Lectures on algebraic D-modules

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CHAPTER 1

D-modules on affine varieties

1. Lecture 1: Analytic continuation of distributions with respect to a parameter and $\mathcal{D}$-modules (01/31/02)

In this course we shall work over the base field k of characteristic 0 (in most case one can assume that $k = \mathbb{C}$). As a motivation for what is going to come let us first look at the following elementary problem.

Analytic problem.
Let $p \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial in n variables ($p : \mathbb{R}^n \to \mathbb{C}$) and let U be a connected component of $\mathbb{R}^n \backslash \{x \mid p(x) = 0\}$. Define

$$p_U(x) = \begin{cases} p(x) & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

Let us take any $\lambda \in \mathbb{C}$ and consider the function $|p_U|^\lambda$.

It is easy to see that if Re $\lambda \geq 0$ then $|p_U(x)|^\lambda$ makes sense as a distribution on $\mathbb{R}^n$, i.e. $\int_U |p_U(x)|^\lambda f(x)dx$ is convergent for any $f(x) \in C_c^\infty(\mathbb{R}^n)$ – a smooth function on $\mathbb{R}^n$ with compact support.

Example. Let $n = 1$, $p(x) = x$ and $U = \mathbb{R}_+$. Then $\int_0^\infty f(x)x^\lambda dx$ is defined for Re $\lambda \geq 0$ (of course the integral is actually well-defined for Re $\lambda > -1$ but we do not need this). We shall denote the corresponding distribution by x_+^λ.

It is easy to see that for Re $\lambda \geq 0$ we have a holomorphic family of distributions $\lambda \mapsto \mid p_U(x) \mid^\lambda$.

Question(Gelfand): Can you extend this family meromorphically in λ to the whole $\mathbb{C}$?

Let $\mathcal{E} \in \mathcal{D}(\mathbb{R}^n)$ be a distribution on $\mathbb{R}^n$. As usual we define $\frac{\partial \mathcal{E}}{\partial x_i}(f) = -\mathcal{E}(\frac{\partial f}{\partial x_i})$.

Example. Let $n = 1$, $p(x) = x$, $U = \mathbb{R}_+$. We have a distribution x_+^λ defined for Re $\lambda \geq 0$. We know that $\frac{d}{dx}(x_+^{\lambda+1}) = (\lambda + 1)x_+^\lambda$. The left hand side is defined for Re $\lambda \geq -1$. Hence the expression

$$x_+^\lambda = \frac{1}{\lambda + 1} \frac{d}{dx}(x_+^{\lambda+1})$$

gives us an extention of x^λ to Re $\lambda \geq -1$, $\lambda \neq -1$. Continuing this process by induction we get the following

Proposition 1.1. *x_+^λ extends to the whole of $\mathbb{C}$ meromorphicaly with poles in negative integers. In particular, for every $f \in C_c^\infty(\mathbb{R})$*

$$\alpha_f(\lambda) = \int_0^\infty f(x)x^\lambda dx$$

has a meromorphic continuation to the whole of $\mathbb{C}$ with poles at $-1, -2, -3, \ldots$.

Example. The proposition works not only for functions with compact support but also for functions which are rapidly decreasing at $+\infty$ together with all derivatives. For example we can take $f(x) = e^{-x}$. In this case we have $\int_0^\infty e^{-x}x^\lambda dx = \Gamma(\lambda + 1)$. The proposition implies that $\Gamma(\lambda)$ has a meromorphic continuation with poles at $0, -1, -2 \ldots$.

Theorem 1.2. *[Atiyah, Bernstein-Gelfand] $\mid p_U \mid^\lambda$ has a meromorphic continuation to the whole $\mathbb{C}$ with poles in a finite number of arithmetic progressions.*

The first proofs of this fact were based on Hironaka's theorem about resolution of singularities. We are going to give a completely algebraic proof of Theorem 1.2 which is due to Bernstein. For this let us first formulate an algebraic statement that implies Theorem 1.2.

Let $\mathcal{D} = \mathcal{D}(\mathbb{A}^n)$ denote the algebra of differential operators with polynomial coefficients acting on $\mathcal{O} = \mathcal{O}(\mathbb{A}^n) = \mathbb{C}[x_1, \ldots, x_n]$. In other words $\mathcal{D}$ is the subalgebra of $\operatorname{End}_{\mathbb{C}} \mathcal{O}$ generated by multiplication by x_i and by $\frac{\partial}{\partial x_j}$.

Theorem 1.3. *There exist $d \in \mathcal{D}[\lambda]$ and $b(\lambda) \in \mathbb{C}[\lambda]$ such that*

$$d(p^{\lambda+1}) = b(\lambda)p^\lambda.$$

Example. Let $n = 1$ and $p(x) = x$. Then we can take $d = \frac{d}{dx}$ and $b(\lambda) = \lambda + 1$.

We claim now that Theorem 1.3 implies Theorem 1.2 (note that Theorem 1.3 is a completely algebraic statement). Indeed, suppose $d(p^{\lambda+1}) = b(\lambda)p^\lambda$. Then $d(\mid p_U \mid^{\lambda+1}) = b(\lambda) \mid p_U \mid^\lambda$. The left hand side is defined for $\mathrm{Re}\lambda \geq -1$, thus the expression

$$|p_U|^\lambda = \frac{1}{b(\lambda)} d(|p_U|^{\lambda+1})$$

gives us a meromorphic continuation of $\mid p_U \mid^\lambda$ to $\mathrm{Re}\ \lambda \geq -1$. So, arguing by induction again, we see that $\mid p_U \mid^\lambda$ can be meromorphicaly extended to the whole of $\mathbb{C}$ with poles at arithmetic progressions α, $\alpha - 1$, $\alpha - 2$, $\ldots$ where α is any root of $b(\lambda)$.

We now want to reformulate Theorem 1.3 once again. Set $\mathcal{D}(\lambda) = \mathcal{D} \otimes_{\mathbb{C}} \mathbb{C}(\lambda)$. Denote by M_p the $\mathcal{D}(\lambda)$-module consisting of all formal expressions $q(x)p^{\lambda-i}$ where $i \in \mathbb{Z}$ and $q(x) \in \mathbb{C}(\lambda)[x_1, \ldots, x_n]$ subject to the relations $qp^{\lambda-i+1} = (qp)p^{\lambda-i}\}$ (the action of $\mathcal{D}(\lambda)$ is defined in the natural way).

Theorem 1.4. *M_p is finitely generated over $\mathcal{D}(\lambda)$.*

Let us show that Theorem 1.3 and Theorem 1.4 are equivalent.

Theorem 1.4 ⇒ Theorem 1.3: Denote by M_i the submodule of M_p generated by $p^{\lambda-i}$. Then $M_i \subset M_{i+1}$ and

$$M_p = \bigcup_i M_i. \tag{1.1}$$

Assume that M_p is finitely generated. Then (1.1) implies that there exists $j \in \mathbb{Z}$ such that $M_p = M_j$. In other words for i as above the module M_p is generated by $p^{\lambda-j}$. Hence there exist $\widetilde{d} \in \mathcal{D}(\lambda)$ such that $\widetilde{d}(p^{\lambda-j}) = p^{\lambda-j-1}$.

Let σ_j be an automorphism of $\mathbb{C}(\lambda)$ sending λ to $\lambda+j-1$. Then σ_j extends to an automorphism of the algebra $\mathcal{D}(\lambda)$ (which we shall denote by the same symbol) and clearly we have $\sigma_j(\widetilde{d})(p^{\lambda+1}) = p^{\lambda}$. But $\sigma_j(\widetilde{d})$ can be written as $\sigma_j(\widetilde{d}) = \frac{d}{b(\lambda)}$, where $d \in \mathcal{D}[\lambda]$ and $b(\lambda) \in \mathbb{C}[\lambda]$. Thus we have $d(p^{\lambda+1}) = b(\lambda)p^{\lambda}$.

Theorem 1.3 ⇒ Theorem 1.4: By shifting λ we see that for every integer $i > 0$ there exists a differential operator $d_i \in \mathcal{D}[\lambda]$ such that

$$d_i(p^{\lambda}) = b(\lambda - i)p^{\lambda-i}.$$

This clearly implies that p^{λ} generates M_p.

We now want to prove Theorem 1.4. To do this we need to develop some maschinery.

1.5. Filtrations. Let A be an associative algebra over k. Recall that *an increasing filtration* on A is the collection of k-subspaces $F_iA \subset A$ (for $i \geq 0$) such that

1) $A = \cup F_iA$ and $\cap F_iA = \{0\}$;

2) We have $F_iA \subseteq F_{i+1}A$ and $F_iA \cdot F_jA \subseteq F_{i+j}A$. It is also convenient to set $F_{-1}A = 0$.

In this case one may define the *associated graded algebra* $\operatorname{gr}^F A$ of A in the following way:

$$\operatorname{gr}^F A = \bigoplus_{i=0}^{\infty} F_iA/F_{i-1}A.$$

We set $\operatorname{gr}_i^F A = F_iA/F_{i-1}A$. Then $gr^F A$ has a natural structure of a graded algebra (i.e. we have $\operatorname{gr}_i^F A \cdot \operatorname{gr}_j^F A \subseteq \operatorname{gr}_{i+j}^F A$). We shall sometimes drop the super-script ^{F} when it does not lead to a confusion.

Similarly let M be a left module over A. Then an increasing filtration on M consists of a collection of k-subspaces $F_jM \subset M$ such that

1) $M = \cup F_jM$ and $\cap F_jM = \{0\}$;

2) We have $F_jM \subseteq F_{j+1}M$ and $F_iA \cdot F_jM \subseteq F_{i+j}M$.

As before one defines

$$\operatorname{gr}^F M = \bigoplus_{i=0}^{\infty} F_j M / F_{j-1} M$$

Thus $\operatorname{gr}^F M$ is a graded $\operatorname{gr} A$-module.

Definition 1.6. (1) *An increasing filtration $F_j M$ is called a good filtration if $\operatorname{gr}^F M$ is finitely generated as $\operatorname{gr} A$-module.*

(2) *Two filtrations $F_j M$ and $F'_j M$ are called equivalent if there exist j_0 and j_1 such that*

$$F'_{j-j_0} M \subseteq F_j M \subseteq F'_{j+j_1} M.$$

Proposition 1.7. (1) *Let $F_j M$ be a good filtration on a left A-module M. Then M is finitely generated over A.*

(2) *If $F_j M$ is a good filtration on M then there exist j_0 such that for any $i \geq 0$ and any $j \geq j_0$ $F_i A \cdot F_j M = F_{i+j} M$.*

(3) *Assume that we have two filtrations F and F' on M such F is good. Assume also that for any $i \geq 0$ the $F_0 A$-module $F_i A$ is finitely generated. Then there exist j_1 such that $F_j M \subset F'_{j+j_1} M$ for any j.*

Corollary 1.8. *Suppose that $F_i A$ is finitely generated over $F_0 A$ as a left module. Then any two good filtrations on a left A-module M are equivalent.*

This clearly follows from the third statement of the theorem.

Proof. (1) By assumption $\operatorname{gr}^F M$ is finitely generated. Let $s_1, \dots s_k$ be the generators of $\operatorname{gr}^F M$, $s_i \in \operatorname{gr}^F_{j_i} M$. For any i choose $t_i \in F_{j_i} M$ which projects to s_i. It is now easy to see that t_i generate M.

(2) If $gr^F M$ is finitely generated over $gr A$ then $gr^F M$ is generated by $\bigoplus_{i=0}^{j_0} gr^F_i M$ for some j_0. Then for any $j \geq j_0$

$$\operatorname{gr}_i A \cdot \operatorname{gr}^F_j M = \operatorname{gr}^F_{i+j} M$$

and hence $F_i A \cdot F_j M + F_{i+j-1} M = F_{i+j} M$.

By induction on i we can assume that $F_{i+j-1} M = F_{i-1} A \cdot F_j M$. Then $F_{i+j} M = F_i A \cdot F_j M + F_{i-1} A \cdot F_j M = F_i A \cdot F_j M$.

(3) First of all we claim that $F_j M$ is finitely generated over $F_0 A$ for all j. It is enough to show that $\operatorname{gr}^F_j M$ is finitely generated over $F_0 A$ for all j. Let $m_1, ..., m_k$ be some generators of $gr^F M$. We may assume that they are homogeneous, i.e. $m_i \in \operatorname{gr}^F_{j_i} M$ for some j_i. Thus for every $j \geq 0$ the map

$$\bigoplus_{i=1}^{k} \operatorname{gr}_{j-j_i} A \to \operatorname{gr}^F_j M; \qquad (a_1, ..., a_k) \mapsto a_1 m_1 + ... a_k m_k$$

is surjective. On the other hand since $\mathrm{gr}_i A$ is a quotient of $F_i A$ (for every i) it follows that every $\mathrm{gr}_i A$ is finitely generated. Hence $\mathrm{gr}^F_j M$ is finitely generated.

Let j_0 be as above. Since F' is a filtration, we have $F_{j_0}M = \bigcup F'_j M \cap F_{j_0}M$ and since $F_{j_0}M$ is a finitely generated F_0A-module it follows that $F_{j_0}M \subseteq F'_{j_0+j_1}M$ for some $j_1 \geq 0$.

Then $F_{i+j_0}M = F_iA \cdot F_{j_0}M \subseteq F_iA \cdot F'_{j_0+j_1}M \subseteq F'_{j_0+j_1+i}M$. So we have proved our proposition for $j \geq j_0$. By increasing j_1 cam make it true for any j.

□

Here is our main example. Let $\mathcal{D}$ be the algebra of polynomial differential operators in n variables. Let's define two filtrations on $\mathcal{D}$:

Bernstein's (or arithmetic) filtration: $F_0\mathcal{D} = k$, $F_1\mathcal{D} = k + \mathrm{span}(x_i, \frac{\partial}{\partial x_j})$, $F_i\mathcal{D}$ is the image of $F_1\mathcal{D}^{\otimes i}$ under the multiplication map.

Geometric filtration (filtration by order of differential operator) denoted by $\mathcal{D}_0 \subset \mathcal{D}_1 \subset ...$:

$\mathcal{D}_0 = \mathcal{O} = k[x_1, \dots, x_n]$

$\mathcal{D}_1 = \mathrm{span}(f \in \mathcal{O}; g\frac{\partial}{\partial x_i}$ where $g \in \mathcal{O})$, $\mathcal{D}_i$ is the image of ${\mathcal{D}_1}^{\otimes i}$ under the multiplication map.

The following lemma describes the algebra $\mathcal{D}$ explicitly as a vector space. The proof is left to the reader.

Lemma 1.9. *For any $d \in \mathcal{D}$ there exists a unique decomposition*

$$d = \sum_{i_1 \leq \dots \leq i_k} p_{i_1, \dots i_k} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}}$$

where $p_{i_1 \dots i_k} \in k[x_1, \dots, x_n]$.

This lemma immediately implies the following

Proposition 1.10. *For both filtrations $\mathrm{gr}\,\mathcal{D} \cong k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$. Here x_i are images of x_i and ξ_j are images of $\frac{\partial}{\partial x_j}$.*

Proof. Let us show, that x_i's and ξ_j's commute in $\mathrm{gr}\,\mathcal{D}$ is for both filtrations. For each $i \neq j$ we have $x_i\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}x_i$ in $\mathcal{D}$ and hence in $gr^F\mathcal{D}$. For $i = j$ we have $x_i\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i}x_i = -1 \in F_0\mathcal{D}$ and hence equal to 0 in $gr^F\mathcal{D}$ for both filtrations.

It follows now easily from the above lemma that $\mathrm{gr}\,\mathcal{D}$ is a polynomial algebra in x_i's and ξ's. □

For Bernstein's filtration the above argument shows a little more – namely that for all i, j we have $[F_i\mathcal{D}, F_j\mathcal{D}] \subset F_{i+j-2}\mathcal{D}$ (note that for the geometric filtration we only have $[\mathcal{D}_i, \mathcal{D}_j] \subset \mathcal{D}_{i+j-1}$). We shall need this fact in the next lecture.

2. Lecture 2: Bernstein's inequality and its applications (02/05/02)

Let $\mathcal{D} = \mathcal{D}(\mathbb{A}^n)$ be the algebra of polynomial differential operators in n variables.

In the last lecture we have introduced two filtrations on $\mathcal{D}$: Bernstein's filtration $F_i\mathcal{D}$ (x_i and $\frac{\partial}{\partial x_j}$ are in $F_1\mathcal{D}$) and geometric filtration $\mathcal{D}_i$ ($x_i \in \mathcal{D}_0$ and $\frac{\partial}{\partial x_j} \in \mathcal{D}_1$). For both of the filtrations $\operatorname{gr}\mathcal{D} \cong k[x_1, \dots x_n, \xi_1 \dots \xi_n]$. Also $\dim_k F_i\mathcal{D} < \infty$.

Let A be a filtered algebra such that $\dim_k F_iA < \infty$ and $grA \cong k[y_1, \dots, y_m]$. Let M be an A-module with a good filtration F. Define $h_F(M, j) = \dim_k F_jM$.

Theorem 2.1. *There exists a polynomial $h_F(M)(t)$ (called the Hilbert polynomial of M with respect to filtration F such that $h_F(M, j) = h_F(M)(j)$ for any $j \gg 0$, $h_F(M)(j)$ has a form $h_F(M)(t) = \frac{ct^d}{d!} + \{$lower order terms$\}$, where $d \leq m$ and $c \in \mathbb{Z}_+$.*

Let us mention that this is actually a theorem from commutative algebra since $h_F(M, j) = h_{\operatorname{gr} F}\operatorname{gr} M, j$ where $\operatorname{gr} F$ is the natural filtration on grM (coming from the grading).

