\vee \to \mathrm{H}_i(F_x; \mathbb{C}) \to \mathrm{H}_i(B \cap X_0; \mathbb{C}) \to \cdots$$

The map from the homology of the local Milnor fiber to that of a neighborhood of x in the singular fiber is the "obvious one", induced by a continuous (not analytic) *specialization map* from a sufficiently nearby fiber X_t to the special fiber X_0 (cf. [GM, 6.2]). Thus the cohomology of the vanishing cycles (or its dual) may be interpretted as saying which cycles "die" in collapsing a nearby fiber to the singular one. This, of course, gives some justification for the terminology.

5.1.3 Monodromy

In this setting there is a *geometric monodromy action* of the fundamental group $\mathbb{Z} \approx \pi_1(D^\times_\epsilon, t)$ as diffeomorphisms of the Milnor fiber F_x. This induces a monodromy operation on the topological nearby and vanishing cycles functors; studying this action (e.g. by decomposing it into eigenspaces) provides a wealth of topological information about the singularity in question.

Suppose $\gamma : I = [0,1] \to D^\times_\epsilon$ is any path. On some neighborhood $U_s \subset D^\times_\epsilon$ of any point $\gamma(s)$, we can trivialize $\mathbb{T}^\times \to D^\times_\epsilon$. Then there is a diffeomorphism $\varphi : f^{-1}(U_s) \to U_s \times F_x$. Fix a point $\widetilde{t}_s \in X_{\gamma(s)} \cap B$ and let $(\gamma(s), y) = \varphi(\widetilde{t})$. Then we can define a lift $\widetilde{\gamma}$ of γ on some interval containing s by $\widetilde{\gamma}(s') = \varphi^{-1}(\gamma(s'), y)$. By patching together finitely many trivializations to cover the compact set $\gamma(I)$, we can lift the whole path γ. This induces a diffeomorphism $T_\gamma : X_{\gamma(0)} \cap B \to X_{\gamma(1)} \cap B$. In the case of a loop γ based at t which

[11] There are, of course, generalizations of Theorem 5.1 for any $\mathcal{F}^\cdot \in \mathscr{C}_X$.

generates $\pi_1(D_\epsilon^\times, t)$, we obtain a diffeomorphism of $X_t \cap B$ (and hence of F_x) by defining $T(\widetilde{t}) = \widetilde{\gamma}(1)$. One can prove that this map is well-defined up to homotopy, independent of the choice of representative path γ and the choices of trivializations used to determine the lift. Consequently, there is a well-defined monodromy *representation* on the level of the singular cohomology $H^\bullet(F_x, \mathbb{C})$ of F_x. In terms of the picture above, the idea is that as we "flow it around" the singularity x, a given cycle in singular homology might change.

The monodromy action T induces (see [Di]) a monodromy operator on the nearby and vanishing cycles.

5.1.4 Perverse sheaves and $\mathcal{D}$-modules

The triangulated category $D_c^b(X)$ contains an abelian subcategory of *perverse sheaves* discovered by Beilinson, Bernstein and Deligne. The surprising theorem is that the shifted functor of vanishing cycles $\varphi_f[-1]$ preserves this subcategory.

A consequence of this is that for regular holonomic $\mathcal{D}$-modules, which by the Riemann-Hilbert correspondence [Ber, 5.9] form a category equivalent to that of perverse sheaves via the DeRham functor DR, one has in addition to the functors Φ_f we consturcted also Deligne's functors $\varphi_f[-1]$.

Problem 5.2. Prove the following comparison theorem, and its generalization to nonunipotent versions of the Beilinson functors Ψ_f, Φ_f, Ξ_f (replace $\mathcal{J}^{0,1}$ in the main construction of this thesis by an irreducible local system of eigenvalue $\alpha \notin \mathbb{Z}$; cf. [B, 3.2]).

Theorem 5.3. *For an object $\mathcal{F}$ in the category of perverse sheaves let $\varphi_f[-1]^{unip}\mathcal{F}$ denote the part of the vanishing cycles on which the monodromy operator T acts unipotently. Then for a regular holonomic $\mathcal{D}$-module $\mathcal{M}$ one has $DR\Phi_f\mathcal{M} = \varphi_f[-1]^{unip}DR\mathcal{M}$.*

5.2 Kashiwara-Malgrange construction

Kashiwara and Malgrange proved a version of Theorem 5.3 for an alternative construction of vanishing cycles for regular holonomic $\mathcal{D}$-modules.

Theorem 5.4. *Let X be smooth and let Y be a hypersurface $f = 0$. The sheaf $\mathcal{D}_X$ has a well-defined exahaustive decreasing $\mathbb{Z}$-indexed filtration $V^\bullet\mathcal{D}_X$ defined in the case where f is smooth as follows. Choose local coordinates $x_1, \ldots, x_{n-1}, f$ on X. Define $V^j\mathcal{D}_X = \sum_{k-\ell\geq j} h_{\alpha,k,\ell}(x)\partial_x^\alpha f^k \partial_f^\ell$ for a multi-index α. There is an analogue of this filtration when f is singular as well.*

Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module. There exists a unique decreasing $\mathbb{Z}$-indexed filtration $V^\bullet\mathcal{M}$ with the following properties:

- $V^i\mathcal{D}_X \cdot V^j\mathcal{M} \subset V^{i+j}\mathcal{M}$ *for all* $i \in \mathbb{Z}$.
- $V^{-n}\mathcal{D}_X \cdot V^{-\ell}\mathcal{M} = V^{-n-\ell}\mathcal{M}$ *and* $V^n\mathcal{D}_X \cdot V^\ell\mathcal{M} = V^{n+\ell}$ *for* $n \geq 0$ *and* $\ell \gg 0$.
- *The eigenvalues of* $f\partial_f$ *acting on* $\mathrm{gr}_V^0\mathcal{M}$ *have real part in* $[0, 1)$. □

We refer the reader to [Sa] for details about this *Kashiwara-Malgrange filtration*, the proof of the theorem, and for the construction of nearby and vanishing cycles functors Ψ_f^{KM} and Φ_f^{KM} in terms of it.

Problem 5.5. Prove a comparison theorem between the Beilinson functors Ψ_f, Φ_f and (the unipotent part of) the Kashiwara-Malgrange functors Ψ_f^{KM}, Φ_f^{KM}.

5.3 Vanishing cycles commute with proper direct image

A property of the functors Ψ_f and Φ_f which is important for applications is their behavior under proper direct image. (Proper direct image is very well-behaved for $\mathcal{D}$-modules: if $h : Z \to X$ is proper then $h_!, h_* : D^b_{\mathrm{coh}}(\mathcal{D}_Z) \to D^b_{\mathrm{coh}}(\mathcal{D}_X)$ coincide [Ber, 3.10].)

Problem 5.6. Prove the following theorem, the analogue of which for the functors Ψ_f^{KM}, Φ_f^{KM} (see §5.2) can be found in [LM].

Theorem 5.7. *Let X, Y, U, f be as before. Let $h : Z \to X$ be a proper morphism. Set $W = Y \underset{X}{\times} Z = (fh)^{-1}(0)$.*

$$\begin{array}{ccccc} W & \overset{\tilde{i}}{\hookrightarrow} & Z & \underset{\tilde{j}}{\hookleftarrow} & V \\ \downarrow & & \downarrow{\scriptstyle h} & & \downarrow \\ Y & \underset{i}{\hookrightarrow} & X & \underset{j}{\hookleftarrow} & U \end{array}$$

Then nearby cycles commute with proper direct image: $h_* \Psi_{fh} \mathcal{M}_Z = \Psi_f h_* \mathcal{M}_Z$, and similarly for vanishing cycles.

A Some category theoretic background

In Section 3 we need some general constructions from category theory.

A.1 Localization of categories

Given a category $\mathscr{C}$ and a nice class of morphisms Σ in $\mathscr{C}$ there is "universal" category $\Sigma^{-1}\mathscr{C}$, equipped with a functor $Q_\Sigma : \mathscr{C} \to \Sigma^{-1}\mathscr{C}$ called the **localization** of $\mathscr{C}$ with respect to Σ, such that for all $\varphi \in \Sigma$ the image $Q_\sigma \varphi$ is an isomorphism.

Definition A.1. $\Sigma \subset \text{Morphisms}(\mathscr{C})$ is **multiplicative** if the following properties hold.

(1) Σ contains all isomorphisms;

(2) Σ is closed under compositions;

(3) For two morphisms $Y \overset{f}{\leftarrow} X \overset{s}{\to} X'$ with $s \in \Sigma$, there exist $Y' \in \text{Objects}(\mathscr{C}), t \in \Sigma \cap \text{Hom}_\mathscr{C}(Y, Y'), g \in \text{Hom}_\mathscr{C}(X', Y')$ such that $gs = tf$; dually, for two morphisms $X \overset{f}{\to} Y \overset{t}{\leftarrow} Y'$ there exists $X' \in \text{Objects}(\mathscr{C}), s \in \Sigma \cap \text{Hom}_\mathscr{C}(X', X), g \in \text{Hom}_\mathscr{C}(X', Y')$ with $tg = fs$;

(4) For two morphisms $f, g \in \text{Hom}_\mathscr{C}(X, Y)$, if there exists $s : W \to X$ in Σ satisfying $fs = gs$, then there exists $t : Y \to Z$ in Σ satisfying $tf = tg$; dually, for two morphisms $f, g \in \text{Hom}_\mathscr{C}(X, Y)$, if there exists $t : Y \to Z$ in Σ satisfying $tf = tg$ then there exists $s : W \to X$ in Σ satisfying $fs = gs$.

Proposition A.2. If Σ is multiplicative, then a localization $Q_\Sigma : \mathscr{C} \to \Sigma^{-1}\mathscr{C}$ exists. If $\mathscr{C}$ is additive then so is $\Sigma^{-1}\mathscr{C}$, and the functor Q_Σ is additive. If $\mathscr{C}$ admits kernels (resp. cokernels, finite products, finite coproducts) then so does $\Sigma^{-1}\mathscr{C}$, and Q_Σ commutes with all these operations.

Proof. We omit the proof; see [KS, Prop. 7.1.22, Ex. 8.4]. □

One way of describing the localization is as follows. Objects in $\Sigma^{-1}\mathscr{C}$ are just objects in $\mathscr{C}$. Morphisms in $\Sigma^{-1}\mathscr{C}$ are defined in the following manner. Construct an directed graph Γ with vertices $\text{Objects}(\mathscr{C})$ and edges $\text{Morphisms}(\mathscr{C}) \cup \{\bar{s} \mid s \in \Sigma\}$, where $\bar{s}$ is an edge that reverses s. Call two paths in Γ from A to B *equivalent* if one can be obtained from the other via a sequence of replacements of the forms: (i) replace $X \overset{f}{\to} Y \overset{g}{\to} Z$ by $X \overset{gf}{\to} Z$ for any composable f, g; or (ii) replace $X \overset{s}{\to} Y \overset{\bar{s}}{\to} X$ by $X \overset{\mathbf{1}}{\to} X$ for any $s \in \Sigma$. Define $\text{Hom}_{\Sigma^{-1}\mathscr{C}}(A, B)$ to be equivalence classes of paths from A to B in Γ.

The canonical functor Q_Σ takes objects to themselves and morphisms to the corresponding equivalence class.

A.2 Exact categories

An additive category $\mathscr{C}$ is called **exact** is if is equipped with a set $\mathcal{E}$ of sequences $A \to B \to C$ of objects and morphisms in $\mathscr{C}$ satisfying the following axioms. We call a morphism *admissible monic* if it occurs as the first arrow in a sequence in $\mathcal{E}$. We call a morphism *admissible epic* if it occurs as the second arrow in a sequence in $\mathcal{E}$.

(0) An admissible monic (resp. epic) is a kernel (resp. cokernel) of any corresponding admissible epic (resp. monic).

(1) Any identify morphism is an admissible monic and an admissible epic.

(2) Admissible monics and admissible epics are both closed under composition.

(3) Arbitrary pushouts (resp. pullbacks) of admissible monics (resp. epics) are admissible monics (resp. epics).

We will need to use the following fact.

Proposition A.3. Let $(\mathscr{C}, \mathcal{E})$ be an exact category, and let $\Sigma^{-1}\mathscr{C}$ be a localization with respect to a multiplicative system of $\mathscr{C}$-morphisms Σ. Define an exact structure on $\Sigma^{-1}\mathscr{C}$ by taking the collection $Q_\Sigma\mathcal{E} = \{Q_\Sigma(A \to B \to C) \mid (A \to B \to C) \in \mathcal{E}\}$ to define the admissible morphisms. The resulting category is exact.

Proof. Follows immediately from Proposition A.2 and the definition above. □

The next criterion will also be useful.

Proposition A.4. A full subcategory $\mathscr{C}$ of an abelian category $\mathscr{A}$ is exact if it is closed under extensions, with respect to the class $\mathcal{E}$ of short exact sequences in $\mathscr{A}$ involving objects in $\mathscr{C}$. □

The "point" of exact categories, at least as they are used in this thesis, is that one has a good notion of exact functors between them. (Namely, those which preserve the distinguished class $\mathcal{E}$ of short sequences.) In particular, given a diagram of exact functors $\mathscr{A} \to \mathscr{C} \to \mathscr{B}$ with $\mathscr{A}$ and $\mathscr{B}$ abelian and $\mathscr{C}$ exact, the composition $\mathscr{A} \to \mathscr{B}$ is exact in the usual sense.