Lemma 2.2. *c and d in the theorem above do not depend on filtration.*

Proof. Let F and F' be good filtrations. Then there exist j_0 and j_1 such that

$$F'_{j-j_0}M \subseteq F_jM \subseteq F'_{j+j_1}M$$

and hence $h_{F'}(j - j_0) \leq h_F(j) \leq h_{F'}(j + j_1)$. This can be true only if h_F and $h_{F'}$ have the same degree and the same leading coefficient. □

Definition 2.3. *For a finitely generated module M $d = d(M)$ as above is called the dimension of M (sometimes it is also called Gelfand-Kirillov of functional dimension of M).*

Theorem 2.4. *[Bernstein's inequality] For any finitely generated module M over $\mathcal{D} = \mathcal{D}(\mathbb{A}^n)$ with Bernstein's filtration we have $d(M) \geq n$.*

Before proving Theorem 2.4 we want to derive some very important corollaries of it (in particular we are going to explain how this theorem implies the results formulated in the previous lecture).

Historical remark. This theorem was first proved in Bernstein's thesis, then a simple proof was given by A Joseph. Then O. Gabber proved a very general theorem which we shall discuss later (this theorem implies that more or less the same is true for geometric filtration). ADD REFERENCES.

2.5. Examples. 1. Let $n = 1$ and M be a finitely generated $\mathcal{D}$-module. Suppose that $d(M) = 0$. This means that $\dim_k M < \infty$. For any two operators on M the trace of their commutator should be 0. But $[\frac{d}{dx}, x] = 1$ in $\mathcal{D}$ and it cannot have zero trace unless $M = 0$. Hence $\dim_k M = \infty$ and $d(M) \geq 1$.
2. $\mathcal{O} = k[x_1, \ldots, x_n]$ is a module over $\mathcal{D}$. Let $F_i\mathcal{O}$ be all polynomials of degree less or equal to i. Then $\dim_k F_i\mathcal{O} = \binom{n+i}{n}$ is a polynomial of degree n with leading term $\frac{i^n}{n!}$. Thus $d(M) = n$ and $c(M) = 1$.

3. Let $n = 1$, fix $a \in \mathbb{A}^n$. Define $\mathcal{D}$-module of δ-functions δ_a as a module with basis $\{\delta_a^{(k)}\}_{k=0}^{\infty}$ and the following action of $\mathcal{D}$:

$$\frac{d}{dx}(\delta_a^{(k)}) = \delta_a^{(k+1)}$$
$$(x-a)\delta_a^{(k)} = (-1)^k k \delta_a^{(k-1)}$$
$$(x-a)\delta_a^{(0)} = 0.$$

It is easy to see that $d(\delta_a) = 1$ and $c(\delta_a) = 1$.

Definition 2.6. *If $d(M) = n$ then M is called holonomic.*

Remark. For a long time it was believed that all irreducible modules are holonomic, but then Stafford found a counterexample and later Bernstein and Lunts constructed a lot of non-holonomic simple modules.

Let A be a filtered algebra such that $\operatorname{gr} A$ is Noetherian and let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of A-modules. Let $\{F_jM_2\}$ be a good filtration on M_2. It induces filtrations on M_1 and M_3, namely $F_jM_1 = F_jM_2 \cap M_1$ and F_jM_3 is the image of F_jM_2. We have a short exact sequence

$$0 \to gr^F M_1 \to gr^F M_2 \to gr^F M_3 \to 0.$$

In fact $\{F_jM_1\}$ and $\{F_jM_3\}$ are good filtrations ($gr^F M_3$ is finitely generated because it is a quotient of $gr^F M_2$, which is finitely generated, to prove that $gr^F M_1$ is finitely generated, we need $gr A$ to be Noetherian).

Proposition 2.7. *Using the same notations as above*

(1) $d(M_2) = \max(d(M_1), d(M_3))$
(2) *If* $d(M_1) = d(M_2) = d(M_3)$, *then* $c(M_2) = c(M_1) + c(M_3)$
(3) *If* $d(M_1) > d(M_3)$, *then* $c(M_2) = c(M_1)$ *and if* $d(M_3) > d(M_1)$, *then* $c(M_2) = c(M_3)$

Proof. Using the exact sequence as above we get

$$h_F(M_2, i) = h_F(M_1, i) + h_F(M_3, i).$$

All the statements of the proposition follow from this fact. □

Corollary 2.8. *Let M be a holonomic module and $c = c(M)$. Then the length of M is less than or equal to c.*

Proof. Let M be a holonomic $\mathcal{D}$-module, $d(M) = n$ and $c(M) = c$.
Suppose we have an exact sequence

$$0 \to N \to M \to N' \to 0.$$

Then we have $d(M) = d(N) = d(N') = n$ (since by Bernstein's inequality $d(N)$, $d(N') \geq n$) and $c(M) = c(N) + c(N')$. Thus $c(N) < c(M)$. Let's take N such that N' is irreducible. Then $\mathit{length}(M) \leq \mathit{length}(N) + 1$. By induction $\mathit{length}(N) \leq c(N)$, then $\mathit{length}(M) \leq c(N) + 1 \leq c(M)$. □

2.9. Example. It is easy to see that very often length of M is actually strictly smaller than $c(M)$. For example, let $n = 1$, $\lambda \in k$ and $M(x^\lambda) = \{q(x)x^{\lambda+i} \mid (qx)x^{\lambda+i} = qx^{\lambda+i+1}\}$ with the natural $\mathcal{D}$-module structure. In this case we have
1. $c(M(x^\lambda)) = 2$
2. $M(x^\lambda)$ is irreducible $\Leftrightarrow \lambda \notin \mathbb{Z}$.

Corollary 2.10. *Let M be any module over $\mathcal{D}$, F_jM – filtration on M (not necessarily good). Assume that there exist $h \in \mathbb{R}[t]$, $h(t) = \frac{ct^n}{n!} + \{$ lower order terms $\}$ where $c \geq 0$, such that $\dim_k F_jM \leq h(j)$. Then M is holonomic and $\mathit{length}(M) \leq c$.*

Proof. Let N be any finitely generated submodule of M. Let's prove, that N is holonomic and $c(N) \leq c$.

Consider F_jN – the induced filtration on N. Let F'_jN be a good filtration on N such that $F'_jN \subset F_jN$ (such a filtration exists: for example choose j such that F_jN generates N and set $F'_iN = F_iN$ for $i \leq j$ and $F'_iN = F_{i-j}\mathcal{D} \cdot F_jN$ for $i > j$). Then we have $\dim F'_jN \leq \dim F_jN$ and hence $h_{F'}(N)(j) \leq h(j) = \frac{cj^n}{n!} + \{$lower order terms$\}$. By Bernstein's inequality $d(N) \geq n$, hence $h_{F'}(N)(j) = \frac{c'j^n}{n!} + \{$lower order terms$\}$, where $c' \leq c$, i.e. N is holonomic and has the length less or equal to c.

Using the same argument as in the proof of the previous corollary, we can prove that M has a finite length, hence M is also holonomic and $\mathit{length}(M) \leq c$. □

Let $p \in k[x_1, \ldots x_n]$ and recall that in the previous lecture we defined a $\mathcal{D}(\lambda)$-module M_p by setting

$$M_p = \{q(x)p^{\lambda+i} \mid (qp)p^{\lambda+1} = qp^{\lambda+1}\}.$$

Theorem 2.11. *M_p is holonomic. In particular, it is finitely generated.*

We have shown last time that the above result implies Theorem 1.2.

Proof. By Corollary 2.10 it is enough to find a filtration F_jM_p for which we have $\dim_k F_jM \leq h(j)$, where $h(x)$ is a polynomial of degree n. Let

$$F_jM_p = \{qp^{\lambda-j} \mid \deg q \leq j(m+1)\}$$

for any $j \geq 0$ (here $m = \deg p$).

Let us show, that this is filtration: first of all $F_{j-1}M \subset F_jM$ and $M = \bigcup F_jM$. It is enough to prove that $F_1\mathcal{D} \cdot F_jM \subset F_{j+1}M$.

For any $i = 1, \dots n$ we have

$$x_i \cdot (qp^{\lambda-j}) = (x_i qp)p^{\lambda-j-1} \in F_{j+1}M,$$

since $\deg(x_i qp) = \deg q + m + 1 \leq j(m+1) + m + 1 = (j+1)(m+1)$ and

$$\frac{\partial}{\partial x_i}(qp^{\lambda-j}) = \frac{\partial q}{\partial x_i}p^{\lambda-j} + (\lambda - j)qp^{\lambda-j-1}\frac{\partial p}{\partial x_i} =$$

$$= \left(p\frac{\partial q}{\partial x_i} + (\lambda - j)q\frac{\partial p}{\partial x_i}\right)p^{\lambda-j-1} \in F_{j+1}M,$$

since $\deg\left(p\frac{\partial q}{\partial x_i} + (\lambda - j)q\frac{\partial p}{\partial x_i}\right) = \deg q + m - 1 \leq (j+1)(m+1)$.

So, F_jM is really filtration.

It is easy to see that $\dim_k F_jM = \binom{j(m+1)+n}{n}$. Thus $\dim_k F_jM$ is a polynomial of degree n in j. By Corollary 2.10 M is holonomic. $\square$

Let us now prove Theorem 2.4. We begin with the following

Lemma 2.12. *Let M be a module over $\mathcal{D}$ with a good filtration F_iM. Then*

$$F_i\mathcal{D} \longrightarrow \mathrm{Hom}(F_iM, F_{2i}M)$$

is an embedding for any i.

Proof. We shall prove Lemma 2.12 by induction on i.

1) For $i = 0$ it is clear, because $F_0\mathcal{D} = k$.

2) Suppose the statement is true for all $i' < i$. Let $a \in F_i\mathcal{D}$ such that

$$a = \sum_{i_1 \leq \dots \leq i_k} p_{i_1,\dots i_k} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}}$$

We may assume that a is not constant. Suppose $\frac{\partial}{\partial x_m}$ occurs in expression for a with a nonzero coefficient. Then $[a, x_m] \neq 0$. Similarly, if x_m occurs in the expression for a with a nonzero coefficient, then $[a, \frac{\partial}{\partial x_m}] \neq 0$.

By the property of Bernstein's filtration $[a, x_m]$ and $[a, \frac{\partial}{\partial x_m}]$ are in $F_{i-1}\mathcal{D}$.

Suppose for example that $[a, x_m] \neq 0$ (the other case is treated similarly). We have to show, that there exists $\alpha \in F_iM$ such that $a(\alpha) \neq 0$. By the induction hypothesis there exists $\alpha' \in F_{i-1}M$ such that $[a, x_m](\alpha') \neq 0$. But if $a(F_iM) = 0$, then

$$[a, x_m](\alpha') = a x_m \alpha' - x_m a \alpha' = a(x_m \alpha') - x_m(a(\alpha')) = 0.$$

Thus we get a contradiction. So $a(F_iM) \neq 0$ and the map

$$F_i\mathcal{D} \longrightarrow \mathrm{Hom}(F_iM, F_{2i}M)$$

is an embedding. $\square$

It remains to explain how the Lemma 2.12 implies Theorem 2.4.
We know that $\dim_k F_i\mathcal{D} = \frac{i^{2n}}{(2n)!} + \{\text{lower order terms}\}$.
But by the previous Lemma $\dim_k F_i\mathcal{D} \leq \dim \mathrm{Hom}(F_iM, F_{2i}M) = h_F(M,i)h_F(M,2i)$, where $h_F(M,i) = \dim F_iM = \frac{ci^d}{d!} + \{\text{lower order terms}\}$. Thus

$$\frac{i^{2n}}{(2n)!} + \{\text{lower order terms}\} \leq c^2\frac{i^d(2i)^d}{(d!)^2} + \{\text{lower order terms}\}.$$

This implies that $n \leq d$.

■

3. Lecture 3 (02/07/02)

Let us study some further properties of the algebra $\mathcal{D}$. We begin by the following

LEMMA 3.1. *$\mathcal{D}$ is both left and right Noetherian.*

PROOF. Let M be a finitely generated left $\mathcal{D}$ module. We have to prove that any submodule N of M is also finitely generated. Since M is finitely generated it admits a good filtration F_iM. On N we have the induced filtration $F_iN = F_iM \cap N$. We already know, that $gr\mathcal{D} \cong k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ is Noetherian, thus F_iN is a good filtration and N is finitely generated. □

Intuitively there exists a correspondence between modules over $\mathcal{D}$ (algebra of polynomial differential operators in n variables) and systems of linear differential equations. Namely assume that we have a system of differential equations of the form

$$\sum_{i=1}^{m} d_{ij}(f_i) = 0 \qquad j = 1, 2, ... \tag{3.1}$$

on m functions (or distributions) on $\mathbb{R}^n$ and d_{ij} are differential operators with polynomial coefficients. Then we can consider a $\mathcal{D}$-module M generated by m elements $\xi_1, ..., xi_m$ with relations given by the same formulas as in (3.1). In this case solutions of the system (3.1) in the space $C^\infty(\mathbb{R}^n)$ (or $Dist(\mathbb{R}^n)$ or any other similar space) are the same as $\mathrm{Hom}_{\mathcal{D}}(M, C^\infty(\mathbb{R}^n))$. In other words, we can think about a system of linear differential equations (with polynomial coefficients) as a $\mathcal{D}$-module together with a choice of generators. In some sense the main point of the theory of $\mathcal{D}$-modules is that particular choice of generators is "irrelevant". Note that the noetherian property of $\mathcal{D}$ implies that it is always enough to consider finitely many equations in (3.1) – these equations are elements of the kernel of the natural map $\mathcal{D}^m \to M$ sending $(d_1, ..., d_m)$ to $d_1(\xi_1) + ... + d_m(\xi_m)$ and this kernel is finitely generated.

It is especially interesting to look at the case when M is generated by one element ξ (such modules are called *cyclic*). In this case we have $M = \mathcal{D}/I$, where I is a left ideal. If I is generated by $d_1, \dots d_k$ then $d(M) \geq n - k$ and when $d_1, ..., d_k$ are in general position we have equality. Thus Bernstein's inequality says us, that in order to have a consistent system of linear differential equations (i.e. a system which has a chance to have non-zero solutions) generically you shouldn't have more than n equations.

In some sense, there are a lot of cyclic modules over $\mathcal{D}$, for example every holonomic module is cyclic. The proof is based on the following observation.

LEMMA 3.2. *$\mathcal{D}$ is a simple algebra, i.e. it has no proper two-sided ideals.*

PROOF. Assume that $I \subset \mathcal{D}$ is a two-sided ideal and $0 \neq d \in I$. Thus there exists $i \geq 0$ such that $d \in F_i\mathcal{D}$. We know that there exists x_α or $\frac{\partial}{\partial x_\beta}$ such that either $[x_\alpha, d] \neq 0$ or $[d, \frac{\partial}{\partial x_\beta}] \neq 0$. Both of these commutators are in I, since I is two-sided ideal and in $F_{i-1}\mathcal{D}$, because of the property of Bernstein's filtration. Proceeding in the same way we'll get $0 \neq d' \in I$, such that $d' \in F_0\mathcal{D} = k$. This means that $I = \mathcal{D}$. □

LEMMA 3.3. *Let A be a simple algebra which has infinite length as a left A-module. Then every A-module of finite length is cyclic.*

Remark. It is easy to see that $\mathcal{D}$ has inifinite length as a module over itself. We proved last time that all holonomic $\mathcal{D}$-modules have finite length. Hence it follows that holonomic modules are cyclic.

PROOF. By induction on the length of M it is enough to show, that if we have an exact sequence of A-modules

$$0 \to K \to M \xrightarrow{\pi} N \to 0,$$

where $K \neq 0$ is simple and N is cyclic of finite length, then M is also cyclic.

Let $n \in N$ be a generator and let $I = \mathrm{Ann}_A(n)$ be the annihillator of n in A. Assume that there is no $m \in \pi^{-1}(n)$ which generates M. We claim that in this situation I annihilates any element of K.

Indeed, choose some $m \in \pi^{-1}(n)$ and let $M' = \mathcal{D} \cdot m$ (not equal to M). Then $\pi : M' \to N$ is an isomorphism, since $M' \cap K = 0$. (If $M' \cap K \neq 0$, then $K \subset M'$ since K is simple and $M = M'$ because of the exactness of the sequence.) Thus $\mathrm{Ann}_A(m) = I$. We see that for any $k \in K$ $\mathrm{Ann}(m + k) = I$, which implies $I \cdot k = 0$ for any $k \in K$.

So, we have proved that $I \subset I' = \bigcap_{k \in K} \mathrm{Ann}(k)$. I' is a two-sided ideal in A. Since A is simple either $I' = A$ or $I' = 0$. If $I' = A$ then $K = 0$ (and we are considering the case, when $K \neq 0$). If $I' = 0$ then we also have $I = 0$. Since N is generated by n it follows and N is a free module of rank 1. Thus N doesn't have finite length by assumption of the lemma. □

COROLLARY 3.4. *Any $\mathcal{D}$-module of finite length is cyclic.*

3.5. The singular support of a $\mathcal{D}$-module. Let M be a $\mathcal{D}$-module and F_iM – good filtration on it. Then $\operatorname{gr}^F M$ can be thought of as a coherent sheaf on $\mathbb{A}^{2n}$. We claim that $\dim(\operatorname{supp} \operatorname{gr}^F M) = d(M)$.

For any graded module M, $\operatorname{supp} \operatorname{gr}^F M$ is invariant under the natural action of $\mathbb{G}_m$ and is canonically defined as a cycle in $\mathbb{A}^{2n}$, i.e. for any irreducible component of it there is a canonically given multiplicity. Thus $\operatorname{supp} gr^F M$ defines a cycle in $\mathbb{P}^{2n-1}$. In fact, the degree of this cycle is $c(M)$.

LEMMA 3.6. *Let M be a $\mathcal{D}$-module, F_iM – good filtration on it and $\operatorname{gr}^F M$ – the corresponding graded module. Let $I_F = \operatorname{Ann}(\operatorname{gr}^F M)$. Then $\sqrt{I_F}$ does not depend on F.*

PROOF. Let F_iM and $F_i^{'}M$ be good filtrations on M. Then they are equivalent, i.e $F_{j-j_0}^{'}M \subset F_jM \subset F_{j+j_1}^{'}M$ for some j_0 and j_1. Let $t = j_0 + j_1 + 1$.

Suppose $\bar{x} \in k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$, $\deg \bar{x} = p$ and $\bar{x} \in \sqrt{I_F}$. Lift $\bar{x}$ to some $x \in F_p\mathcal{D}$. Since $\bar{x} \in \sqrt{I_F}$ there exist q such that $x^q \cdot F_iM \subset F_{i+pq-1}M$. Then $x^{qt} \cdot F_i^{'} \subset F_{i+tpq-1}^{'}M$ (proof is left to the reader). This means that $\bar{x}^{qt} \in I_{F'}$ and hence $\bar{x} \in \sqrt{I_{F'}}$. □

In fact, the same argument proves the following more general result:

PROPOSITION 3.7. *Let A be a filtered algebra such that $\operatorname{gr} A$ is commutative and F_iA is a Noetherian module over F_0A for any i. Then for any finitely generated A-module M $\operatorname{supp} \operatorname{gr}^F M$ is canonically defined, i.e. for any good filtration F_iM on M $\sqrt{\operatorname{Ann}(gr^F M)}$ does not depend on F_iM.*

Consider now the geometric filtration on $\mathcal{D}$ and let M be any finitely generated $\mathcal{D}$-module.

DEFINITION 3.8.
(1) *supp $gr^F M$ with respect to the geometric filtration is called the geometric singular support of M. It will be denoted by $s.s.(M)$.*
(2) *supp $gr^F M$ with respect to Bernstein's filtration is called the arithmetic singular support of M. It will be denoted by $s.s.^a(M)$.*

We shall later define $s.s.(M)$ as a cycle in $\mathbb{A}^{2n}$ (i.e. we shall assign a multiplicity to every irreducible component of $s.s.(M)$).

As we have already pointed out, $s.s.^a(M)$ is invariant under the standard action of $\mathbb{G}_m$. The geometric singular support $s.s.(M)$ is invariant under the following action of $\mathbb{G}_m$ on $\mathbb{A}^{2n}$: $\lambda(x_i) = x_i$ and $\lambda(\xi_j) = \lambda\xi_j$.

The following theorem is non-trivial and it will be proved in Lecture 5.

THEOREM 3.9. $\dim s.s.(M) = \dim s.s.^a(M)$.

(if either of them is equal to n, then M is holonomic).

3.10. $\mathcal{O}$-coherent modules. Let $\mathcal{O} = k[x_1, \ldots, x_n] \subset \mathcal{D}$. Let M be $\mathcal{D}$-module. Then $s.s.(M) = \{\xi_1 = \cdots = \xi_n = 0\} \Leftrightarrow M$ is finitely generated over $\mathcal{O}$. On the level of $\operatorname{gr}^F M$ it is equivalent to the fact, that ξ_j act locally nilpotently. We shall say that M is $\mathcal{O}$-coherent in this case.

It turns out that all $\mathcal{O}$-coherent $\mathcal{D}$-modules are quite simple as $\mathcal{O}$-modules. Namely we have the following

THEOREM 3.11. *If M is $\mathcal{O}$-coherent then M is locally free over $\mathcal{O}$ ($\Leftrightarrow$ M is the module of sections of a vector bundle on $\mathbb{A}^n$)*

PROOF. Let $x \in \mathbb{A}^n$ and $\mathcal{O}_x$ be the local ring of x. Let $m_x \subset \mathcal{O}_x$ be the maximal ideal. As usual, define $M_x = \mathcal{O}_x \otimes_{\mathcal{O}} M$. Since M is $\mathcal{O}$-coherent $\dim_k M_x/(m_x \cdot M_x) < \infty$. Let $\bar{s}_1, \ldots, \bar{s}_k$ be a basis of this space. Lift this basis to $s_1, \ldots, s_k$ in M. By Nakayama lemma $s_1, \ldots, s_k$ generate M_x. We have to show that these elements are linearly independent over $\mathcal{O}_x$. So, assume that we have $\sum_{i=1}^k \varphi_i s_i = 0$, where not all of $\varphi_i = 0$.

We say, that $\operatorname{ord}_x \varphi = n$ if $\varphi \in m_x^n$ and $\varphi \notin m_x^{n+1}$. Define $\nu = \min_i(\operatorname{ord}_x \varphi_i)$. Without loss of generality we may assume that $\operatorname{ord}_x \varphi_1 = \nu$.

Then there exists a vector field η (defined locally around x), that $\eta(\varphi_1) \neq 0$ and $\operatorname{ord}_x \eta(\varphi_1) < \nu$.

Let us apply such an η to the expression $\sum_i \varphi_i s_i = 0$. We will get

$$0 = \sum_i \eta(\varphi_i) s_i + \sum_i \varphi_i \eta(s_i).$$

Since s_i generates M_x we have $\eta(s_i) = \sum_j a_{ij} s_j$ for some a_{ij}. So, we have

$$0 = \sum_i \left(\eta(\varphi_i) + \sum_j \varphi_j a_{ji} \right) s_i.$$

The coefficient of s_1 in this sum is $\eta(\varphi_1) + \sum_j \varphi_j a_{j1}$. Since $\operatorname{ord}_x \eta(\varphi_1) < \nu$ and $\operatorname{ord}_x(\sum_j \varphi_j a_{ij}) \geq \nu$, this coefficient is non-zero and has order less than ν at x. By continuing the same process we shall get a relation between the s_i's with $\nu = 0$, and this means that there exist nontrivial linear relation between $\bar{s}_1, \ldots, \bar{s}_k$ ($\sum_i \bar{\varphi}_i \bar{s}_i = 0$, where $\bar{\varphi}_i$ is the image of φ_i in $\mathcal{O}_x/m_x$). □

Here is a very important corollary of this result.

COROLLARY 3.12. *Let M be a finitely generated $\mathcal{D}$-module. Then M is $\mathcal{O}$-coherent if and only if*

$$s.s.(M) = \{(x, 0) |\ x \in \mathbb{A}^n\}.$$

More canonically, if we identify $gr\mathcal{D}$ (with respect to the geometric filtration) with $\mathcal{O}(T^*\mathbb{A}^n)$ then M is $\mathcal{O}$-coherent if and only if $s.s.(M)$ is equal to the zero section in $T^*\mathbb{A}^n$.

Proof. Assume that $s.s.(M)$ is as above. Then it follows that for every good filtration F on M (with respect to the geometric filtration on $\mathcal{D}$) the module $gr^F M$ is finitely generated over $k[x_1, ..., x_n, \xi_1, ..., \xi_n]$ and all ξ_i act locally nilpotently on it. This implies that $gr^F M$ is finitely generated over $k[x_1, ..., x_n] = \mathcal{O}$. Hence M is also finitely generated over $\mathcal{O}$.

Conversely, assume that M is $\mathcal{O}$-coherent. Define a filtration on M by setting

$$F_j M = M \text{ for every } j \geq 0.$$

Then $gr^F M = M$ as an $\mathcal{O}$-module and all ξ_i act on $gr^F M$ by 0. By Theorem 3.11 we know that M is locally free over $\mathcal{O}$ hence $s.s.(M)$ is equal to the zero section. □

Here is an example:

Lemma 3.13. *Let δ_b be module of δ-functions at some $b \in \mathbb{A}^1$. Then $s.s.(\delta_b) = \{(x, \xi) \mid x = b\}$ and $s.s.^a(\delta_b) = \{(0, \xi)\}$.*

One can think about $\mathcal{D}$-modules as quasi-coherent sheaves on $\mathbb{A}^n$ ($\mathcal{O}$-modules) with an additional structure. Singular support somehow "measures" singularities of M.

3.14. Flat connections. Let us now study more carefully what kind of additional structure we need to introduce on a quasi-coherent sheaf so that it becomes a $\mathcal{D}$-module.

So let M be an $\mathcal{O}$-module. Any $\mathcal{D}$-module on M structure gives us a map $\nabla : M \to M \otimes_{\mathcal{O}} \Omega^1(\mathbb{A}^n)$, where $\Omega^1(\mathbb{A}^n)$ is the module of differential 1-forms. Namely, for eny vector field v on $\mathbb{A}^n$ and any $m \in M$ we have $\nabla(m)(v) = v(m)$. This map satisfies the following condition $\nabla(fm) = m \otimes df + f \cdot \nabla(m)$ for any $f \in \mathcal{O}$ and $m \in M$. Such a map ∇ is called a connection.

To formulate which connections arise from a $\mathcal{D}$-module structure let us define the notion of flat connection.

Given a connection ∇ define a map $\nabla^2 : M \to M \otimes_{\mathcal{O}} \Omega^2(\mathbb{A}^n)$ in the following way:

$$\nabla^2(m) = (\nabla \otimes 1)(\nabla(m)) + \nabla(m) \otimes d(w)$$

where d denotes the standard de Rham differential. It is easy to see that ∇^2 is an $\mathcal{O}$-linear map and thus can be thought of an element of $\mathrm{End}_{\mathcal{O}}(M) \underset{\mathcal{O}}{\otimes} \Omega^2$.

Definition 3.15. *Connection ∇ is called flat if $\nabla^2 = 0$.*

The following lemma is well-known and it is left to the reader.

Lemma 3.16. *Let M be an $\mathcal{O}$-module. Then a flat connection on M is the same as a $\mathcal{D}$-module structure.*

Let V be a finite dimensional vector space over k and $M = V \otimes_k \mathcal{O}$ as an $\mathcal{O}$-module. Let $d : M \to M \otimes \Omega^1$ be the de Rham differential. Then any connection ∇

has the form $\nabla = d + \omega$, where $\omega \in \mathrm{End}(V) \otimes \Omega^1$.

$$\omega : M \to M \otimes_{\mathcal{O}} \Omega^1 = V \otimes_k \Omega^1$$

Let $\omega = T \otimes \alpha$, where $T : V \to V$. Then for any $m = v \otimes f$ we have $\omega(m) = T(v) \otimes f\alpha$.

LEMMA 3.17. *∇ is a flat connection iff $d\omega + [\omega, \omega] = 0$.*

3.18. Poisson structures.

DEFINITION 3.19. *Let R be a commutative algebra. Poisson bracket $\{\cdot, \cdot\} : R \times R \to R$ on R is a Lie bracket satisfying the following condition*

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\}$$

for every $f_1, f_2, f_3 \in R$.

Let X be a smooth affine algebraic variety over k. If we forget about Jacobi identity, then a Poisson bracket on $\mathcal{O}(X) = R$ corresponds to $\eta \in \bigwedge^2 T_X$. (Jacobi identity gives us some identity on η). Any $\eta \in \bigwedge^2 T_X$ defines a map $\eta : T_X^* \to T_X$. If this map is an isomorphism, then there exists $\omega \in \bigwedge^2 T_X^* = \Omega_X^2$, corresponding to η. It is well-known that η satisfies the Jacobi identity if and only if ω is closed.

DEFINITION 3.20. *A closed non-degenerate 2-form ω is called a symplectic form.*

Example. Let Y be any smooth variety. Then $X = T^*Y$ is symplectic. For example let $Y = \mathbb{A}^n$, then $X = \mathbb{A}^{2n}$ with coordinates $x_1, \dots, x_n, \xi_1, \dots \xi_n$. In this case symplectic form is $\omega = \sum_i dx_i \wedge d\xi_i$ and for Poisson bracket we have $\{\xi_i, \xi_j\} = 0$, $\{x_i, x_j\} = 0$ and $\{\xi_i, x_j\} = \delta_{i,j}$.

Assume that A is a filtered algebra and there exists a fixed $l > 0$ such that $[F_i A, F_j A] \subset F_{i+j-l} A$. Then $\mathrm{gr}\, A$ is commutative and is endowed with a canonical Poisson bracket. Namely, for $\bar{x} \in \mathrm{gr}_i A$ and $\bar{y} \in \mathrm{gr}_j A$ let $x \in F_i A$ and $y \in F_j A$ be their preimages. Define $\{\bar{x}, \bar{y}\}$ as the image $\bar{z}$ of $[x, y]$ in $\mathrm{gr}_{i+j-l} A$.

The proof of the following lemma is left to the reader.

LEMMA 3.21.
(1) *$\bar{z}$ depends only on $\overline{x}$ and $\overline{y}$ (and does not depend on x and y).*
(2) *The assignment $\overline{x}, \overline{y} \mapsto \overline{z}$ is a Poisson bracket on $\mathrm{gr}\, A$.*

Examples.

(1) Bernstein's filtration.

In this case we can set $l = 2$, since $[F_i\mathcal{D}, f_j\mathcal{D}] \subset F_{i+j-2}\mathcal{D}$. The Poisson bracket defined as above coincide with the standard Poisson bracket on

$$\mathbb{A}^{2n} = \mathrm{Spec}(k[x_1, \dots x_n, \xi_1, \dots, \xi_n])$$

coming from the identification $\mathbb{A}^{2n} \simeq T^*\mathbb{A}^n$.

(2) Geometric filtration.

In this case $[\mathcal{D}_i, \mathcal{D}_j] \subset \mathcal{D}_{i+j-1}$ and $l = 1$. The Poisson bracket on $\mathbb{A}^{2n}$ is the same as before.

DEFINITION 3.22. *Assume X is a Poisson affine algebraic variety and $Z \subset X$ – a closed subvariety. Let $\mathcal{I}(Z) \subset \mathcal{O}(X)$ be the ideal of Z. Then Z is called coisotropic if*

$$\{\mathcal{I}(Z), \mathcal{I}(Z)\} \subset \mathcal{I}(Z).$$

LEMMA 3.23. *Let X be a Poisson affine algebraic variety, $Z \subset X$ – closed subvariety. Let $\eta : T^*X \to TX$ map corresponding to the Poisson bracket. For any $z \in Z$ we have $T_zZ \subset T_zX$ and $T_zZ^\perp \subset T_z^*X$.*

Then Z is coisotropic if and only if then $\eta(T_zZ^\perp) \subset T_zZ$ for any smooth point $z \in Z$.

Proof of this lemma is left to the reader.

LEMMA 3.24. *Suppose X is symplectic and $Z \subset X$ is coisotropic. Then $T(Z) \supset T(Z)^\perp$.*

Using this lemma, we get the following fact: if X is symplectic and $Z \subset X$ is coisotropic, then for any irreducible component Z_α of Z we have $\dim Z_\alpha \geq \frac{1}{2}\dim X$.

THEOREM 3.25. **[Gabber]** *Let A be a filtered algebra such that $[F_iA, F_jA] \subset F_{i+j-l}A$ for some l. Let M be an A-module with a good filtration F_iM. Let $I_F = \mathrm{Ann}(\mathrm{gr}^F M)$ (the annihillator of $\mathrm{gr}^F M$) in $\mathrm{gr}\, A$ and let $J(M) = \sqrt{I_F}$. Then*

(1) $\{I_F, I_F\} \subset I_F$

(2) *If $\mathrm{gr}\, A$ is noetherian then $\{J(M), J(M)\} \subset J(M)$.*

Let us note that the first assertion of the theorem is basically trivial. The second assertion is highly non-trivial and it will not be proved in these lectures.

Let us derive some corollaries from Gabber's theorem.

COROLLARY 3.26. *Let A be a filtered algebra such that $[F_iA, F_jA] \subset F_{i+j-l}A$ for some $l > 0$. Assume that $grA$is isomorphic to the algebra of functions on a smooth affine symplectic variety X (as a Poisson algebra).*

Let M be a finitely generated A-module endowed with a good filtration F. Then the dimension of every irreducible component of $\mathrm{supp}\,\mathrm{gr}^F M$ is greater or equal to $\frac{1}{2}\dim X$.

Example. Take $A = \mathcal{D}$ with Bernstein's filtration. Last time we've showed that $\dim(\mathrm{supp}\, gr^F M) \geq n = \frac{1}{2}\dim \mathbb{A}^{2n}$ (Bernstein's inequality). The above Corollary, however, is clearly a stronger statement. Also applying the same argument for the geometric filtration we also see that the dimension of every component of $s.s.(M)$ is greater or equal to n.

Remark. Gabber's theorem for $\mathcal{D}$ with geometric filtration was proved earlier by Malgrange.

Definition 3.27. *Assume that X is a smooth symplectic variety , $Z \subset X$ is coisotropic and $\dim Z_\alpha = \frac{1}{2}\dim X$ where Z_α is any irreducible component of Z. Then Z is called Lagrangian.*

Corollary 3.28. *Let M be a holonomic $\mathcal{D}$-module. Then both $s.s.(M)$ and $s.s.^a(M)$ are Lagrangian.*

REFERENCES?