B Basics of $\mathcal{D}$-modules

In this appendix we give a brief overview, with examples, of the essentials of $\mathcal{D}$-modules. A more comprehensive, but not very detailed, treatment can be found in [Ber]; other references include [G] and [HTT].

B.1 Differential operators and $\mathcal{D}$-modules

Fix an algebraically closed field $\Bbbk$ of characteristic 0. Let X be a smooth algebraic variety over $\Bbbk$ with structure sheaf $\mathcal{O}_X$. Consider the sheaf $\mathcal{E}nd_{\Bbbk}(\mathcal{O}_X)$ of endomorphisms of $\mathcal{O}_X$ viewed as a sheaf of $\Bbbk$-vector spaces. We regard $\mathcal{O}_X$ as a subsheaf (of rings) of $\mathcal{E}nd_{\Bbbk}(\mathcal{O}_X)$ via its action of left multiplication. Another subsheaf of $\mathcal{E}nd_{\Bbbk}(\mathcal{O}_X)$ is the tangent sheaf Θ_X of $\Bbbk$-derivations of $\mathcal{O}_X$.

Definition B.1. The sheaf $\mathcal{D}_X$ of **differential operators** on X is the $\mathcal{O}_X$-subalgebra of $\mathcal{E}nd_{\Bbbk}(\mathcal{O}_X)$ generated by Θ_X.

Proposition B.2. $\mathcal{D}_X$ is a locally-free $\mathcal{O}_X$-module. (In particular, it is quasi-coherent.)

Proof. We work in a sufficiently small affine neighborhood U of a point $p \in X$. Since X is smooth, we can choose regular functions (local *analytic* coordinates) $x_1, \dots, x_n$ near p, so that the differentials $\mathrm{d}\, x_i$ are a basis for the free $\mathcal{O}_U$-module Ω^1_U of Kähler differentials, and we can consider the dual basis $\partial_i = \frac{\partial}{\partial x_i}$ for Θ_U. Then by definition

$$\mathcal{D}_X(U) = \mathcal{D}_U = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial^\alpha,$$

where we have used multi-index notation, $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. In the sequel we will refer to x_is and ∂_is chosen as above as a **local coordinate system**. □

The global sections $D_X = \Gamma(X, \mathcal{D}_X)$ are called the **Weyl algebra** when $X = \mathbb{A}^n$ is affine space, and is traditionally denoted by A_n. This is the polynomial algebra in the coordinates t_i on $\mathbb{A}^n$ and the corresponding partial derivatives $\partial_i = \frac{\partial}{\partial t_i}$, which satisfy the commutation relations $[t_i, t_j] = [\partial_i, \partial_j] = 0, [t_i, \partial_j] = \delta_{ij}$.

It is not difficult to show that $\mathcal{D}_X$ can also be characterized inductively in a coordinate-free manner, using the notion of the *order* of a differential operator.

Definition B.3. The sheaf of differential operators of **order** $\leq k$ on X is defined inductively as

$$\mathcal{D}_X^{\leq -1} = 0, \qquad \mathcal{D}_X^{\leq k} = \{d \in \mathcal{E}nd_{\Bbbk}(\mathcal{O}_X) \mid [d, f] \in \mathcal{D}_X^{\leq k-1}, \text{ for all} f \in \mathcal{O}_X\} \quad (k \geq 0).$$

It is a simple exercise to prove that $\mathcal{D}_X$ has an exhaustive filtration by the $\mathcal{D}_X^{\leq k}$, and that this filtration makes $\mathcal{D}_X$ into a sheaf of (non-commutative) filtered $\mathcal{O}_X$-algebras (i.e. $\mathcal{D}_X^{\leq k}\mathcal{D}_X^{\leq \ell} \subset \mathcal{D}_X^{\leq k+\ell}$). By the proof of Proposition B.2, the sheaf of associated graded rings is locally generated as an $\mathcal{O}_X$-algebra by the symbols ξ_i of the tangent vectors ∂_i. The ξ_i are coordinate functions on the cotangent space of X, which yields a canonical identification

$$\operatorname{gr} \mathcal{D}_X \cong \pi.\mathcal{O}_{\mathrm{T}^*X},$$

where $\pi : \mathrm{T}^*X \to X$ is the cotangent bundle. We will return to the order filtration below.

Definition B.4. A $\mathcal{D}_X$**-module** is a quasicoherent sheaf of $\mathcal{O}_X$-modules with a left action of $\mathcal{D}_X$ respecting the $\mathcal{D}_X \supset \mathcal{O}_X$-module structure.

Remark B.5. Since $\mathcal{D}_X$ is generated by $\mathcal{O}_X$ and Θ_X and their sections satisfy $[\xi, f] = \xi(f)$, it is not difficult to check that a $\mathcal{D}_X$-module structure on a quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ is equivalent to a $\Bbbk$-linear map $\nabla : \Theta_X \to \mathcal{E}nd_{\Bbbk}(\mathcal{M})$ satisfying $\nabla_{f\xi} = f\nabla_\xi$, $\nabla_\xi f = \xi(f) + f\nabla_\xi$ and $\nabla_{[\xi,\xi']} = [\nabla_\xi, \nabla'_\xi]$. (The equivalence is given by setting $\xi \cdot m = \nabla_\xi(m)$.) So if $\mathcal{M}$ is locally $\mathcal{O}$-free of finite rank, and thus corresponds to an algebraic vector bundle $E \to X$, a $\mathcal{D}_X$-module structure on $\mathcal{M}$ is equivalent to the datum of a flat (a.k.a. integrable) connection on E. For this reason, we shall refer to $\mathcal{D}$-modules which are locally $\mathcal{O}$-free of finite rank as **integrable connections**. One can show that this condition not only implies, but is *equivalent* to, the condition on a $\mathcal{D}$-module of being $\mathcal{O}$-coherent (see, e.g., [G, Prop. 3.5.1]).

Example B.6. It is useful to have a few simple of examples of $\mathcal{D}$-modules in mind. Obviously $\mathcal{D}_X$ itself is a $\mathcal{D}_X$-module, as is $\mathcal{O}_X$, and more generally any sheaf of smooth functions, such as the sheaf of complex analytic functions on X in the case where $\Bbbk = \mathbb{C}$.

When $X = \mathbb{A}^1$, we can describe $\mathcal{D}_X$-modules (= modules over the Weyl algebra $A_1 = D_{\mathbb{A}^1} = \Bbbk\langle t, \partial\rangle$) very explicitly; here we give several key examples.