4. Lecture 4 (02/12/02): Functional dimension and homological algebra

Let us pass to a different subject. Let M be a finitely generated $\mathcal{D}(\mathbb{A}^n)$-module, $d_a(M) = dim(s.s.^a(M))$ and $d_g(M) = dim(s.s.(M))$.

Theorem 4.1. $d_a(M) = d_g(M)$

In order to prove this theorem we shall need some results and constructions from homological algebra which we now briefly recall. We shall study these things in much more detail later when we discuss derived categories.

4.2. Complexes. Let A be any ring. Recall that a *complex* of (left) A-modules is the following data:

- A (left) A-module M^i for each $i \in \mathbb{Z}$
- A homomorphism $\partial_i : M_{i-1} \to M_i$ for each $i \in \mathbb{Z}$ such that for all i we have

$$\partial_i \circ \partial_{i-1} = 0.$$

When it does not lead to a confusion we shall write ∂ instead of ∂_i.

One can also define a *bicomplex* as a collection M^{ij} $(i, j \in \mathbb{Z})$ of A-modules with differentials $\partial^1_{ij} : M^{ij} \to M^{i+1,j}$ and $\partial^2_{ij} : M^{ij} \to M^{i,j+1}$ satisfying $\partial^1_{ij} \circ \partial^1_{i-1,j} = 0$, $\partial^2_{ij} \circ \partial^2_{i,j-1} = 0$ and $\partial^2_{i+1,j} \circ \partial^1_{ij} = \partial^1_{i,j+1} \circ \partial^2 ij$. In this case one can define the *total complex* $\mathrm{Tot}(M^\bullet)$ of the bicomplex $M^\bullet$ by setting

$$\mathrm{Tot}^k(M^\bullet) = \bigoplus_{i+j=k} M^{ij}, \quad \partial_k = \bigoplus_{i+j=k} \partial^1_{ij} + (-1)^j \partial^2_{ij}.$$

4.3. Left exact and right exact functors. Let A-mod denote the category of left A-modules. Let $F : A\text{-mod} \to \mathfrak{Ab}$ be a an additive functor (here $\mathfrak{Ab}$ denotes the category of abelian groups).

Definition 4.4.
1. *A functor F is called left exact if for any short exact sequence of A-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ the sequence $0 \to F(M_1) \to F(M_2) \to F(M_3)$ is exact. (If F is a contravariant left exact functor, then the short exact sequence of modules goes to the exact sequence $0 \to F(M_3) \to F(M_2) \to F(M_1)$.)*
2. *Functor F is called right exact if for any short exact sequence of A-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ we have an exact sequence of $F(M_1) \to F(M_2) \to F(M_3) \to 0$.*

(3) *A functor F is called exact if it both left and right exact.*

Example. The functor $N \mapsto \mathrm{Hom}_a(M, N)$ is left exact. Also the functor $N \mapsto \mathrm{Hom}_A(N, M)$ is a contravariant left exact functor.

In some sense the main point of homological algebra is to "correct" non-exactness of certain functors. This is usually done by means of the following construction.

DEFINITION 4.5. (1) *A-module P is called projective if $Hom(P, \cdot)$ is an exact functor.*

(2) *An A-module I is called injective if $Hom(\cdot, I)$ is exact.*

It is easy to see that free modules are projective.

DEFINITION 4.6. *Projective resolution of a module M is a complex $P^\bullet$ of projective modules*

$$\cdots \to P^{-2} \xrightarrow{\partial_{-1}} P^{-1} \xrightarrow{\partial_0} P^0 \to 0$$

such that $P^0/Im(\partial_0) = M$ and $H^{-i}(P^\bullet) = Ker(\partial_{-i+1})/Im(\partial_{-i}) = 0$ for any $i > 0$.

Recall that $\mathrm{Ext}^i(M, N)$ is defined as follows: let $P^\bullet$ be a projective resolution of M. Then $\mathrm{Hom}(P^\bullet, N)$ is also a complex

$$0 \to \mathrm{Hom}(P^0, N) \to \mathrm{Hom}(P^{-1}, N) \to \mathrm{Hom}(P^{-2}, N) \to \dots$$

By definition $\mathrm{Ext}^i(M, N)$ is the i-th cohomology of this complex.

THEOREM 4.7. (1) *$Ext^i(M, N)$ is a functor in both variables (in particular, it does not depend on the choice of $P^\bullet$).*

(2) *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of modules. Then we have a long exact sequence of Ext groups:

$$... \to \mathrm{Ext}^i(M_3, N) \to \mathrm{Ext}^i(M_2, N) \to \mathrm{Ext}^i(M_1, N) \to \mathrm{Ext}^{i+1}(M_3, N) \to ...$$

Similarly, for a short exact sequence

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

we have a long exact sequence

$$... \to \mathrm{Ext}^i(M, N_1) \to \mathrm{Ext}^i(M, N_2) \to \mathrm{Ext}^i(M, N_3) \to \mathrm{Ext}^{i+1}(M, N_1) \to ...$$

One can think about the second assertion of Theorem 4.7 as the statement which "compensates" the non-exactness of the Hom functor.

In fact $\mathrm{Ext}(M, N)$ makes sense not only for modules but also for complexes. Namely, suppose $M^\bullet$ and $N^\bullet$ are bounded complexes (i.e. collection of $\{M^i\}$ $i \in \mathbb{Z}$, where $M^i = 0$ for $\mid i \mid \gg 0$ and maps $\partial : M^i \to M^{i+1}$). There exists a complex of projective modules $P^\bullet$ and a map of complexes $\alpha : P^\bullet \to M^\bullet$ such that α induces isomorphism on cohomologies.

Define a bicomplex $\mathrm{Hom}(P^\bullet, N^\bullet)$:

$$\begin{array}{ccccccc} & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & \mathrm{Hom}(P^{-i}, N^{j+1}) & \longrightarrow & \mathrm{Hom}(P^{-i-1}, N^{j+1}) & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & \mathrm{Hom}(P^{-i}, N^{j}) & \longrightarrow & \mathrm{Hom}(P^{-i-1}, N^{j}) & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & \mathrm{Hom}(P^{-i}, N^{j-1}) & \longrightarrow & \mathrm{Hom}(P^{-i-1}, N^{j-1}) & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \end{array}$$

(by the definition $\mathrm{Hom}^{ij}(P^\bullet, N^\bullet) = \mathrm{Hom}(P^{-i}, N^j)$).

If we have any bicomplex $\{C^{ij}\ i, j \in \mathbb{Z}\}$ with differentials ∂^1 and ∂^2 which commutes, we can define a total complex $K^\bullet = \mathrm{Tot}(C^{ij})$, where $K^p = \bigoplus_{i+j=p} C^{ij}$ (here we assume that in each summation only finitely many summands are non-zero). There is a way to define differential on this complex. By definition $\mathrm{Ext}^i(M^\bullet, N^\bullet) = H^i(\mathrm{Tot}(\mathrm{Hom}(P^\bullet, N^\bullet)))$.

THEOREM 4.8. *Let A be a filtered algebra over k such that grA is a finitely generated commutative regular algebra of dimension m (regular means the algebra of functions on a smooth affine algebraic variety). Let M be any finitely generated left A-module. Define $d(M) = \dim(\mathrm{supp}\, gr^F M)$, where F is any good filtration on M and $j(M) = \min(j \mid \mathrm{Ext}^j(M, A) \neq 0)$. Then*

(1) $d(M) + j(M) = m$;
(2) $\mathrm{Ext}^j(M, A)$ *is a finitely generated right A-module and* $\dim(\mathrm{Ext}^j(M, A)) \leq m - j$;
(3) *for $j = j(M)$ we have an equality in 2.*

Let's consider $A = \mathcal{D}$ with either filtration.

COROLLARY 4.9. *$d_a(M) = d_g(M) = 2n - j(M)$ for any finitely generated $\mathcal{D}$-module M.*

From now on we set $d(M) = d_a(M) = d_g(M)$.

COROLLARY 4.10. *M is holonomic $\Leftrightarrow$ $\mathrm{Ext}^j(M, \mathcal{D}) \neq 0$ only for $j = n$.*

PROOF. If M is holonomic, then $d(M) = n$ and hence $j(M) = n$. So we have $\mathrm{Ext}^j(M, \mathcal{D}) = 0$ for $j < n$ be definition of $j(M)$. For $j > n$ $\mathrm{Ext}^j(M, \mathcal{D}) = 0$ (by the second part of the theorem it's dimension is less or equal to $2n - j < n$ and by Bernstein's inequality it's 0).

□

As we have seen before $\mathrm{Ext}^n(M, \mathcal{D})$ has a natural structure of right $\mathcal{D}$-module. In fact $\mathcal{D} = \mathcal{D}(\mathbb{A}^{2n})$ has a natural antiinvolution σ (i.e. $\sigma : \mathcal{D} \to \mathcal{D}$ such that $\sigma(d_1 d_2) = \sigma(d_2)\sigma(d_1)$) namely $x_i \mapsto x_i$ and $\frac{\partial}{\partial x_j} \mapsto -\frac{\partial}{\partial x_j}$. And hence every right module is also a left module.

Remark. The existence of the above involution is an "accidental" fact, i.e. when we replace $\mathbb{A}^n$ by a general algebraic variety it will not exist anymore. However, we shall see later (when we discuss general varieties) that the existence of a canonical equivalence between the categories of left and right $\mathcal{D}$-modules is not accidental.

COROLLARY 4.11. *For any holonomic module M let $\mathbb{D}(M)$ be $\mathrm{Ext}^n(M,\mathcal{D})$ considered as a left module. Then $M \mapsto \mathbb{D}(M)$ is an exact contravariant functor from the category of holonomic modules to itself and $\mathbb{D}(\mathbb{D}(M)) = M$.*

PROOF. The fact that $\mathbb{D}$ is exact follows immediately from the long exact sequence of Ext's.

Let us show that $\mathbb{D}^2 \simeq \mathrm{Id}$. Let P be a finitely generated projective $\mathcal{D}$-module. Let $P^\vee = \mathrm{Hom}(P,\mathcal{D})$ be the dual module (considered as a left module as before). It is clear that $P^\vee$ is projective. If $P^\bullet$ is a complex of projective $\mathcal{D}$-modules then we shall denote by $(P^\vee)^\bullet$ the complex defined by

$$(P^\vee)^i = (P^{-i})^\vee$$

with the obvious differential.

Let M be a holonomic $\mathcal{D}$-module and let $P^\bullet$ be its projective resolution. Then it is clear that $(P^\vee)^\bullet[n]$ is a projective resolution of $\mathbb{D}(M)$. Thus $(P^\vee[n])^\vee[n] = P$ is a projective resolution of $\mathbb{D}(\mathbb{D}(M))$. Hence $\mathbb{D}(\mathbb{D}(M)) = M$. □

COROLLARY 4.12. *Let M be $\mathcal{O}$-coherent. Then $\mathbb{D}(M) = \mathrm{Hom}_{\mathcal{O}}(M,\mathcal{O}) = M^\vee$ as an $\mathcal{O}$-module (i.e. $\mathbb{D}(M)$ is a dual vector bundle). The dual connection is described in the following way: Let $\nabla_M : M \to M \otimes_{\mathcal{O}} \Omega^1$. Then for $\nabla_{M^\vee} : M^\vee \to M^\vee \otimes_{\mathcal{O}} \Omega^1$ we have $\nabla_{M^\vee}(\xi)(m) = -\xi(\nabla_M(m)) \in \Omega^1$ for every $\xi \in M^\vee$ and $m \in M$.*

The proof will be given next time.

Example. Let $M = V \otimes \mathcal{O}$ be a trivial $\mathcal{O}$-module and ∇_M is given by $\omega_M \in \mathrm{End}(V) \otimes \Omega^1$. Then $\omega_{M^\vee} \in \mathrm{End}(V^*) \otimes \Omega^1$ is $(-\omega_M)^T$ (here the T-superscript denotes the transposed matrix).

5. Lecture 5 (02/14/02)

In this lecture we want to prove Theorem 4.8 and Corollary 4.12. Let us start with the latter. To do this we'll need to develope some technology.

To prove this statement let us introduce the de Rham complex. Let M be any left $\mathcal{D}$-module. Let us form a complex

$$\begin{array}{ccccccccccc} 0 & \to & M & \to & M \otimes_{\mathcal{O}} \Omega^1 & \to & M \otimes_{\mathcal{O}} \Omega^2 & \to & \dots & \to & M \otimes_{\mathcal{O}} \Omega^n & \to & 0 \\ & & -n & & -n+1 & & -n+2 & & & & 0 & & \end{array}$$

(the second row shows the cohomological degress). Differentials in this complex $d : M \otimes_{\mathcal{O}} \Omega^k \to M \otimes_{\mathcal{O}} \Omega^{k+1}$ are defined as follows. We have a flat connection $\nabla : M \to M \otimes_{\mathcal{O}} \Omega^1$. For any $m \otimes \omega \in M \otimes_{\mathcal{O}} \Omega^k$ we set

$$d(m \otimes \omega) = m \otimes d(\omega) + \nabla(m) \wedge \omega.$$

From the condition of flatness it follows, that this is really a complex, i.e. $d^2 = 0$.
Remark. Let $n = 1$. Then for any M the corresponding de Rham complex $dR(M)$ has a form $0 \to M \xrightarrow{\nabla} M \otimes_{\mathcal{O}} \Omega^1 \to 0$.

In particular one can consider the complex $dR(\mathcal{D})$. Note that $dR(\mathcal{D})$ is a complex of right $\mathcal{D}$-modules ($\mathrm{End}_{\mathcal{D}} M$ acts on $dR(\mathcal{D})$ and in particular $\mathcal{D}^{op}$ acts on $dR(\mathcal{D})$).

LEMMA 5.1. $H^i(dR(\mathcal{D})) = \begin{cases} 0 & \textit{if } i \neq 0; \\ \Omega^n & \textit{if } i = 0. \end{cases}$

To prove this lemma let us look at the complex $dR(\mathcal{D})$ more closely. For any $0 \leq i \leq n \quad (dR(\mathcal{D}))^{-i} = \mathcal{D} \otimes_{\mathcal{O}} \Omega^{n-i}$. In turn, $\Omega^j = \mathcal{O} \otimes \Lambda^j(k^n)$. So the de Rahm complex for $\mathcal{D}$ has a form

$$0 \to \mathcal{D} \to \mathcal{D} \otimes_k k^n \to \cdots \to \mathcal{D} \otimes_k \Lambda^j(k^n) \to \cdots \to \mathcal{D} \otimes_k \Lambda^n(k^n) \to 0$$

In case of $\mathbb{A}^n$ this complex coincides with Koszul complex of M. Let us briefly recall this notion.

5.2. Koszul complex. Let N be module over $k[y_1, \ldots, y_n]$. There exists a natural complex, associated with N, called Koszul complex

$$0 \to N \to N \otimes k^n \to \cdots \to N \otimes \Lambda^j(k^n) \to \cdots \to N \otimes \Lambda^n(k^n) \to 0.$$

The map $N \to N \otimes k^n$ is given by $n \mapsto \oplus_i y_i(n)$, the map $N \otimes k^n \to N \otimes \Lambda^2(k^n)$ is given by $(p_1, \ldots, p_n) \mapsto \oplus(y_i p_j - y_j p_i)$, all the other maps are defined in the same way. We shall denote this complex by $Kos(M)$.

LEMMA 5.3. *If M is free over $\mathcal{O}$ then Koszul complex, corresponding to M then $Kos(M)$ is a free resolution of $M/(< y_1, \ldots, y_n > \cdot M)$, i.e. for $i \neq 0$ $H^i(Kos(M)) = 0$ and $H^0(Kos(M)) = M/(< y_1, \ldots, y_n > \cdot M)$.*

To see, that $H^0(Koszul(M)) = M/(< y_1, \ldots, y_n > \cdot M)$, one should look at the map $N \otimes \Lambda^{n-1}(k^n) \xrightarrow{d} N \otimes \Lambda^n(k^n) \to 0$. $\Lambda^{n-1}(k^n)$ is isomorphic to k^n and d maps $(p_1, \ldots, p_n)$ to $\sum_{i=1}^n y_i p_i$.

Observation. Let M be any $\mathcal{D}$-module, $dR(M)$ – de Rahm complex of M. Then M has a structure of $k[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$-module, since $\frac{\partial}{\partial x_j} \in \mathcal{D}$. Koszul complex, corresponding to this module structure is the same as $dR(M)$.

Example. Let $n = 2$. The de Rham complex has a form

$$0 \to M \to M \otimes \Omega^1 \to M \otimes \Omega^2 \to 0.$$

Here the first map is $m \mapsto \frac{\partial m}{\partial x_1} dx_1 + \frac{\partial m}{\partial x_2} dx_2$, and the second $p_1 dx_1 + p_2 dx_2 \mapsto \left(\frac{\partial p_2}{\partial x_1} - \frac{\partial p_1}{\partial x_2} \right) dx_1 \wedge dx_2$. It's easy to see that these maps coincide with differentials in Koszul complex.

Corollary 5.4. *If M is free over $k[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]$ then $H^i(dR(M)) = 0$ for any $i \neq 0$ and $H^0(dR(M)) = M/(< \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} > \cdot M)$.*

In particular we can take $M = \mathcal{D}$.
Then $H^0(dR(\mathcal{D})) = \mathcal{D}/(< \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} > \cdot \mathcal{D})$. We can naturally identify it with Ω^n, but since we have chosen coordinates in $\mathbb{A}^n$ this is the same as $\mathcal{O}$. So $dR(\mathcal{D})$ is a free resolution of $\mathcal{O}$.

5.5. Projective resolution of any $\mathcal{O}$-coherent module. Let M and N be left $\mathcal{D}$-modules. Then $M \otimes_{\mathcal{O}} N$ is also a left $\mathcal{D}$-module. Derivatives are acting on $M \otimes_{\mathcal{O}} N$ by Leibnitz rule: $\frac{\partial}{\partial x_i}(m \otimes n) = \frac{\partial m}{\partial x_i} \otimes n + m \otimes \frac{\partial n}{\partial x_i}$.

Given M we can consider $M \otimes_{\mathcal{O}} dR(\mathcal{D})$. Since M is $\mathcal{O}$-coherent, it's locally free and hence projective.

Fact (from commutative algebra). Let R be any commutative ring, P – projective module over R. Then $N \mapsto P \otimes_R N$ is an exact functor.

Using this fact, we get $H^i(M \otimes_{\mathcal{O}} dR(\mathcal{D})) = 0$ for $i \neq 0$ and $H^0(M \otimes_{\mathcal{O}} dR(\mathcal{D})) = M \otimes_{\mathcal{O}} \mathcal{O} = M$. So, $M \otimes_{\mathcal{O}} dR(\mathcal{D})$ is a projective (over $\mathcal{D}$) resolution of M. (It is projective over $\mathcal{D}$, because if M is projective over $\mathcal{O}$, then $M \otimes \mathcal{D}$ is projective over $\mathcal{D}$ since $\mathrm{Hom}_{\mathcal{D}}(M \otimes_{\mathcal{O}} \mathcal{D}, N) = \mathrm{Hom}_{\mathcal{O}}(M, N)$ and exactness of one of these functors is equivalent to the exactness of the other.)

Let us compute duality using this resolution.
Consider thr complex $\mathrm{Hom}_{\mathcal{D}}(M \otimes dR(\mathcal{D}), \mathcal{D})$ which looks like

$$0 \to \mathrm{Hom}_{\mathcal{D}}(M \otimes dR^0(\mathcal{D}), \mathcal{D}) \to \cdots \to \mathrm{Hom}_{\mathcal{D}}(M \otimes dR^{-n}(\mathcal{D}), \mathcal{D}) \to 0$$

We claim that $\mathrm{Hom}_{\mathcal{D}}(M \otimes dR(\mathcal{D}), \mathcal{D}) = M^\vee \otimes dR(\mathcal{D})[-n]$ (by $dR(\mathcal{D})[-n]$ we mean the complex, shifted by n to the right).

$\mathrm{Hom}_{\mathcal{D}}(M \otimes dR(\mathcal{D}), \mathcal{D}) = \mathrm{Hom}_{\mathcal{D}}(M \otimes_{\mathcal{O}} \mathcal{D} \otimes \Omega^{n-i}, \mathcal{D}) = M^\vee \otimes_{\mathcal{O}} \mathcal{D} \otimes \mathrm{Hom}(\Omega^{n-i}, \mathcal{O})$.
But $\mathrm{Hom}(\Omega^{n-i}, \mathcal{O}) \cong \Omega^i$, if an identification of $\mathcal{O}$ and Ω^n is chosen (as in our case). And the differential of this complex is exactly the one of $M^\vee \otimes dR(\mathcal{D})$.

Example. Let $n = 1$, M be $\mathcal{O}$-coherent sheaf of rank 1 ($M \cong \mathcal{O}$ as an $\mathcal{O}$-module). Let α be a one form, such that the flat connection, corresponding to the $\mathcal{D}$-module structure, has a form $\nabla(f) = df + f\alpha$.

The corresponding de Rham complex has the form $0 \to \mathcal{D} \to \mathcal{D} \to 0$, where the map $\mathcal{D} \to \mathcal{D}$ is given by right multiplication by $d - \alpha$ (i.e. if we've chosen coordinate x and in this coordinate $\alpha = g(x)dx$ then $d - \alpha$ means $\frac{d}{dx} - g(x)$).

As $\mathcal{O}$-modules M and $M^\vee$ are the same. As $\mathcal{D}$-modules they correspond to forms α and $-\alpha$. Thus the above complex shows that $\mathbb{D}(M) = M^\vee$. **Exercise.** Let $M = \delta_a$, where $a \in k$. Prove that $\mathbb{D}(\delta_a) = \delta_a$.

Let us prove Theorem 4.8. We shall need the following fact from commutative algebra.

Theorem 5.6. *[Serre] Let R be a regular algebra of dimension m and let M be a finitely generated R-module. Set $d(M) = \dim(\operatorname{supp} M)$ and $j(M) = \min(j \mid \operatorname{Ext}^j(M, R) \neq 0)$. Then*

1. *$d(M) + j(M) = m$;*
2. *$\operatorname{Ext}^j(M, R)$ is a finitely generated R-module and $\dim(\operatorname{Ext}^j(M, R)) \leq m - j$;*
3. *for $j = j(M)$ we have an equality in 2.*

We are going to apply this theorem to $R = \operatorname{gr} A$.

Lemma 5.7. *For any filtered A-module M there exists a filtered resolution $P^\bullet$ of M such that $\operatorname{gr} P^\bullet$ is a free resolution of $\operatorname{gr} M$.*

Proof. Choose homogeneous generators $(\bar{m}_i)_{i \in I}$ of $\operatorname{gr} M$ (so that each m_i is of some degree k_i). The map $R^I \to M$ is a surjective map of graded modules if we shift the grading on i-th component by k_i.

Lift $\bar{m}_1, \ldots, \bar{m}_k$ to $m_1, \ldots, m_k$, where $m_j \in F_{i_j} M$. Earlier we've proved that $\{m_i\}$ generates M. So, we get a map $A^k \to M$ which is also a map of filtered modules if we shift the filtration on j-th component by i_j.

We have two surjective maps $\alpha_0 : A^I \twoheadrightarrow M$ and $\beta_0 : R^I \twoheadrightarrow M$. By construction gr $\alpha_0 = \beta_0$.

Let N be the kernel of α_0. We can repeat the same construction for N and get $\alpha_1 : A^J \twoheadrightarrow N$ such that gr $\alpha_1 = \beta_1 : R^J \twoheadrightarrow N$. So we shall have two exact sequences $A^l \xrightarrow{\alpha_1} A^k \xrightarrow{\alpha_0} M \to 0$ and $B^l \xrightarrow{\beta_1} B^k \xrightarrow{\beta_0} grM \to 0$.

Repeating the same construction we shall get eventually a filtered resolution of M whose associated graded complex is a free resolution of $\operatorname{gr} M$.

Let $K^\bullet$ be a complex with filtration $\{F_i K^\bullet\}$ (this means that $d(F_i K^p) \subset F_i K^{p+1}$). Then $gr^F K^\bullet$ is also a graded complex: $gr^F K^\bullet = \oplus gr^F_j K^\bullet$, where $gr^F_j K^\bullet$ is a complex $\{\cdots \to gr^F_j K^p \to gr^F_j K^{p+1} \to \ldots\}$. So we have a set of groups $H^i(\operatorname{gr}^F_j K^\bullet)$.
Homologies of the original complex $\{\cdots \to K^{i-1} \xrightarrow{d} K^i \xrightarrow{d} K^{i+1} \to \ldots\}$ also inherits a filtration. So we have another set of groups $\operatorname{gr}^F_j(H^i(K^\bullet))$.

The following lemma is well-known in homological algebra.

Lemma 5.8.

1. *$\operatorname{gr}^F_j(H^i(K^\bullet))$ is a subquotient (i.e. a quotient of the subspace) of $H^i(\operatorname{gr}^F_j K^\bullet)$.*
2. *Assume that there exists some $n \in \mathbb{Z}$ such that $H^i(\operatorname{gr} K^\bullet) = 0$ for $i > n$. Then $\operatorname{gr}^F_j(H^n(K^\bullet) = H^n(\operatorname{gr}_j K^\bullet)$ for all j.*

It is clear that Lemma 5.8 and Theorem 5.6 imply Theorem 4.8.

6. Lecture 7 (02/21/02): $\mathcal{D}$-modules on general affine varieties

Let X be an affine algebraic variety over a field k of characteristic 0. We would like to define the algebra of differential operators on X which we denote by $\mathcal{D}_X$.

Definition 6.1. (1) *Let R be a commutative ring. Define the filtered algebra of differential operators on R inductively: $D_0(R) = R$ acting by multiplication. For $d : R \to R$ we say that $d \in D_n(R)$ if for any element $r \in R$ $[d, r] \in D_{n-1}(R)$; $D(R) = \cup D_n(R)$. Elements of $D_n(R)$ are called differential operators of order less or equal to n.*

(2) *Let M and N be two R-modules. Define the space of differential operators $Diff(M, N) = \cup Diff_n(M, N)$ from M to N in the following way:*
- $Diff_0(M, N) = \mathrm{Hom}_R(M, N)$
- *$d \in Diff_n(M, N)$ iff for any $r \in R$ we have $[d, r](m) = d(rm) - r(dm) \in Diff_{n-1}(M, N)$.*

We set $\mathcal{D}_X = \mathcal{D}(\mathcal{O}_X)$.

This definition makes sense for arbitrary X but we shall work with it only when X is a smooth variety.

Note that $\mathcal{D}_X$ has two natural structures of a $\mathcal{O}_X$-module. Let U be an affine open subset of X. It is easy to see that $\mathcal{D}_U \simeq \mathcal{O}_U \otimes \mathcal{D}_X \simeq \mathcal{D}_X \otimes \mathcal{O}_U$. In other words the two quasi-coherent sheaves on X (coming from $\mathcal{D}_X$ considered as an $\mathcal{O}_X$-module with either $\mathcal{O}_X$-module structure) are canonically isomorphic. Therefore, it makes sense to talk about one sheaf of differential operators on X (this remark will become especially important when we start discussing non-affine varieties).

Remark.

- For $X = \mathbb{A}^n$ we get the previous definition of $\mathcal{D}$ with geometric filtration.
- There exists a correct notion of $\mathcal{D}$-modules on singular variety X. We shall discuss it in the next lecture.

Let $U \subset X$ be an open subset of X and $\dim X = n$.

Definition 6.2. *A coordinate system on U is a set of functions $x_1, \ldots, x_n$, $x_i \in \mathcal{O}_U$ and a set of vector fields $\partial_1, \ldots, \partial_n$ such that $\partial_i(x_j) = \delta_{i,j}$ (i.e. an etale map $U \to \mathbb{A}^n$).*

Lemma 6.3. *For any point $x \in X$ there exist a neighborhood $U \ni x$ such that on U there exist a coordinate system.*

Assume that there exist a coordinate system on all of X.

Lemma 6.4. *We have $\mathcal{D}_X = \oplus_\alpha \mathcal{O}_X \partial^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{Z}_+$ and $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$.*

Corollary 6.5. *$\mathcal{D}_X$ is generated by $\mathcal{O}_X$ and Vec_X, where Vec_X is the $\mathcal{O}_X$-module of vector fields on X.*

Proof. This is true locally and thus globally since X is affine. □

Corollary 6.6. *$\mathcal{D}_X$ is filtered and $\mathrm{gr}(\mathcal{D}_x) \cong \mathcal{O}_{T^*X}$.*

Proof. Let us define a map $\varphi : \mathrm{gr}(\mathcal{D}_X) \to \mathcal{O}_{T^*X}$. It's easy to see that $\mathrm{gr}(\mathcal{D}_X)$ is generated by $gr_1(\mathcal{D}_X)$ over $\mathrm{gr}_0(\mathcal{D}_X)$. Let $gr_0(\mathcal{D}_X) = \mathcal{O}_X \hookrightarrow \mathcal{O}_{T^*X}$, $gr_1(\mathcal{D}_X) = Vec_X \to \mathcal{O}_{T^*X}$ be natural maps. Locally φ is defined by these maps and is an isomorphism. Since X is affine, this is true globally. □

6.7. Left and right modules over $\mathcal{D}_X$. Let $\mathcal{M}^l(\mathcal{D}_X)$ denote left $\mathcal{D}_X$-modules and $\mathcal{M}^r(\mathcal{D}_X)$ – right $\mathcal{D}_X$-modules.

Let M be an $\mathcal{O}_X$-module. To define the left $\mathcal{D}_X$-module structure on M we have to describe the action of the Lie algebra Vec_X on M such that for any $\partial \in Vec_X$ and any $f \in \mathcal{O}_X$ we have $(f\partial)(m) = f(\partial(m))$.

The right $\mathcal{D}_X$-module structure is the same as an action of vector fields satisfying the following conditions:

(1) $(\partial_1 \cdot \partial_2 - \partial_2 \cdot \partial_1)(m) = -[\partial_1, \partial_2](m)$ for any vector fields ∂_1 and ∂_2
(2) $(f\partial)(m) = \partial(fm)$ for any $f \in \mathcal{O}_X$ and any $\partial \in \mathcal{D}_X$.

Lemma 6.8. *$\Omega^n(X)$ has canonical structure of right $\mathcal{D}_X$-module structure.*

Proof. In fact we claim that the action of vector fields on $\Omega^n(X)$ by $-Lie_\partial$ satisfies the properties listed above.

By Cartan formula $Lie_\partial(\omega) = d(\iota_\partial(\omega)) - \iota_\partial(d(\omega))$. For $\omega \in \Omega^n(X)$ the last term is equal to 0 and since $d(\iota_{f\partial})(\omega) = d(\iota_\partial(f\omega))$ $\Omega^n(X)$ is a right $\mathcal{D}_X$-module. Functions on X form a left $\mathcal{D}_X$-module, because vector field acts by $Lie_\partial(g) = \iota_\partial(dg)$ (by Cartan formula) and $\iota_{f\partial}(dg) = f\iota_\partial(dg)$. □

Lemma 6.9. *Let M be a left $\mathcal{D}_X$-module. Then $M \otimes \Omega^n(X)$ has a structure of right $\mathcal{D}_X$-module given by*

$$\partial(m \otimes \omega) = \partial m \otimes \omega - m \otimes Lie_\partial(\omega).$$

Similarly if M is a right $\mathcal{D}_X$-module, then $M \otimes (\Omega^n(X))^{-1}$ has a structure of left $\mathcal{D}_X$-module.

Exercise. Describe the action of ∂ on $M \otimes (\Omega^n(X))^{-1}$.

Corollary 6.10. *The categories $\mathcal{M}^l(\mathcal{D}_X)$ and $\mathcal{M}^r(\mathcal{D}_X)$ are canonically equivalent. The equivalence is given by $M \mapsto M \otimes \Omega^n(X)$.*

We shall use the notation $\mathcal{M}(\mathcal{D}_X)$ for this category.

Let $M \in \mathcal{M}(\mathcal{D}_X)$ be finitely generated. There exist a good filtration on M and $s.s.(M) = supp(\mathrm{gr}\, M)$ is a closed subset of T^*X which doesn't depend on the filtration. By Gabber's theorem $\dim s.s.(M) \geq n$. As in the case of $\mathbb{A}^n$ a $\mathcal{D}_X$-module M is called holonomic if $\dim s.s.(\mathrm{gr}\, M) = n$.

Assume that M is holonomic. In this case we can define a cycle $s.c.(M)$ (the singular cycle of M) in the following way: assume that $Z_1, ..., Z_k$ are the irreducible components of $supp(grM)$ (note that Gabber's theorem implies that the dimension of each Z_i is equal to n). Define m_i to be the rank of $\mathrm{gr}\, M$ at the generic point of

Z_i. Then we define $s.c.(M) = \sum m_i Z_i$. It is easy to see that $s.c.(M)$ doesn't depend on the choice of a good filtration. Moreover, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of holonomic modules then $s.c.(M_2) = s.c.(M_1) + s.c.(M_3)$.

LEMMA 6.11. *Holonomic modules have finite length.*

PROOF. It is easy to see by induction that if $s.c.(M) = \sum m_i Z_i$ then the length of M does not exceed $\sum m_i$. □

Inverse and direct image functors.

6.12. Inverse image. Let $\pi : X \to Y$ be a morphism of affine algebraic varieties. Then for any $f \in C^\infty(Y)$ we can consider a new function $f \circ \pi \in C^\infty(X)$. We would like to have an analogous construction for $\mathcal{D}$-modules.

DEFINITION 6.13. *Let M be a left $\mathcal{D}_Y$-module. The inverse image of M is the left $(\mathcal{D}_X)$-module $\pi^0(M) = \mathcal{O}_x \otimes_{\mathcal{O}_Y} M$ as an $\mathcal{O}_X$-module with the following action of vector fields: for any $\partial \in Vec_X$ $\quad \partial(f \otimes m) = \partial(f) \otimes m + f \otimes \pi_*(\partial)m$.*

Let $\mathcal{D}_{X\to Y}$ be the inverse image of $\mathcal{D}_Y$ This is a left $\mathcal{D}_X$-module and right $\mathcal{D}_Y$-module.

LEMMA 6.14. $\pi^0 M = \mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} M$.

PROOF. For any module M there exist k and l (each of them may by equal to infinity), such that the sequence $\mathcal{D}_Y^l \to \mathcal{D}_Y^k \to M \to 0$ is exact. The functor π^0 is right exact, so we have an exact sequence

$$\pi^0 \mathcal{D}_Y^l \to \pi^0 \mathcal{D}_Y^k \to \pi^0 M \to 0.$$

For $M = \mathcal{D}_Y$ the claim is true. Thus it is true also for an arbitrary M.