- Corresponding to $\mathcal{O}_X$ we have the polynomial ring $\Bbbk[t]$. It is a simple exercise to deduce from the commutation relations in the Weyl algebra that $\Bbbk[t] \approx A_1/A_1\partial$ as left A_1-modules.
- $\Bbbk[t]$ is an A_1-submodule of $\Bbbk[t, t^{-1}]$, which is itself generated as an A_1-module t^{-1}, satisfying $\partial t \cdot t^{-1} = 0$ (since the successive derivatives of t^{-1} give all the negative powers of t). It follows that $\Bbbk[t, t^{-1}] = A_1 \cdot t^{-1} \approx A_1/A_1\partial t$ as A_1-modules.
- With respect to the preceding identification the submodule $\Bbbk[t] = A_1 t \cdot t^{-1} \approx A_1 t/A_1\partial t$. So the quotient module $\Bbbk[t, t^{-1}]/\Bbbk[t]$, which we denote by δ_0, is isomorphic to $A_1/A_1 t$. This is called the δ**-function** module, because the generator 1 of this quotient, which we might suggestively denote by δ, satisfies $t\delta = 0$, much like the Dirac δ "function" from calculus.

A general recipe for producing $\mathcal{D}_X$-modules, which provides much motivation for the theory, is to make them from (systems of) linear partial differential equations on X with polynomial coefficients. Indeed, in the affine case, such a system of p equations in q unknown functions is given by a collection $P_{ij} \in D_X$, for $1 \le i \le p, 1 \le j \le q$, corresponding to the system of equations

$$\sum_{j=1}^{q} P_{ij} f_j = 0, \qquad (i = 1, \ldots, p). \tag{S}$$

Corresponding to this equation we can consider the $\mathcal{D}_X$-module

$$\mathcal{M}_S = \frac{\bigoplus_{j=1}^{q} \mathcal{D}_X \cdot e_j}{\sum_{i=1}^{p} \mathcal{D}_X \left(\sum_{j=1}^{q} P_{ij} e_j \right)}.$$

The existence of a *solution* to the given system of PDEs by sections of a sheaf of functions $\mathcal{F}$ (e.g. analytic functions $\mathcal{O}_X^{\mathrm{an}}$) with a $\mathcal{D}_X$-module structure (e.g. the obvious one on $\mathcal{O}_X^{\mathrm{an}}$)

is equivalent to the existence of a morphism of $\mathcal{D}_X$-modules $\mathcal{M}_S \to \mathcal{F}$. In some sense it is reasonable to equate a function defined by a differential equation with the corresponding $\mathcal{D}$-module.

For a concrete example of this with $X = \mathbb{A}^1$, consider the differential equation

$$\frac{\partial}{\partial t} t \frac{\partial}{\partial t} f = 0, \quad \text{or equivalently} \quad \{t\frac{\partial}{\partial t} f_1 = 0, \quad t\frac{\partial}{\partial t} f_2 = f_1\}.$$

The local solutions $(f_1, f_2) = c(1, \log(t))$ to this equation in the sheaf of analytic functions on the complex plane are given up to scale by branches of the logarithm. So we abusively denote by log the generator of the A_1-module

$$A_1 \cdot \log = A_1/A_1\partial t\partial \approx (A_1 e_1 \oplus A_1 e_2)/(A_1 t\partial e_1 + A_1(t\partial e_2 - e_1))$$

(or the corresponding $\mathcal{D}_X$-module). Observe that the action of $t\partial$ on the $\mathcal{D}$-module $A_1 \cdot \log$ is given (in terms of the generators e_1, e_2) by the nilpotent matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. The exponential $\exp(2\pi i t\partial) = \left(\begin{smallmatrix} 1 & 2\pi i \\ 0 & 1 \end{smallmatrix}\right)$ gives the (unipotent) monodromy action on the fundamental solution $(1, \log)$ to the differential equation.

B.2 Left and right $\mathcal{D}$-modules

In addition to left $\mathcal{D}$-modules, we can consider right $\mathcal{D}$-modules. The most important example of a right $\mathcal{D}_X$-module is the sheaf ω_X of top-degree differential forms on X, which has a right action by vector fields given by $\omega \cdot \xi = -\mathrm{Lie}_\xi \omega$, where

$$\mathrm{Lie}_\xi \omega(\xi_1, \ldots, \xi_n) = \xi \cdot \omega(\xi_1, \ldots, \xi_n) - \sum \omega(\xi_1, \ldots, [\xi, \xi_i], \ldots, \xi_n).$$

This extends, the reader can check, to a right $\mathcal{D}_X$-module structure as claimed. Also, the construction can be relativized to give a right $\mathcal{D}_Y$-module structure on $\omega_{Y/X}$ for a morphism $Y \to X$.

This construction yields a functor $\omega_X \underset{\mathcal{O}_X}{\otimes} \bullet$ from left $\mathcal{D}_X$-modules to right $\mathcal{D}_X$-modules, where the module structure on the tensor product $\omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{F}$ is given by

$$(\omega \otimes u)\xi = \omega \cdot \xi \otimes u - \omega \otimes \xi \cdot u.$$

Proposition B.7. [Ber, 1.4] This functor gives an equivalence of categories between left and and right $\mathcal{D}_X$-modules; the inverse functor is $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \bullet)$, where the left $\mathcal{D}_X$-module structure on this $\mathcal{H}om$ is given by $(\xi \cdot \psi)(\omega) = -\psi(\omega) \cdot \xi + \psi(\omega \cdot \xi)$. □

These *side changing functors* allow one to pass freely from left to right $\mathcal{D}$-modules.

B.3 Direct and inverse image functors

B.3.1 Definitions

Given a morphism $\alpha : Y \to X$ one can define a naïve inverse image functor $\alpha^\Delta : \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathcal{D}_Y)$, coinciding with the inverse image $a^{\cdot} = \mathcal{O}_Y \underset{\alpha^{-1}\mathcal{O}_X}{\otimes} \alpha^{-1}(\bullet)$ for $\mathcal{O}$-modules equipped with the $\mathcal{D}$-action defined in a local coordinate system by

$$\xi(f \otimes m) = \xi f \otimes m + \sum f\xi(x_i) \otimes \partial_i m.$$

It is more convenient to work in the derived category of bounded complexes of $\mathcal{D}$-modules, $D^b(\mathcal{D}_X)$, and to add a homological shift by the codimension of Y relative to X. So we define the **inverse image** functor $D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y)$ by

$$\alpha^! = \mathrm{L}\alpha^\Delta[\dim Y - \dim X].$$

The definition of direct images is more involved; it is easier to define them first for right $\mathcal{D}$-modules and then use the side changing functors to obtain the definition for left $\mathcal{D}$-modules. The **direct image** functor $j_* : D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X)$ is defined in terms of *transfer bimodules*. If we set

$$\mathcal{D}_{Y\to X} = \alpha^\Delta(\mathcal{D}_X),$$

a $(\mathcal{D}_Y, \alpha^{-1}\mathcal{D}_X)$-bimodule, then a naïve guess for transfering a *right* $\mathcal{D}_Y$-module structure on $\mathcal{G}$ to X is to take the sheaf $\alpha.(\mathcal{G} \underset{\mathcal{D}_Y}{\otimes} \mathcal{D}_{Y\to X})$. Using the side changing functors, one sets

$$\mathcal{D}_{X\leftarrow Y} = \mathcal{D}_{Y\to X} \underset{\mathcal{O}_Y}{\otimes} \omega_{Y/X},$$

a $(\alpha^{-1}\mathcal{D}_X, \mathcal{D}_Y)$-bimodule, and considers the functor $\alpha.(\mathcal{D}_{X\leftarrow Y} \underset{\mathcal{D}_Y}{\otimes} \bullet)$. This proves ill-behaved, so one passes to the derived categories as above and defines the direct image as

$$\alpha_*\mathcal{G}^{\cdot} = \mathrm{R}\alpha.(\mathcal{D}_{X\leftarrow Y} \overset{\mathrm{L}}{\underset{\mathcal{D}_y}{\otimes}} \mathcal{G}^{\cdot})$$

for a complex of $\mathcal{D}_Y$-modules $\mathcal{G}^{\cdot}$. With these slightly odd-looking definitions, certain nice properties hold, such the facts that $\alpha^!\beta^! = (\beta\alpha)^!$ and $\alpha_*\beta_* = (\alpha\beta)_*$, and that $\mathcal{H}^0\alpha_*$ is left adjoint to $\mathcal{H}^0\alpha^!$ when α is a closed embedding.

Example B.8. For an example computation, consider an open embedding $U \overset{j}{\hookrightarrow} X$. In this case the $\mathcal{O}$-module inverse image j^Δ is simply the exact functor j^{-1} of restriction, so the same is true of $j^!$. We will sometimes write $\mathcal{F}|_U$ for $j^!\mathcal{F}$ in this case.

What about the direct image j_*? The transfer bimodules are

$$\mathcal{D}_{U\to X} = \mathcal{O}_U \underset{j^{-1}\mathcal{O}_X}{\otimes} j^{-1}\mathcal{D}_X = \mathcal{D}_X|_U = \mathcal{D}_U$$

, which is a two-sided $\mathcal{D}_U$-module, and $\mathcal{D}_{X\leftarrow U} = \mathcal{D}_U \underset{\mathcal{O}_U}{\otimes} \omega_{U/X}$. Since $j^{-1}\omega_X^\vee \cong \omega_U^\vee$ we have $\omega_{U/X} \cong \mathcal{O}_U$; thus $\mathcal{D}_{X\leftarrow U} = \mathcal{D}_U$ as well. So the direct image functor is

$$j_* = \mathrm{R}j.(\mathcal{D}_U \overset{\mathrm{L}}{\underset{\mathcal{D}_U}{\otimes}} \bullet) = \mathrm{R}j.,$$

where $j.$ is the ordinary direct image for $\mathcal{O}$-modules. For an open *affine* embedding $j.$ is exact, so $j_* = j.$. For example, when $\mathbb{A}^1 - \{0\} = U \overset{j}{\hookrightarrow} X = \mathbb{A}^1$ we have $j_*\mathcal{O}_U = j.\mathcal{O}_U$. So the corresponding A_1-module is $\Gamma(X, j.\mathcal{O}_U) = \Gamma(U, \mathcal{O}_U) = \Bbbk[t, t^{-1}]$. That is, if we regard $\Bbbk[t, t^{-1}]$ as a D_U-module and take its direct image, we obtain the same abelian group regarded as an A_1-module, in this case.

Now let $\{0\} = Y \overset{i}{\hookrightarrow} X = \mathbb{A}^1$ be the complementary closed embedding. Note that $\mathcal{D}_Y = \mathbb{k}$, so $\mathcal{D}_Y$-modules are just vector spaces. The direct image $i_*\mathcal{O}_Y = \mathrm{R}i_.(\mathcal{D}_{X\leftarrow Y} \overset{\mathrm{L}}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{O}_Y)$. Since $\mathcal{O}_Y = \mathcal{D}_Y = \mathbb{k}$ and since the sheaf-theoretic direct image $i_.$ is exact for the closed embedding i, this is just $i_.(\mathcal{D}_{X\leftarrow Y}) = i_.(\mathcal{D}_{Y\to X} \underset{\mathcal{O}_Y}{\otimes} \omega_{Y/X}) = i_.(i^{-1}\mathcal{D}_X \underset{i^{-1}\mathcal{O}_X}{\otimes} \omega_Y)$. The global sections of this sheaf are

$$\mathcal{D}_{X,0} \underset{\mathcal{O}_{X,0}}{\otimes} \mathcal{O}_Y = A_1 \underset{\mathbb{k}}{\otimes} \mathcal{O}_{X,0} \underset{\mathcal{O}_{X,0}}{\otimes} \mathbb{k} = A_1 \underset{\mathbb{k}[t]}{\otimes} \mathbb{k}$$

where $\mathbb{k}$ is functions on the point Y, viewed as an $\mathbb{k}[t]$-module with trivial action of t. Since $\mathbb{k} \cong \mathbb{k}[t]/t\mathbb{k}[t]$ as $\mathbb{k}[t]$-modules, we find $\Gamma(X, i_*\mathcal{O}_Y) = A_1/A_1t$, so $i_*\mathcal{O}_Y$ is the δ-function module δ_0 defined earlier.

B.3.2 Kashiwara's theorem

One of the most important facts about direct images concerns the special case of a closed embedding (such as $Y \overset{i}{\hookrightarrow} X$ in the main situation studied in the body of this thesis). For a closed embedding $\alpha : Y \to X$, the direct image has no higher cohomology; the functor $\mathcal{H}^0\alpha_*$ is exact.

Theorem B.9 (Kashiwara)**.** *The functor* $\mathcal{H}^0\alpha_*$ *induces an equivalence of categories* $\mathrm{Coh}(\mathcal{D}_Y) \to \mathrm{Coh}_Y(\mathcal{D}_X)$ *between coherent* $\mathcal{D}_Y$*-modules and the full subcategory of coherent* $\mathcal{D}_X$*-modules supported on* Y*. The inverse functor is* $\mathcal{H}^0\alpha^!(= \alpha^\Delta[\dim Y - \dim X])$. □

A proof can be found in [Ber, 1.10].