Another way to see it:

$$\mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} M = (\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y) \otimes_{\mathcal{D}_Y} M = \mathcal{O}_X \otimes_{\mathcal{O}_Y} M.$$

□

Suppose we have a coordinate system $y_1, \dots y_m, \quad \partial_1, \dots, \partial_m$ on Y. Then $\mathcal{D}_Y = \oplus_\alpha \mathcal{O}_Y \partial^\alpha$ and $\mathcal{D}_{X\to Y} = \oplus_\alpha \mathcal{O}_X \partial^\alpha$, where α is a multi-index.

6.15. Direct image. Let $\pi : X \to Y$. For every distribution f with compact support on X we can construct $\pi_0(f)$, taking integral of f over the fibers of π. Let us define an analogous operation on modules. Since distributions intuitively correspond to right $\mathcal{D}$-modules it will be easier to spell out the definition of direct image for right modules. Since we have canonical equivalence between the categories of left and right modules the definition will make sense for left modules as well.

DEFINITION 6.16. *For any $\pi : X \to Y$ we define the functor $\pi_0 : \mathcal{M}^r(\mathcal{D}_X) \to \mathcal{M}^r(\mathcal{D}_Y)$ defined by $\pi_0(M) = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$.*

Remark. "0" means that the functors are not yet derived.

Example. Let $X = \{0\}$ and $Y = \mathbb{A}^1$ and $\pi : X \hookrightarrow Y$. Consider k as a $\mathcal{D}_X$-module.

$$\pi_0 k = \mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y = \bigoplus_{n\geq 0} k\left(\frac{\partial}{\partial x}\right)^n = \mathcal{D}_{\mathbb{A}^1}/(x\cdot \mathcal{D}_{\mathbb{A}^1}) = \delta_0$$

as a right $\mathcal{D}_Y$-module.

Since the categories of right and left $\mathcal{D}$-modules are canonically equivalent, we can define direct and inverse images for both left and right modules. For example, let M be the left $\mathcal{D}_X$-module. Then

$$\pi_0(M) = (M \otimes \Omega^n(X) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \otimes (\Omega^m(Y))^{-1}.$$

Example. Let $X = \mathbb{A}^1$ and $Y = pt$. Then for any right $\mathcal{D}_X$-module M we have $\pi_0 M = M/\left(M\cdot \frac{\partial}{\partial x}\right)$. Indeed, $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_k k = \mathcal{O}_X$, and $\mathcal{O}_X$ is generated by 1 with relations $\partial(1) = 0$ for any $\partial \in Vec_X$. Hence $\pi_0 M = M \otimes_{\mathcal{D}_X} \mathcal{O}_X = M/\left(M\cdot\frac{\partial}{\partial x}\right)$.

More generally we have the following lemma.

Lemma 6.17. *Let X be a smooth affine variety, $\pi : X \to pt$. Then for every right $\mathcal{D}_X$-module M $\quad \pi_0 M = M/\operatorname{span}(M\cdot\partial)$ (coinvariants of Vec_X on M).*

For functions $\quad f, g \in C^\infty(X)$ we also can consider their product. The corresponding operation on modules is the tensor product: let M and N be left $\mathcal{D}_X$-modules, then $M \otimes N = M \otimes_{\mathcal{O}_X} N$ with the action of vector fields by Leibnitz rule.

For modules there exist also an operation called exterior product: for $M \in \mathcal{M}(\mathcal{D}_X)$ and $N \in \mathcal{M}(\mathcal{D}_Y)$ we can consider $\quad M \boxtimes N \in \mathcal{M}(\mathcal{D}_{X\times Y})$. By the definition this is $M \underset{k}{\otimes} N$ with the natural structure of a $\mathcal{D}_{X\times Y} = \mathcal{D}_X \otimes_k \mathcal{D}_Y$-module.

Lemma 6.18. *$M \otimes N = \Delta^0(M \boxtimes N)$, where $\Delta : X \to X \times X$ is the diagonal embedding.*

Let's formulate some results about inverse and direct images.

Theorem 6.19. *Let $X \xrightarrow{\pi} Y \xrightarrow{\tau} Z$ be the morphisms of affine algebraic varieties. Then*

1. *$(\tau\cdot\pi)_0 = \tau_0 \cdot \pi_0$ and $(\tau\cdot\pi)^0 = \pi^0\cdot\tau^0$.*
2. *Functors π_0 and π^0 maps holonomic modules to holonomic ones. The same is true for their derived functors (since π_0 and π^0 are right exact, $L^i\pi_0$ and $L^i\pi^0$ are defined).*

Theorem 6.20. *[Kashiwara] Let $\pi : X \to Y$ be a closed embedding. Then π_0 is an equivalence between $\mathcal{M}(\mathcal{D}_x)$ and $\mathcal{M}_X(\mathcal{D}_Y)$, where $\mathcal{M}_X(\mathcal{D}_Y)$ is a category of $\mathcal{D}_Y$-modules, which are set-theoretically supported on X.*

Let us recall that a module M is set-theoretically supported on X if for $f \in \mathcal{I}_X \subset \mathcal{O}_Y$ acts locally nilpotently on M.

Example. Let $X = 0$ and $Y = \mathbb{A}^1$. Let $\delta_0 = \mathcal{D}_Y/(x \cdot \mathcal{D}_Y)$ and M be any $\mathcal{D}_Y$-module, supported at 0. Then $\mathrm{Hom}(\delta_o, M) = \{m \in M \mid xm = 0\}$.Since x acts locally nilpotently, there exist $m \in M$, $(m \neq 0)$ such that $x(m) = 0$. If M is irreducible , then $M = \delta_0$. Thus Kashiwara's theorem in this case says that any module supported at 0 is a direct sum of δ_0's which is equivalent to saying that $Ext^1(\delta_0, \delta_0) = 0$. This may be computed explicitly.

7. Lecture 8 (02/26/02): Proof of Kashiwara's theorem and its corollaries

We now want to prove Theorem 6.20. Theorem 6.19 will be proved in the next lecture.

Let us show first of all that the image of i_0 is contained in $\mathcal{M}_X(\mathcal{D}_Y)$. By definition, $i_0(M) = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$, and $\mathcal{D}_{X\to Y} = \mathcal{D}_Y/\mathcal{J}\mathcal{D}_Y$, where $\mathcal{J}$ is the ideal of X. Every element of $\mathcal{D}_{X\to Y}$ is killed by a large power of $\mathcal{J}$, i.e. for every $d \in \mathcal{D}_Y$ there exists n, such that $d\mathcal{J}^n \subset \mathcal{J}\mathcal{D}_Y$ (actually, one can take $n = \mathrm{ord}(d) + 1$). (It' s enough to prove that for $d = \partial_1 \dots \partial_k$, where $\partial_j \in Vec_Y$.) And this means that $i_0(M)$ is set-theoretically supported on X.

In order to prove Kashiwara's theorem we shall construct a functor $i^! : \mathcal{M}(\mathcal{D}_Y) \to \mathcal{M}(\mathcal{D}_X)$, which will be the inverse of i_0, when restricted on $\mathcal{M}_X(\mathcal{D}_Y)$. For every $M \in \mathcal{M}(\mathcal{D}_Y)$, define $i^!M = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, M) = \{$ all m in M, killed by $\mathcal{J}\}$. The structure of $\mathcal{D}_X$-module is given as follows: any vector field $\partial \in Vec_X$ can be extended locally to a vector field $\tilde{\partial} \in Vec_Y$, which preserves $\mathcal{J}$. For any $m \in i^!M$ define $\partial m = \tilde{\partial} m$. This is an element of $i^!M$, since $\tilde{\partial}$ preserves $\mathcal{J}$. Let us show, that this definition does not depend on the choice of the extension. Suppose we have two such extensions $\tilde{\partial}$ and $\tilde{\partial}'$. Then $\tilde{\partial} - \tilde{\partial}' = 0$ on X, i.e. $v = \tilde{\partial} - \tilde{\partial}' \in \mathcal{J} \cdot Vec_X$ and thus $v(m) = m \cdot v = 0$, since m is killed by $\mathcal{J}$. But $v(m)$ is supported on X, so $(\tilde{\partial} - \tilde{\partial}')(m) = 0$.

Recall the following definitions.

Definition 7.1. *Suppose we have two functors $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_1$. Then F is called left adjoint to G (or G – right adjoint to F), if for any $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$ there exist a functorial isomorphism $\alpha_{A,B} : Hom(F(A), B) \to Hom(A, G(B))$.*

In this case we have canonical maps

$$FG \to \mathrm{Id}_{\mathcal{C}_1}, \quad \mathrm{Id}_{\mathcal{C}_2} \to GF,$$

called adjunction morphisms.

Remark. For a given F, if G exists, it is unique up to canonical isomorphism.

Theorem 7.2. (1) *$i^!$ is right adjoint to i_0.*

(2) *The functors $\mathcal{M}(\mathcal{D}_X) \underset{i^!}{\overset{i_0}{\rightleftarrows}} \mathcal{M}_X(\mathcal{D}_Y)$ are mutually inverse.*

It is clear that Theorem 7.2 implies Theorem 6.20.

PROOF. Let $i : X \hookrightarrow Y$ be a closed embedding. We want to prove that $i^!$ is right adjoint to i^0, i.e. that for any $\mathcal{D}_X$-module M and $\mathcal{D}_Y$-module N we have $\mathrm{Hom}(i_0M, N) \cong \mathrm{Hom}(M, i^!N)$.

There exists a map $M \hookrightarrow i_0M = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$, given by $m \mapsto m \otimes 1$. Given $f \in \mathrm{Hom}(i_0M, N)$, restrict it to M. In fact, we'll get a map $M \to i^!N$. Given $g \in \mathrm{Hom}(M, i^!N)$, we want to construct $\tilde{g} : M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \to N$. We know that $\mathcal{D}_{X\to Y} = \mathcal{D}_Y/\mathcal{J}\mathcal{D}_Y$. So a map $m \otimes d \mapsto g(m)d$ is well-defined, since $g(m)$ is killed by $\mathcal{J}$. The first part of Theorem 7.2 is proved.

Since i_0 and $i^!$ are adjoint, we have canonical adjunction morphisms $i_0i^! \to \mathrm{Id}$ and $\mathrm{Id} \to i^!i_0$. In order to prove the second part of the theorem, we have to prove that these morphisms are in fact isomorphisms. It is enough to show this locally.

By induction on codimension of X in Y it is enough to assume that X is a smooth hypersurface in Y, given by an equation $f = 0$. Locally we can choose a coordinate system $y_1, \ldots, y_m \quad \partial_1, \ldots, \partial_m$ on Y, such that $y_m = f$.

We claim that $\mathcal{D}_{X\to Y}$ is free over $\mathcal{D}_X$:

$$\mathcal{D}_{X\to Y} = \bigoplus_n \mathcal{D}_X \partial^n,$$

where ∂ is a vector field on Y, such that $\partial(f) = 1$. Indeed

$$\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes \mathcal{D}_Y = \bigoplus \mathcal{O}_X \partial_1^{\alpha_1} \ldots \partial_m^{\alpha_m}$$

and $\bigoplus_{\alpha_1,\ldots,\alpha_{m-1}} \mathcal{O}_X \partial_1^{\alpha_1} \ldots \partial_{m-1}^{\alpha_{m-1}}$ is $\mathcal{D}_X$.

Let us prove that $\mathrm{Id} \to i^!i_0$ is isomorphism. Let M be $\mathcal{D}_X$-module. Then

$$i_0M = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} = M \otimes_{\mathcal{D}_X} \left(\bigoplus_j \mathcal{D}_X \partial^j \right) = \bigoplus_j M\partial^j.$$

We have a map $f : M\partial^j \to M\partial^{j-1}$. It is isomorphism for $i > 0$ (it follows from the fact, that $\partial \circ f$ acts on $M\partial^j$ by j, and this can be proved by induction).

So $\mathrm{Ker} f \mid_{i_0M} = M$, and this means that $i^!i_0M = M$.

Now we have to show that $i_0i^! \to \mathrm{Id}$ is an isomorphism.

Let N be $\mathcal{D}_Y$ module supported on X and $\mathrm{Ker} f = S \subset N$. By definition $S = i^!N$. We'll prove that $N = i_0S$, i.e. $N = \bigoplus S\partial^j$.

Consider $\tilde{N} = \sum_j S\partial^j \subset N$ and $d = f\partial$. On $S\partial^j \quad d$ acts by eigenvalue j. This can be proved by induction: if $nd = \lambda n$, then $n\partial d = n(f\partial + 1)\partial = (nd)\partial + n\partial = (\lambda + 1)(n\partial)$. Since $\partial f = f\partial + 1$, it acts by $j + 1$ on $S\partial^j$. This means that $\tilde{N}$ is in fact the direct sum and $f : S\partial^j \to S\partial^{j-1}$ is surjective on $\tilde{N}$.

Consider $L = N/\tilde{N}$. To show that $L = 0$, it is enough to show that f has zero kernel on L (since we know that f acts locally nilpotently on L).

Let $l \in N$, such that $lf \in \tilde{N}$. We want to show that $l \in \tilde{N}$. Since f is surjective, there exist $n \in \tilde{N}$ such that $nf = lf$. This means that $(n-l)f = 0$. Since $n \in \tilde{N}$ and $n - l \in S \subset \tilde{N}$, $l \in \tilde{N}$. □

Let us see some applications of Kashiwara's theorem (we shall see more applications in the future).

7.3. $\mathcal{D}$-modules on singular varieties. Let X be any affine variety. There exists a closed embedding $X \hookrightarrow Y$, where Y is smooth. Define $\mathcal{M}(\mathcal{D}_X) = \mathcal{M}_X(\mathcal{D}_Y)$. By Kashiwara's theorem this definition for smooth varieties is the same, as we already have. We claim that this definition does not depend on the embedding.

Suppose $i_1 : X \hookrightarrow Y_1$ and $i_2 : X \hookrightarrow Y$ are two such embeddings. Then there exist Y_3, such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{i_1} & Y_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ Y_2 & \xrightarrow{j_2} & Y_3 \end{array}$$

where all maps are closed embeddings, is commutative.

It follows clearly from Kashiwara's theorem that for $k = 1$ or 2 we have $\mathcal{M}_X(\mathcal{D}_{Y_k}) \subset \mathcal{M}_X(\mathcal{D}_{Y_3})$. This defines an equivalence $\mathcal{M}_X(\mathcal{D}_{Y_1}) \simeq \mathcal{M}_X(\mathcal{D}_{Y_2}$. Let us check that that this equivalence does not depend on the choice of Y_3.

We have a functor $F : \mathcal{M}_X(\mathcal{D}_Y) \to \mathcal{M}(\mathcal{O}_X)$ defined by $M \mapsto \{m \in M \mid \mathcal{J} \cdot m = 0\}$. It is easy to see that this functor is faithful. If we have two embeddings $i_1 : X \hookrightarrow Y_1$, $i_2 : X \hookrightarrow Y_2$ then it is easy to see that the above equivalence commutes with the corresponding functors $F_i : \mathcal{M}_X(\mathcal{D}_{Y_i}) \to \mathcal{M}(\mathcal{O}_X)$. This implies that this equivalence does not depend on the choice of Y_3. Moreover, it shows that we have a well-defined faithful functor $\mathcal{M}(\mathcal{D}_X) \to \mathcal{M}(\mathcal{O}_X)$ for an arbitrary affine variety. This shows that we can think of an object of $\mathcal{M}(\mathcal{D}_X)$, defined above, as an object of $\mathcal{M}(\mathcal{O}_X)$ plus some additional structure. If $M \in \mathcal{M}(\mathcal{O}_X)$ is a "$\mathcal{D}_X$-module" in our definition, then one can show that $\mathcal{D}_X$ acts on M (cf. problem set 4), but a "$\mathcal{D}_X$-module" structure is not recovered from $\mathcal{D}_X$-action.

Historical remark. Kashiwara proved his theorem before Bernstein's inequality was stated. In fact, the first proof of this inequality, which was in Bernstein's thesis, used Kashiwara's theorem.

Exercises

In this collection of problems all varieties are assumed to be affine.

1. Let X be an algebraic variety, $\Delta : X \to X \times X$ the diagonal embedding. Let J denote the ideal of $\Delta(X)$ in $\mathcal{O}_{X\times X}$ and let $X^{(n)}$ denote the closed subscheme of $X \times X$ corresponding to the ideal J^n.

2. Let $\eta : A \to B$ be a homomorphism of commutative algebras such that B is finite over A. Recall that in this case we have the functor $\eta^! : A - mod \to B - mod$

defined as

$$\eta^!(M) = \mathrm{Hom}_A(B, M).$$

Let X be a scheme over our base field k (the definitions below makes sense (and are interesting) when k has arbitrary characteristic but we shall consider only the case when it has characteritic 0). Recall that a *nilpotent extension* of X is a closed embedding $i : X \to Y$ where Y is another scheme and the ideal of X in Y is nilpotent. If Y and Z are two nilpotent extensions of X we say that $\eta : Y \to Z$ is a morphism of extensions it it is a morphism of schemes which is equal to identity on X.

A *!-crystal* on X is a collection of the following data:

(1) An $\mathcal{O}_Y$-module M_Y for every nilpotent extension Y of X.
(2) An isomorphism $\alpha_\eta : M_Y \simeq \eta^! M_Z$ for every finite map of extensions $\eta : Y \to Z$.