B.3.3 Tensor products

Given $\mathcal{D}_X$ and $\mathcal{D}_Y$ -modules $\mathcal{F}$ and $\mathcal{G}$, one can form their **external tensor product** $\mathcal{F}\boxtimes\mathcal{G} \in \mathrm{Mod}(\mathcal{D}_{X\times Y})$ using the projection maps $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$. Set

$$\mathcal{F} \boxtimes \mathcal{G} = \mathcal{D}_{X\times Y} \underset{p_X^{-1}\mathcal{D}_X \otimes_{\mathbb{k}} p_Y^{-1}\mathcal{D}_Y}{\otimes} (p_X^{-1}\mathcal{F} \underset{\mathbb{k}}{\otimes} p_Y^{-1}\mathcal{G}).$$

Given two $\mathcal{D}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ one can also define the **internal tensor product** $\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{G}$ as a $\mathcal{D}_X$-module, using the Leibnitz rule: the action of vector fields given by

$$\xi(f \otimes g) = \xi(f) \otimes g + f \otimes \xi(g)$$

defines the $\mathcal{D}$-module structure. Of course when working with *complexes* of $\mathcal{D}_X$-modules (i.e., in the derived category) one must derive the tensor product. It is not hard to show that there is a canonical isomorphism $\mathcal{F}^{\cdot} \overset{\mathrm{L}}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{G}^{\cdot} \cong \Delta^!(\mathcal{F}^{\cdot} \boxtimes \mathcal{G}^{\cdot})$, where $\Delta : X \to X \times X$ is the diagonal map. We will use this isomorphism below when sketching the proof of a lemma used in the body of the thesis.

B.4 Duals

A crucial tool in the theory of quasicoherent $\mathcal{O}$-modules is the notion of duality; any $\mathcal{O}_X$-module $\mathcal{M}$ has a dual $\mathcal{M}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$. The natural definition turns out to take complexes of left $\mathcal{D}$-modules to complexes of right $\mathcal{D}$-modules, which we turn into left $\mathcal{D}$-modules via the side-changing functor discussed above. Hence we define

$$\mathbb{D}_X \mathcal{M}^\cdot = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\cdot, \mathcal{D}_X) \underset{\mathcal{O}_X}{\otimes} \omega_X^\vee[\dim X].$$

This gives a duality (exact, contravariant autoequivalence of categories) on the bounded derived category $D^b_{\mathrm{coh}}(\mathcal{D}_X)$ of complexes of $\mathcal{D}_X$-modules with $\mathcal{D}_X$-*coherent* cohomology.[12] It has additional nice properties which we will return to below, when we have defined holonomic $\mathcal{D}$-modules.

B.5 Holonomicity

The category of $\mathcal{D}$-modules admits an Artinian subcategory of *holonomic* $\mathcal{D}$-modules, which are in some sense the "smallest" ones.

B.5.1 Singular support and holonomicity

Recall that $\mathcal{D}_X$ has an increasing filtration by the order of differential operators, and that the associated graded sheaf $\mathrm{gr}\, \mathcal{D}_X \cong \mathcal{O}_{\mathrm{T}^*X}$.

Definition B.10. An increasing filtration $\mathcal{M}^0 \subset \mathcal{M}^1 \subset \cdots$ of a quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ making it into a filtered $\mathcal{D}_X$-module (with respect to the order filtration) is call **good** if the following equivalent conditions hold: (i) The associated graded $\mathrm{gr}\, \mathcal{D}_X$-module $\mathrm{gr}\, \mathcal{M}$ is $\mathrm{gr}\, \mathcal{D}_X$-coherent; (ii) Each $\mathcal{M}^i$ is $\mathcal{O}_X$-coherent and $\mathcal{D}_X^{\leq 1} \mathcal{M}^i = \mathcal{M}^{i+1}$ for all $i \gg 0$.

Using the isomorphism $\mathrm{gr}\, \mathcal{D}_X \cong \mathcal{O}_{\mathrm{T}^*X}$ we can make the following defnition.

Definition B.11. The **characteristic variety** (a.k.a. **singular support**) of a $\mathcal{D}$-module $\mathcal{M}$ with a good filtration is the closed subvariety

$$\mathrm{S.S.}(\mathcal{M}) = \mathrm{supp}\, \mathrm{gr}\, \mathcal{M} \subset \mathrm{T}^* \mathcal{M}.$$

Proposition B.12 (Bernstein)**.** Any $\mathcal{D}$-coherent $\mathcal{D}_X$-module admits a good filtration, and its singular support does not depend upon which good filtration one chooses. □

The most important fact about singular supports is the following.

Theorem B.13 (Bernstein's inequality)**.** *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then the dimension of any irreducible component of* $\mathrm{S.S.}(\mathcal{M})$ *is* $\geq \dim X$*, unless* $\mathcal{M} = 0$*.* □

[12] There is something to prove here, of course; cf. [Ber, 3.5].

This is essentially an algebraic fact. One reduces to the affine case $X = \operatorname{Spec} R$, and observes the notion of singular support makes sense for any module over a (not necessarily commutative) filtered R-algebra A. The theorem follows from Kashiwara's Theorem B.3.2 plus the general fact (which holds subject to suitable hypotheses on R and A, satisfied for rings of differential operators on a smooth variety) that a finitely-generated A-module $\mathcal{M}$ has a canonical filtration of length $2 \dim R$, the subquotients of which satisfy codim $\mathrm{S.S.}(C^i/C^{i+1}) = i$. For details see [HTT, 2.3].

Definition B.14. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is called **holonomic** if $\mathcal{M} = 0$ or $\dim \mathrm{S.S.}(\mathcal{M}) = \dim X$.

One can show that a $\mathcal{D}_X$-module $\mathcal{M}$ is $\mathcal{O}_X$-coherent (i.e. an integrable connection) if and only if $\mathrm{S.S.}(\mathcal{M})$ is actually the zero section of $\mathrm{T}^*X \to X$ [Ber, 2.8]. So $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules are holonomic. In fact, any holonomic $\mathcal{D}$-module is $\mathcal{O}$-coherent on a dense open subset [Ber, 2.11].

Key properties of holonomic modules include:

- Holonomic modules form an abelian subcategory $\mathrm{Hol}(\mathcal{D})$ of the category of all $\mathcal{D}$-modules, closed under subquotients and extensions. This follows essentially from the fact that $\mathrm{S.S.}(\mathcal{G}) = \mathrm{S.S.}(\mathcal{F}) \cup \mathrm{S.S.}(\mathcal{H})$ for an extension $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of coherent $\mathcal{D}$-modules.

- Holonomic modules have finite length.

- The bounded derived category $D^b_{\mathrm{hol}}(\mathcal{D}_X)$ of complexes of $\mathcal{D}_X$-modules with holonomic cohomology, is canonically equivalent to the bounded derived category $D^b(\mathrm{Hol}(\mathcal{D}_X))$ of complexes of holonomic modules.

- The functors α_* and $\alpha^!$ take holonomic complexes to holonomic complexes.

- Since $\mathcal{F} \overset{\mathrm{L}}{\underset{\mathcal{O}}{\otimes}} \mathcal{G} = \Delta^!(\mathcal{F} \boxtimes \mathcal{G})$ up to a homological shift, it follows that internal tensor product preserves holonomicity as well.

Remark B.15. In the case of $\mathcal{D}$-modules on affine space, there is an alternative criterion of holonomicity, which uses an additional *Bernstein filtration* on the Weyl algebra that gives degree 1 to all the generators t_i and ∂_j; see Remark 2.6.

B.5.2 Extension

One nice property of holonomic $\mathcal{D}$-modules which is used in the body of the thesis (specifically, for the proof of the b-function lemma in Section 2) is the following *extension lemma.*

Lemma B.16. [Ber, 3.7] Let $\mathcal{F}$ be a $\mathcal{D}_X$-module, U an open subset of X, and $\mathcal{H} \subset \mathcal{F}|_U$ a holonomic submodule. Then there exists a holonomic submodule $\mathcal{H}' \subset \mathcal{F}$ extending $\mathcal{H}$, i.e. such that $\mathcal{H}'|_U = \mathcal{H}$. □

B.5.3 Duality

The duality functor $\mathbb{D}$ defined in §B.4 is well-behaved not only for the derived category $D^b_{\mathrm{coh}}(\mathcal{D}_X)$, but also for $D^b_{\mathrm{hol}}(\mathcal{D}_X)$ and for the subcategory of holonomic *modules* (not complexes). The important fact is the following.

Proposition B.17. [Ber, 3.5] $\mathcal{M}$ is holonomic if and only if $\mathbb{D}\mathcal{M}$ is a module (concentrated in cohomological degree 0). Moreover, $\mathbb{D}$ restricts to an exact, contravariant autoequivalence of the category $\mathrm{Hol}(\mathcal{D}_X)$. □

B.5.4 Functors for holonomic modules

The previous result enables the definition of dualized versions of direct and inverse image on the derived category of holonomic modules, analogous to exceptional inverse image and direct image with compact support in the category of abelian sheaves.

Definition B.18. Let $\alpha : Y \to X$ be a morphism. The functors $\alpha_! : D^b_{\mathrm{hol}}(\mathcal{D}_Y) \to D^b_{\mathrm{hol}}(\mathcal{D}_X)$ and $\alpha^* : D^b_{\mathrm{hol}}(\mathcal{D}_X) \to D^b_{\mathrm{hol}}(\mathcal{D}_Y)$ are defined by

$$\alpha^*\mathcal{F} = \mathbb{D}_Y \alpha^! \mathbb{D}_X \mathcal{F}, \qquad \alpha_! \mathcal{G} = \mathbb{D}_X \alpha_* \mathbb{D}_Y \mathcal{G}.$$

Of the properties of these functors, the most important for this thesis are [Ber, 3.9]:

- $\alpha_!$ is left adjoint to $\alpha^!$.
- α^* is left adjoint to α_*.
- $\alpha^! = \alpha^*[2(\dim Y - \dim X)]$ if α is smooth.
- There exists a canonical **Goresky-MacPherson map** $\alpha_!\mathcal{G} \to \alpha_*\mathcal{G}$. Its definition for α an open immersion is given in §3.3.3. It is an isomorphism for α proper. See also Theorem 2.3.

We finish with a result about duality and internal tensor product for holonomic modules and local systems.

Proposition B.19. Let $\mathcal{M}$ and $\mathcal{L}$ be holonomic $\mathcal{D}$-modules, with $\mathcal{L}$ assumed $\mathcal{O}$-coherent (i.e. an integrable connection). Then

$$\mathbb{D}(\mathcal{M} \underset{\mathcal{O}}{\otimes} \mathcal{L}) \cong \mathbb{D}(\mathcal{M}) \underset{\mathcal{O}}{\otimes} \mathbb{D}(\mathcal{L})$$

and moreover

$$\mathbb{D}(\mathcal{L}) \cong \mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$$

with the canonical $\mathcal{D}$-module struction on the $\mathcal{H}om$ defined by $(\xi\psi)(\ell) = \xi\psi(\ell) - \psi(\xi\ell)$ for vector fields ξ.

Proof. See [G, Prop. 4.4.6]. □

C $\mathcal{D}$-modules Quick Reference / List of notation

In the following, X (resp. Y) is a variety over $\Bbbk$ of dimension n (resp. m), $\mathcal{F}$ (resp. $\mathcal{G}$) is a $\mathcal{D}_X$ (resp. $\mathcal{D}_Y$) -module, and $\alpha : Y \to X$ is a map of varieties.

- Categories:

(§§B.1,B.5)
- $\mathrm{Mod}(\mathcal{D}_X)$ (resp. $\mathrm{Hol}(\mathcal{D}_X), \mathrm{Coh}(\mathcal{D}_X)$) = all (resp. holonomic, resp. coherent) left $\mathcal{D}_X$-modules.
- $D^b(\mathscr{A})$ = derived category of bounded complexes of objects in the abelian category $\mathscr{A}$. We abbreviate $D^b(\mathrm{Mod}(\mathcal{D}_X))$ by $D^b(\mathcal{D}_X)$.
- $D^b_{\mathrm{Hol}}(\mathcal{D}_X)$ (resp. $D^b_{\mathrm{Coh}}(\mathcal{D}_X)$) = derived category of bounded complexes of $\mathcal{D}_X$-modules with holonomic (resp. coherent) cohomology.

- Some $\mathcal{D}$-modules and functors between them:

(§B.1)
- $\mathcal{D}_X$ = sheaf of algebraic differential operators on X.
- A_n = **Weyl algebra**, ring of differential operators on $\mathbb{A}^n$.

(§B.2)
- ω_X (resp. $\omega_{Y/X}$) = the sheaf of n-forms on X (resp. relative $(n-m)$-forms on Y), a locally-free $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) -module of rank 1, endowed with a *right* $\mathcal{D}_X$ (resp. $\mathcal{D}_Y$) -module structure defined by the action of vector fields via Lie derivative, i.e. $\omega \cdot \xi = -\mathrm{Lie}_\xi \omega$.