This data should satisfy the following compatibility condition: for every chain $Y \xrightarrow{\eta} Z \xrightarrow{\rho} W$ of finite morphisms of nilpotent extensions of X we have $\alpha_{\eta\circ\rho} = \alpha_\eta \circ \alpha_\rho$ (I hope that the meaning of the right hand side is clear.

We denote by $Crys(X)$ the category of !-crystals on X.

Let X be a (not necessarily smooth) algebraic variety. $M \in calM(\mathcal{D}_X)$ (defined via right modules). Let also $X \to Y$ be a nilpotent extension of X. We may imbed Y into some smooth variety Z. In this case M gives rise to a $\mathcal{D}_Z$-module M_Z on Z supported on X. Define M_Y to be the set of all elements of M_Z which are scheme-theoretically supported on Y.

a) Show that the collection $\{M_Y\}$ has a natural structure of a !-crystal.

b) Show that the resulting functor $\mathcal{M}(\mathcal{D}_X) \to Crys(X)$ is an equivalence of categories (hint: do it first for smooth X using problem 1).

8. Lecture 9 (02/28/02): Direct and inverse images preserve holonomicity

Last time we have defined two functors of inverse image $i^!$ and i^0. The first is left exact and the second is right exact. What is the relation between those functors?

Lemma 8.1.

$$i^! = L^{\dim X - \dim Y} i^0 \qquad (*)$$

$$i^0 = R^{\dim Y - \dim X} i^! \qquad (**)$$

(This is true even for $\mathcal{O}$-modules.)

Proof. As in the proof of Kashiwara's theorem, we can assume that X has codimension 1 in Y and is given by equation $f = 0$. By definition $i^0 M = \mathcal{O}_X \otimes_{\mathcal{O}_Y} M$.

$$0 \to \mathcal{O}_Y \xrightarrow{f} \mathcal{O}_Y \to \mathcal{O}_X \to 0$$

is a free resolution of $\mathcal{O}_X$. Taking tensor product with M we get

$$M \xrightarrow{f} M \to \mathcal{O}_X \otimes_{\mathcal{O}_Y} M \to 0.$$

By definition $L^{-1}i^0M = \mathrm{Ker} f = i^!M$ and $i^0M = \mathrm{Coker} f = R^1i^!M$. □

Lemma 8.2. *Let $i : X \hookrightarrow Y$ be smooth embedding. Then $i^!$ maps holonomic modules in $\mathcal{M}_X(\mathcal{D}_Y)$ to holonomic modules in $\mathcal{M}(\mathcal{D}_X)$.*

Proof. As before, we can assume that X has codimension 1 in Y and is given by equation $f = 0$.

In the proof of Kashiwara's theorem we have shown that $i_0M = \bigoplus_j M\partial^j$, where ∂ is a vector field on Y such that $\partial f = 1$. It's easy to see that in this case $d(i_0M) = d(M) + 1$.

By Kashiwara's theorem $i_0i^!N = N$ if $N \in \mathcal{M}_X(\mathcal{D}_Y)$. So $d(N) = d(i_0i^!N) = d(i^!N) + 1$ and $d(i^!N) = d(N) - 1$. If N is holonomic, so is $i^!N$. □

Definition 8.3. *Suppose $X \subset Y$ is singular. Then $M \in \mathcal{M}(\mathcal{D}_X) = \mathcal{M}_X(\mathcal{D}_Y)$ is holonomic if it is holonomic as $\mathcal{D}_Y$-module.*

Lemma 8.4. *$i^!$ and i_0 define inverse equivalences of $\mathcal{M}_X^{hol}(\mathcal{D}_Y)$ and $\mathcal{M}^{hol}(\mathcal{D}_X)$.*

By this lemma the definition above does not depend on Y.

Theorem 8.5. *Let $\pi : \mathbb{A}^n \to \mathbb{A}^m$ be an affine map (i.e. a composition of a linear map and a translation). Then π_0, π^0 and their derived functors map holonomic modules to holonomic. For any holonomic $\mathcal{D}_{\mathbb{A}^n}$-module N and any holonomic $\mathcal{D}_{\mathbb{A}^m}$-module M*

$$\sum_i c(L^i\pi_0N) \leq c(N) \quad \textit{and} \quad \sum_i c(L^i\pi^0M) \leq c(M). \tag{1}$$

Theorem 8.5 implies that if π is an affine map then the functors π_0, π^0 and their derived functors preserve holonomicity. We shall se later that this statement is true in general (i.e. when π is an arbitrary map of algebraic varieties).

Example. Let $\pi : X \to pt$. Then for every right $\mathcal{D}_X$-module M $\quad \pi_0M = M/(M \cdot Vec_X)$. For any left module M $\quad \pi_0M = M \otimes \Omega^n(X)/(M \otimes \Omega^n(X) \cdot Vec_X)$. (We have not multiplied by $\Omega^{-n}(pt)$, since $\Omega^{-n}(pt) = 1$).

We claim that $L^i\pi_0M = H_{dR}^{n+i}(M)$. In our case $\mathcal{D}_{X\to Y} = \mathcal{O}_X$. This means that $\pi_0M = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} = M \otimes_{\mathcal{D}_X} \mathcal{O}_X$ for right modules and $\pi_0M = \Omega^n(X) \otimes_{\mathcal{D}_X} M$ for left modules.

Earlier we have proved that $dR(\mathcal{D}_X)$ is a projective resolution of $\Omega^n(X)$ as a right $\mathcal{D}_X$-module. To compute $L^\bullet\pi_0M$, we have to compute $H^\bullet(dR(\mathcal{D}_X)) = dR(\mathcal{D}_X) \otimes_{\mathcal{D}_X} M = dR(M)$. Thus $L^i\pi_0M = H_{dR}^{n+i}(M)$.

Corollary 8.6. *If M is holonomic,* $\dim H_{dR}^i(M) < \infty$.

Proof. Let be a map $\pi : X \to pt$. In the example above we have seen that $H_{dR}^i(M) = L^{i-n}\pi_0M$. Hence, since M is holonomic, $H_{dR}^i(M)$ is holonomic by theorem Theorem 8.5. And any holonomic module over $\mathcal{D}_{pt}$ is finite dimensional. □

Proof of Theorem 8.5. Any affine map is a composition of a standard projection and an affine embedding. So, it's enough to prove the theorem for embeddings and for projections.

If π is a projection, then π^0 is exact. By induction we can assume that $m = n-1$ and $\pi : \mathbb{A}^n \to \mathbb{A}^{n-1}$. In this case $\mathcal{O}_{\mathbb{A}^n} = k[x_1, \dots, x_n]$, $\quad \mathcal{O}_{\mathbb{A}^{n-1}} = k[x_1, \dots, x_{n-1}]$ and thus

$$\pi^0 M = \mathcal{O}_{\mathbb{A}^n} \otimes_{\mathcal{O}_{\mathbb{A}^{n-1}}} M = \bigoplus_i M x_n^i.$$

Let $F_i M$ be a good filtration on M. It induces filtration on $\pi^0 M$, namely

$$F_k \pi^0 M = \sum_{i+j=k} F_j M \cdot x_n^i.$$

Since M is holonomic $\dim F_j M = \frac{c(M) j^{n-1}}{(n-1)!} + \dots$. This means that

$$\dim F_k \pi^0 M = \frac{c(M) k^n}{n!} + \dots,$$

hence $\pi^0 M$ is holonomic and $c(\pi^0 M) \leq c(M)$ (we have discussed it in lecture 2). Since π^0 is exact, we proved the theorem in this case.

Suppose π is an embedding. By induction we can assume $m = n+1$, $\quad \pi : \mathbb{A}^n \hookrightarrow \mathbb{A}^{n+1}$. Let $\mathbb{A}^n = \{x = 0\}$, where $x = x_n$ and M be a holonomic module on $\mathbb{A}^{n+1}$.

Choose a good filtration $F_i M$ on M (with respect to Bernstein's filtration on $\mathcal{D}_{\mathbb{A}^{n+1}}$). Let $N \subset M$ be the part of M on which x acts nilpotently. Then $N = \pi_0 \pi^! M$. In the proof of Kashiwara's theorem we've showed, that x acts surjectively on N. This means that $\operatorname{Coker}(x)$ on M is the same as $\operatorname{Coker}(x)$ on M/N. On M/N $\quad x$ has no kernel and M/N is holonomic as a quotient of holonomic module. Thus to prove that $\pi^0 M = \operatorname{Coker}(x)$ is holonomic, it is enough to assume that $\operatorname{Ker}(x)$ on M is 0.

The module $\pi^0 M = M/xM$ inherits filtration from M which might be not good. Since M is holonomic we know that

$$\dim F_i M = \frac{c(M) i^{n+1}}{(n+1)!} + \dots$$

By definition the map $F_i M/(x \cdot F_{i-1} M) \twoheadrightarrow F_i(M/xM)$ is surjective. Since x has no kernel it follows that $\dim x \cdot F_{i-1} M = \dim F_{i-1} M$ and hence

$$\dim F_i(M/xM) = \frac{c(M) i^n}{n!} + \dots$$

In lecture 2 we' discussed that such an equality for an arbitrary filtration implies that M/xM is holonomic and $c(M/xM) \leq c(M)$. Hence $\pi^0 M$ is holonomic.

Now let's prove inequality (1). Let M_1 be maximal submodule of M supported on $\mathbb{A}^{n-1}$. From the following exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0$$

we know, that $c(M) = c(M_1) + c(M/M_1)$. By construction $\pi^0 M_1 = 0 = \pi^!(M/M_1)$. Thus $c(\pi^! M_1) = c(M_1)$ and $c(\pi^0 M/M_1) \leq c(M/M_1)$. Since $\pi^! = L^{-1}\pi^0$, we have $c(L^{-1}\pi^0 M) + c(\pi^0 M) \leq c(M)$. Our theorem is proved for π^0.

In order to prove this statement for π_0 we shall introduce the *Fourier transform* $F : \mathcal{M}(\mathcal{D}_V) \to \mathcal{M}(\mathcal{D}_{V^*})$, where V is vector space and V^* – its dual.

Since $\mathcal{D}_V$ is generated by $V^* \subset \mathcal{O}(V)$ and $V \subset Vec_V$, $\mathcal{D}_{V^*}$ is generated by $V \subset \mathcal{O}(V^*)$ and $V^* \subset Vec_{V^*}$, $\mathcal{D}_V$ and $\mathcal{D}_{V^*}$ are isomorphic via $V \leftrightarrow V$ and $V^* \leftrightarrow V^*$. Obviously, this isomorfhism preserves Bernstein's filtration. So F maps holonomic modules to holonomic.

Example. In case $V = \mathbb{A}^1$ isomorphism between $\mathcal{D}_V$ and $\mathcal{D}_V^*$ is given by $x \mapsto \frac{d}{dx}$ and $\frac{d}{dx} \mapsto -x$.

Let $\pi : V \to W$ be a linear map and $\widetilde{\pi} : W^* \to V^*$ be its dual.

$$F_W(\pi_0 M) = \widetilde{\pi}^0(F_V M)$$

Thus, if the theorem is true for π^0, it's true for π_0 also.

□

Corollary 8.7. *If M is holonomic on $\mathbb{A}^n$ then*

$$\sum \dim H^i_{dR}(M) \leq c(M).$$

CHAPTER 2

D-modules on general algebraic varieties

1. Lectures 10 and 11 (03/5/02 and 03/7/02): $\mathcal{D}$-modules for arbitrary varieties

Let X be any algebraic variety (for simplicity we shall assume that all the varieties in question are quasi-projective; this, however, doesn't affect the statements but only simplifies the exposition). Then there exists unique sheaf $\mathcal{D}_X$ of differential operators on X such that for any affine subset $U \subset X \quad \Gamma(U, \mathcal{D}_X) = \mathcal{D}_U$. This sheaf has both right and left $\mathcal{O}_X$-modules structures.

Lemma 1.1. *$\mathcal{D}_X$ is quasi-coherent with respect to either $\mathcal{O}_X$-module structure.*

Suppose X is affine, $f \in \mathcal{O}_X$ and $U = U_f = \{x \in X \mid f(x) \neq 0\}$ – an open affine subset of X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then

$$\Gamma(U, \mathcal{F}) = \mathcal{O}_U \otimes_{\mathcal{O}_X} \Gamma(X, \mathcal{F}).$$

$\mathcal{D}_X$ has two $\mathcal{O}_X$-module structures and

$$\mathcal{O}_U \otimes_{\mathcal{O}_X} \mathcal{D}_X \simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_U \simeq \mathcal{D}_U.$$

As in the case of affine varieties, here we have quasi-coherent sheaves of left $\mathcal{D}_X$-modules $\mathcal{M}^l(\mathcal{D}_X)$ and quasi-coherent sheaves of right $\mathcal{D}_X$-modules $\mathcal{M}^r(\mathcal{D}_X)$. And these two categories are isomorphic via $M \mapsto M \otimes \Omega^n(X)$, where $n = \dim X$.

Some statements that we proved for affine varieties remain true (with the same proof) in the general case. For example, since Kashiwara's theorem is a local statement, it is still true in general case. If X is singular, we can cover it by U_i such that each U_i can be embedded into a smooth variety. In this case $\mathcal{D}$-module on X is the set $\{M_i\}$ of $\mathcal{D}$-modules on U_i (for each i), such that for any i and $j \quad M_i \mid_{U_j} = M_j \mid_{U_i}$, and for any triple i, j and k compatibility condition holds.

Let X be smooth. Then $\mathcal{D}_X$ is a filtered sheaf of algebras and

$$gr\mathcal{D}_X = p_*\mathcal{O}(T^*X),$$

where $p : T^*X \to X$ is a standard projection. In this case we also have a functor of singular support: $M \mapsto s.s.(M) \subset T^*X$. Let's denote the cycle, given by singular support of M, by $s.c.(M)$.

Definition. Module M is called holonomic iff $d(s.s.(M)) = n$.

In this case holonomic modules also have finite length.

Inverse and direct images in case of arbitrary varieties.

Let $\pi : X \to Y$ and $N \in \mathcal{M}^l(\mathcal{D}_Y)$. Then we can define the inverse image $\pi^0 N = \mathcal{O}_X \otimes_{\pi^*\mathcal{O}_Y} \pi^*(N)$ – sheaf of $\pi^*\mathcal{O}_Y$-modules (since $\pi^*\mathcal{O}_Y \hookrightarrow \mathcal{O}_X$). This definition is totally analogous to the case of affine varieties.

Let's try to define direct image, as we've done for affine varieties. First of all define $\mathcal{D}_{X\to Y} = \pi^0\mathcal{D}_Y$. Let $M \in \mathcal{M}^r(\mathcal{D}_X)$ be right $\mathcal{D}_X$-module. If we set $\pi_0 M = \pi_*(M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$, then π_0 is neither left nor right exact, since $\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}$ is right exact and π_* is left exact. Moreover, π_0 defined in this way is not compatible with composition.

In fact, direct image is defined correctly only in derived category. We'll discuss this definition later.

Remark. If π is closed embedding, this definition is still good.

1.2. $\mathcal{D}$-affine varieties. Definition. An algebraic variety X is called $\mathcal{D}$-affine if the functor of global sections $M \mapsto \Gamma(X, M)$ is an equivalence between $\mathcal{M}^l(\mathcal{D}_X)$ and the category of modules over $\mathcal{D}_X^{glob} = \Gamma(X, \mathcal{D}_X)$).

Remark. If we replace $\mathcal{D}$ by $\mathcal{O}$ in definition, by Serre we'll just get affine varieties.

THEOREM 1.3. *$\mathbb{P}^n$ is $\mathcal{D}$-affine.*

PROOF. We claim, that any variety X is $\mathcal{D}$-affine if and only if functor of global sections Γ is exact on $\mathcal{D}_X$-modules and for any nonzero module $M \quad \Gamma(X, M) \neq 0$.

If X is $\mathcal{D}$-affine it is easy to see that these two properties are satisfied. The other implication follows from the more general fact, to formulate which we need the following definition.

Definition. Let $\mathcal{A}$ be an abelian category with infinite direct limits. On object $P \in \mathcal{A}$ is called projective generator, if P is projective and $\mathrm{Hom}(P, X) \neq 0$ for any $X \neq 0$. The following result is well-known.

LEMMA 1.4. *Let $\Lambda = \mathrm{End}\, P$ and $F : \mathcal{A} \to \{$right Λ-modules$\}$ be the functor defined by $F(X) = \mathrm{Hom}(P, X)$. Then F is an equivalence of categories.*

If Γ is exact on $\mathcal{D}_X$-modules and $\Gamma(X, M) \neq 0$ for any $M \neq 0$, then $\mathcal{D}_X$ is a projective generator of $\mathcal{M}(\mathcal{D}_X)$ (since $\Gamma(X, M) = \mathrm{Hom}(\mathcal{D}_X, M)$). In this case $\Lambda = \mathrm{End}\, \mathcal{D}_X = (\mathcal{D}_X^{glob})^{op}$. So, X is $\mathcal{D}$-affine by the lemma above.

Now let us prove the theorem Theorem 1.3. In our case $X = \mathbb{P}^n = \mathbb{P}(V)$. Let $\widetilde{V} = V \backslash \{0\}, \quad j : \widetilde{V} \to V$ be natural inclusion and $\pi : \widetilde{V} \to \mathbb{P}(V)$ be projection.