(§B.3)
- $\alpha_\cdot \mathcal{G}$ (resp. $\alpha^\cdot \mathcal{F}$) = direct (resp. inverse) image functor for $\mathcal{O}$-modules.
- $\alpha^\Delta \mathcal{F} = \alpha^\cdot \mathcal{F}$ endowed with its canonical $\mathcal{D}_Y$-module structure.
- $\mathcal{D}_{Y\to X} = \alpha^\Delta(\mathcal{D}_X)$ (**transfer** $(\mathcal{D}_Y, \alpha^{-1}\mathcal{D}_X)$ **bimodule**).
  - $\alpha^\Delta \mathcal{F} = \mathcal{D}_{Y\to X} \underset{\alpha^{-1}\mathcal{D}_X}{\otimes} \alpha^{-1}\mathcal{F}$; $\mathrm{L}\alpha^\Delta \mathcal{F}^\cdot = \mathcal{D}_{Y\to X} \overset{\mathrm{L}}{\underset{\alpha^{-1}\mathcal{D}_X}{\otimes}} \alpha^{-1}\mathcal{F}^\cdot$
- $\alpha^! \mathcal{F}^\cdot = \mathrm{L}\alpha^\Delta \mathcal{F}^\cdot[m-n]$ (**inverse image**: $D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y)$).
- $\mathcal{D}_{X\leftarrow Y} = \mathcal{D}_{Y\to X} \underset{\mathcal{O}_Y}{\otimes} \omega_{Y/X}$ (**transfer** $(\alpha^{-1}\mathcal{D}_X, \mathcal{D}_Y)$ **bimodule**).
- $\alpha_* \mathcal{G}^\cdot = \mathrm{R}\alpha_\cdot(\mathcal{D}_{X\leftarrow Y} \overset{\mathrm{L}}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{G}^\cdot)$ (**direct image**: $D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X)$).

(§B.4)
- $\mathbb{D}\mathcal{F}^\cdot = \mathbb{D}_X \mathcal{F}^\cdot = \mathrm{R}\mathcal{H}om^\cdot_{\mathcal{D}_X}(\mathcal{F}^\cdot, \mathcal{D}_X) \underset{\mathcal{O}_X}{\otimes} \omega_X^\vee[n]$ (**duality**: $D^b_{\mathrm{Coh}}(\mathcal{D}_X)^{\mathrm{op}} \to D^b_{\mathrm{Coh}}(\mathcal{D}_X)$).
  - Preserves holonomicity, on objects and complexes.
  - Coincides with $\mathcal{H}om_{\mathcal{O}_X}(\bullet, \mathcal{O}_X)$ for integrable connections.

(§B.5.3)
- $\alpha^* \mathcal{F}^\cdot = \mathbb{D}_Y \alpha^! \mathbb{D}_X \mathcal{F}^\cdot$ (**$*$ inverse image**).
  - $\alpha^! = \alpha^*[2(n-m)]$ for smooth α [Ber, 3.9].
- $\alpha_! \mathcal{G}^\cdot = \mathbb{D}_X \alpha_* \mathbb{D}_Y \mathcal{G}^\cdot$ (**! direct image**).
  - $\exists$ canonical morphism $\alpha_! \to \alpha_*$, isomorphism for α proper [Ber, 3.9].

(Thm. 2.3)
- $\alpha_{!*}\mathcal{G} = \mathrm{im}(\mathcal{H}^0(\alpha_! \mathcal{G}) \to \mathcal{H}^0(\alpha_* \mathcal{G}))$, **Goresky-MacPherson extension**.

- Adjunctions: [Ber, 3.9]
  - $\alpha_!$ is left adjoint to $\alpha^!$.
  - α^* is left adjoint to α_*.

(§A.2) • Exact Categories:

- $\hookrightarrow$ = admissible monomorphism
- $\twoheadrightarrow$ = admissible epimorphism

(§3.2.1) • Pro-ind Category:

- $\Pi = \{(a,b) \in \mathbb{Z}^2 \mid a \le b\}$ as a poset under $(a,b) \le (a',b') \Leftrightarrow a \ge a', b \ge b'$.
- $\mathscr{A}^\Pi$ = category of "Pi-shaped diagrams" in the abelian category $\mathscr{A}$.
- $\mathscr{A}^\Pi_\leftrightarrow$ = admissible diagrams in $\mathscr{A}^\Pi$.
- $\widetilde{\varphi}\mathcal{F}^{a,b}$ = diagram $\mathcal{F}^{\varphi(a),\varphi(b)}$ induced from $\mathcal{F}^{\cdot,\cdot} \in \mathscr{A}^\Pi$.
- $\lim\limits_{\leftrightarrow}\mathscr{A}$ = localization of $\mathscr{A}^\Pi_\leftrightarrow$ by the class Σ of morphisms induced by natural transformations $\widetilde{\varphi}\mathcal{F}^{a,b} \to \widetilde{\psi}\mathcal{F}^{a,b}$, $\varphi \ge \psi$ well-behaved maps $\mathbb{Z} \to \mathbb{Z}$.
- $\lim\limits_{\leftrightarrow}$ = localization functor $\mathscr{A}^\Pi_\leftrightarrow \to \lim\limits_{\leftrightarrow}\mathscr{A}$.
- $\mathcal{F}_k^{\cdot,\cdot}$ = truncated diagram $\mathcal{F}_k^{a,b} = \mathcal{F}^{\max(a,k),\max(b,k)}$ for $\mathcal{F}^{\cdot,\cdot} \in \mathscr{A}^\Pi$.
- ${}_\ell\mathcal{F}^{\cdot,\cdot}$ = reverse truncated diagram $\mathcal{F}^{\min(a,\ell),\min(b,\ell)}$.

• Pro-ind holonomic $\mathcal{D}$-modules:

(§3.3.1) • $\mathcal{M}^{-\infty,\infty} = \lim\limits_{\leftrightarrow}(\mathcal{M} \otimes \mathcal{J}_f^{a,b})$.

- $\mathcal{M}_k^{-\infty,\infty} = \lim\limits_{\leftrightarrow}(\mathcal{M} \otimes \mathcal{J}_{f,k}^{a,b})$.

• More functors:

(§§3.3.3, 3.4) • $\Pi^!\mathcal{M} = \Pi^*\mathcal{M} = j_!\mathcal{M}^{-\infty,\infty} = j_*\mathcal{M}^{-\infty,\infty} = \lim\limits_{\leftrightarrow} j_!(\mathcal{M} \otimes \mathcal{J}_f^{a,b})$.

- $\Pi_k^!\mathcal{M}$ (resp. $\Pi_k^*\mathcal{M}$) $= j_!\mathcal{M}_k^{-\infty,\infty}$ (resp. $j_*\mathcal{M}_k^{-\infty,\infty}$).

(§3.4) • $\Pi_{!*}^{k,\ell}\mathcal{M} = \operatorname{coker}(\Pi_\ell^!\mathcal{M} \hookrightarrow \Pi_\ell^*\mathcal{M} \hookrightarrow \Pi_k^*\mathcal{M})$, for $\ell \ge k$.

(§3.5) • $\Psi_f^{(i)}\mathcal{M} = \Pi_{!*}^{i,i}\mathcal{M}$ (**nearby cycles**). Case $i = 0$ is denoted $\Psi_f\mathcal{M}$.

(§3.6.2) • $\Xi_f^{(i)}\mathcal{M} = \Pi_{!*}^{i,i+1}\mathcal{M}$ (**maximal extension**). Case $i = 0$ is denoted $\Xi_f\mathcal{M}$.

(§3.7) • $\Phi_f\mathcal{M}$ = nonzero homology of complex (3) in §3.7 (**vanishing cycles**).

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