Let M be $\mathcal{D}_{\mathbb{P}^n}$-module. Then

$$\Gamma(\widetilde{V}, \pi^0 M) = \bigoplus_{k \geq 0} \Gamma(\mathbb{P}^n, M \otimes \mathcal{O}(k)).$$

Let us introduce the Euler vector field $\mathcal{E} = \sum x_i \frac{\partial}{\partial x_i}$.This vector field acts on $\Gamma(\mathbb{P}^n, M \otimes \mathcal{O}(k))$ by multiplication by k. Thus we have $\Gamma(\mathbb{P}^n, M) = \Gamma(\widetilde{V}, \pi^0 M)^{\mathcal{E}}$ (note that $\mathcal{E}$ acts semi-simply on $\Gamma(\widetilde{V}, \pi^0 M)$, hence taking $\mathcal{E}$ invariants doesn't influence the exactness).

Consider the following sequence of functors:

$$M \mapsto \pi^0 M \mapsto \Gamma(\widetilde{V}, \pi^0 M).$$

Here the first functor is exact, but the second may be not exact. In fact you can again decompose $\Gamma(\widetilde{V}, \pi^0 M)$ as a composition of two functors: first we replace M by $j_0 M$ and then take global sections on V. Since V is affine the latter is exact.

For any open embedding $j : U \to X$ the functor of direct image j_0 is left exact and all higher derived functors are supported on $X \backslash U$. Consider the embedding $j : \widetilde{V} \hookrightarrow V$. For any short exact sequence of $\mathcal{D}_{\mathbb{P}^n}$-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

we have the long exact sequence

$$0 \to j_0\pi^0 M_1 \to j_0\pi^0 M_2 \to j_0\pi^0 M_3 \to R^1 j_0 \pi^0 M_1 \to \dots .$$

By Kashiwara's theorem $R^1 j_0 \pi^0 M_1$ is a direct sum of δ_0 – modules of δ-functions (since it is supported at 0).

The eigenvalues of $\mathcal{E}$ on $\Gamma(V, \delta_0)$ are $-1, -2, -3, \dots$. And since the eigenvalues of $\mathcal{E}$ on $\Gamma(V, j_0\pi^0 M_3)$ are nonnegative, $\alpha : \Gamma(V, j_0\pi^0 M_3) \to \Gamma(V, R^1 j_0 \pi^0 M_1)$ should be zero. Hence the functor $M \mapsto \Gamma(\widetilde{V}, \pi^0 M)$ is exact and hence the functor $M \mapsto \Gamma(\mathbb{P}(V), M) = \Gamma(\widetilde{V}, \pi^0 M)^{\mathcal{E}}$ is also exact.

Let us prove that $\Gamma(\mathbb{P}^n, M) \neq 0$ if $M \neq 0$. There exists $k \geq 0$ such that $\Gamma(\mathbb{P}^n, M \otimes \mathcal{O}(k)) \neq 0$. Hence there exists $0 \neq m \in \Gamma(\widetilde{V}, \pi^0 M)$ such that $\mathcal{E} m = km$. Since $\mathcal{E}(\frac{\partial}{\partial x_i} m) = (k-1)\frac{\partial}{\partial x_i} m$, if there exist i such that $\frac{\partial}{\partial x_i} m \neq 0$, then there exist $l \in \Gamma(\widetilde{V}, \pi^0 M)$ with eigenvalue $k-1$. If $\frac{\partial}{\partial x_i} m = 0$ for all i, then $\mathcal{E} m = 0$. Hence (by induction on k) $\Gamma(\widetilde{V}, \pi^0 M)^{\mathcal{E}} \neq 0$. □

2. Derived categories.

2.1. Motivation of studying derived categories. Let $\pi : X \to Y$. There is no way to define direct image on the level of abelian categories $\mathcal{M}(\mathcal{D}_X) \to \mathcal{M}(\mathcal{D}_Y)$ so, that it will be compatible with composition. In order to define such a functor, we need to work in derived category.

Let $\mathcal{A}$ be abelian category and $\mathcal{C}(\mathcal{A})$ be category of all complexes. Define $\mathcal{C}^+(\mathcal{A})$ as a category of complexes $K^\bullet$, such that $K^i = 0$ for $i \ll 0$. In the same way define categories $\mathcal{C}^-(\mathcal{A})$ of complexes bounded from above and $\mathcal{C}^b(\mathcal{A})$ of bounded complexes.

Let $\mathcal{C}_0(\mathcal{A})$ be the category of complexes with zero differential. We have a map $H : \mathcal{C}(\mathcal{A}) \to \mathcal{C}_0(\mathcal{A}) = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}$.

Theorem 2.2. *There exist unique up to canonical equivalence pair category $\mathcal{D}(\mathcal{A})$ (called derived category) and functor $Q : \mathcal{C}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ such that*

1. *If $f : K^\bullet \to L^\bullet$ is a quasi isomorphism, then $Q(f)$ is isomorphism.*

2. *This pair is universal with the property 1, i.e. for any functor $F : \mathcal{C}(\mathcal{A}) \to \mathcal{D}'$ satisfying 1, there exist unique functor $G : \mathcal{D}(\mathcal{A}) \to \mathcal{D}'$ such that $F = G \circ Q$.*

More generally, let $\mathcal{C}$ be any category and $\mathcal{S}$ – any class of morphisms. Then there exist category $\mathcal{C}[\mathcal{S}^{-1}]$ and a morphism $\mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}]$ which satisfy conditions *1* and *2* of the theorem above.

2.3. Structures on derived categories. 1. Shift functors.
For any i there exist a functor of shift to the left $K^\bullet \mapsto K^\bullet[i]$, where $K^j[i] = K^{j+i}$ for any j.

2. Distinguished triangles.
Motivation of introducing the notion of distinguished triangle:

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Then the functor $RF : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ is well defined. To say, that this functor is exact, we need to have some analog of short exact sequences which is distinguished triangles.

First of all let us define the notion of cone of a morphism. Let $K^\bullet \xrightarrow{f} L^\bullet$ be the map of complexes. Define a complex $C(f)^\bullet$, called cone of f, as follows: $C(f)^\bullet = K^\bullet[1] \oplus L^\bullet$, i.e. $C(f)^i = K^{i+1} \oplus L^i$ with differential $d(k^{i+1}, l^i) = (-d_K k^{i+1}, f(k^{i+1}) + d_L(l^i))$.

Exercise. Show that if f is an embedding, then $C(f)$ is quasi isomorphic to $L^\bullet/K^\bullet$.

Lemma 2.4. *The sequence*

$$H^i(K) \to H^i(L) \to H^i(C(f)) \to H^{i+1}(K) \to \dots$$

is exact.

Proof. If the sequence $K^\bullet \to L^\bullet \to C(f)$ were exact, this lemma would be the well known statement from algebraic topology. In our case it is not exact, but we can replace $L^\bullet$ by a quasi isomorphic complex such that the sequence will become exact.

Let's define a complex, called cylinder of f by $Cyl(f) = K^\bullet \oplus K^\bullet[1] \oplus L^\bullet$ with the following action of differential

$$d : (k^i, k^{i+1}, l^i) \mapsto (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1}) - d_L l^i).$$

The natural inclusion $L^\bullet \to Cyl(f)$ is a quasi isomorphism and the sequence $K^\bullet \to Cyl(f) \to C(f)$ is exact. □

Definition. Distinguished triangle in $\mathcal{D}(\mathcal{A})$ is a "triangle" $X \to Y \to Z \to X[1]$ which is the image under Q of $K^\bullet \to L^\bullet \to C(f) \to K^\bullet[1]$.

Main problem: given a morphism $f : X \to Y$ in $\mathcal{D}(\mathcal{A})$ there is no canonical way to complete it to exact triangle. (There is a map $X \to Y \to Z$ and it's unique, but up to noncanonical isomorphism.)

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A functor $F : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ is called exact, if it maps distinguished triangles to distinguished triangles.

Definition. An object $X \in \mathcal{D}(\mathcal{A})$ is called H^0-complex if $H^i(X) = 0$ for $i \neq 0$.

Lemma 2.5. *$\mathcal{A} \to \mathcal{D}(\mathcal{A})$ induces an equivalence between $\mathcal{A}$ and the full subcategory of H^0-complexes.*

Let X and Y be the objects of $\mathcal{A}$. Then we can redefine notions Ext functors: $\mathrm{Ext}^i(X,Y) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X^\bullet, Y^\bullet[i])$ ($\mathrm{Hom}_{\mathcal{A}}(X,Y) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X^\bullet, Y^\bullet)$). If $\mathcal{A}$ has enough projectives or injectives, this definition coincides with previous one.

2.6. Another way of thinking about derived category. Let $\mathcal{S}$ be any class of morphisms in category $\mathcal{A}$.

Definition. $\mathcal{S}$ is called localizible class of morphisms if

1) If $s, t \in \mathcal{S}$ then $s \circ t$ is also in $\mathcal{S}$, if it's defined.
2) For any given morphisms $s \in \mathcal{S}$ and f there exist an object W and morphisms $t \in \mathcal{S}$ and g. such that the following diagrams are commutative

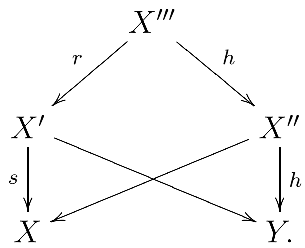
$$\begin{array}{ccc} W & \overset{g}{\dashrightarrow} & Z \\ {\scriptstyle t\in\mathcal{S}}\downarrow & & \downarrow{\scriptstyle s\in\mathcal{S}} \\ X & \underset{f}{\longrightarrow} & Y \end{array} \qquad\qquad \begin{array}{ccc} W & \overset{g}{\dashleftarrow} & Z \\ {\scriptstyle t\in\mathcal{S}}\uparrow & & \uparrow{\scriptstyle s\in\mathcal{S}} \\ X & \underset{f}{\longleftarrow} & Y \end{array}$$

3) Let $f, g : X \to Y$. Then there exist $s \in \mathcal{S}$ such that $sf = sg$ iff there exist $t \in \mathcal{S}$ such that $ft = gt$.

If $\mathcal{S}$ is a localizible class, then $\mathcal{C}[\mathcal{S}^{-1}]$ has a nice description. Morphisms in this category are given by diagrams

$$\begin{array}{ccccc} & & X' & & \\ & {\scriptstyle s\in\mathcal{S}}\swarrow & & \searrow{\scriptstyle f} & \\ X & & & & Y. \end{array}$$

Two diagrams $\begin{array}{ccccc} & & X' & & \\ & {\scriptstyle s\in\mathcal{S}}\swarrow & & \searrow{\scriptstyle f} & \\ X & & & & Y \end{array}$ and $\begin{array}{ccccc} & & X'' & & \\ & {\scriptstyle t\in\mathcal{S}}\swarrow & & \searrow{\scriptstyle g} & \\ X & & & & Y \end{array}$ define the same morphism if there exist an object X''' and morphisms $\mathcal{S} \ni r : X''' \to X'$, $f : X''' \to X''$, such that the following diagram is commutative



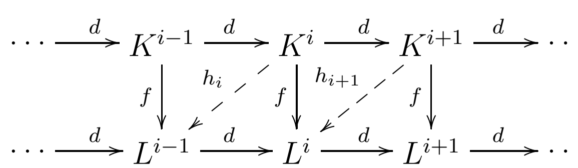
Lemma 2.7. *If $\mathcal{S}$ is a localizible class, then $\mathcal{C}[\mathcal{S}^{-1}]$ is the category with objects – objects of $\mathcal{C}$ and morphisms – equivalence classes of the diagrams above.*

We want to define derived category $\mathcal{D}(\mathcal{A})$ as localization of $\mathcal{C}(\mathcal{A})$ by quasi-isomorphisms. Unfortunately they do not form a localizible class.

Example. Let $\mathcal{A}$ be the category of abelian groups. Consider the complex $K^\bullet = \{0 \to \underset{-1}{\mathbb{Z}} \xrightarrow{*2} \underset{0}{\mathbb{Z}} \to 0\}$ and the complex $0 \to \mathbb{Z}/2\mathbb{Z} \to 0$, quasi isomorphic to $K^\bullet$ via s. Let $f : K^\bullet \to K^\bullet$ be multiplication by 2. Then $sf = 0$ which contradicts to the condition 3 of definition, i.e. there is no quasi isomorphism $t : L^\bullet \to K^\bullet$, such that $ft = 0$ (since for any such quasi-isomorphism $t(L^0) \neq 0$ and hence $2t(L^0) \neq 0$).

So, we shall replace $\mathcal{C}(\mathcal{A})$ by homotopy category $\mathcal{K}(\mathcal{A})$, where quasi-isomorphisms form a localizible class.

Let $f : K^\bullet \to L^\bullet$ be a map between complexes. Then f is homotopic to 0 if there exist $h_i : K^i \to L^{i-1}$



such that $f = dh + hd$. If $f, g : K^\bullet \to L^\bullet$ are two such morphisms, they are said to be homotopic if there exist a map h as above, with $f - g = dh + hd$.

Lemma 2.8. *If f is homotopic to 0 then it is equal to 0 in derived category.*

This is true because if f is homotopic to 0, it can be factorized through the cone of $id : K^\bullet \to K^\bullet$.

Definition. Let $\mathcal{A}$ be any abelian category. Then the homotopy category $\mathcal{K}(\mathcal{A})$ is the category with objects $Ob(\mathcal{K}(\mathcal{A})) = Ob(\mathcal{C}(\mathcal{A}))$ and morphisms $Mor(\mathcal{K}(\mathcal{A})) = Mor(\mathcal{C}(\mathcal{A}))/\{f \mid f \text{ is homotopic to } 0\}$.

Defined in this way, $\mathcal{K}(\mathcal{A})$ is an additive category. Cohomology is well defined in $\mathcal{K}(\mathcal{A})$, so quasi-isomorphisms are defined.

Theorem 2.9. *Quasi-isomorphisms form a localizible class in $\mathcal{K}(\mathcal{A})$.*

In the example above the map f itself was homotopic to 0 so the third condition of definition of localizible class is satisfied.

Claim. Derived category $\mathcal{D}(\mathcal{A})$ is the localization of $\mathcal{K}(\mathcal{A})$ by quasi-isomorphisms.

Corollary 2.10. *$\mathcal{D}(\mathcal{A})$ is additive.*

To define addition we need to define the common denominator for two morphisms $X \xleftarrow{s\in\mathcal{S}} X' \xrightarrow{f} Y$ and $X \xleftarrow{t\in\mathcal{S}} X'' \xrightarrow{g} Y.$ By definition this is a pair of diagram $X \xleftarrow{p\in\mathcal{S}} X''' \xrightarrow{\tilde{f}} Y$ and $X \xleftarrow{p\in\mathcal{S}} X''' \xrightarrow{\tilde{g}} Y$, equivalent to the given two. The sum of the last two is given by $\tilde{f} + \tilde{g}$.

As before, we can define $\mathcal{D}^b(\mathcal{A}), \quad \mathcal{D}^+(\mathcal{A}), \quad \mathcal{D}^-(\mathcal{A})$. Assume $\mathcal{A}$ has enough projectives, i.e. for any $X \in \mathcal{A}$ there exist projective P, which maps surjectively on X. Let $\mathcal{K}^-(\mathcal{P})$ denote the category of complexes $\dots \to P^i \to P^{i+1} \to \dots$, where all P^j are projective, with morphisms up to homotopy. Then the natural functor $\mathcal{K}^-(\mathcal{P}) \to \mathcal{D}^-(\mathcal{A})$ is an equivalence of categories. From this easily follows that for any $X, Y \in \mathcal{A} \quad \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y[i]) = \mathrm{Ext}^i(X, Y)$. To see it one has to replace X and Y by their projective resolutions.

If $\mathcal{A}$ has enough injectives, then $\mathcal{K}^+(\mathcal{I}) \to \mathcal{D}^+(\mathcal{A})$ is an equivalence of categories.

2.11. Derived functors. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive and left exact functor. We would like to define the derived functor $RF : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{A})$ by universal properties. We denote by $\mathcal{K}^+(F)$ the natural extension of F to the functor $\mathcal{K}^+(\mathcal{A}) \to \mathcal{K}^+(\mathcal{B})$.

Definition.THe derived functor of F is a pair (RF, ε_F) where $RF : \mathcal{D}+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ is an exact functor and $\varepsilon_F : \mathcal{Q}_{\mathcal{B}} \circ \mathcal{K}^+(F) \to RF \circ \mathcal{Q}_A$ is a morphism of functors satisfying the following universality condition:

For every exact functor $G : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ and a morphism of functors $\varepsilon : \mathcal{Q}_{\mathcal{B}} \circ \mathcal{K}^+(F) \to G \circ \mathcal{Q}_A$ there exists unique morphism $\eta : RF \to G$ for which the following diagram is commutative:

$$\begin{array}{ccccc} & & \mathcal{Q}_B \circ \mathcal{K}^+(F) & & \\ & \swarrow^{\varepsilon_F} & & \searrow^{\varepsilon} & \\ G \circ \mathcal{Q}_{\mathcal{A}} & & \xleftarrow{\eta\circ\mathcal{Q}_A} & & RF \circ \mathcal{Q}_{\mathcal{A}} \end{array}$$

It is easy to see that if (RF, ε_F) exists then it is unique up to canonical isomorphism. The problem in general is to show existence as well as to compute RF. For this we have to introduce some additional definitions.

Definition. Let $\mathcal{R}$ be some class of objects of $\mathcal{A}$. We say that $\mathcal{R}$ is admissible with respect to F, if there exist an acyclic complex $\cdots \to K^i \to K^{i+1} \to \dots$ in $\mathcal{C}^+(\mathcal{A})$ with $K^i \in \mathcal{R}$ such that $F(K^\bullet)$ is also acyclic and for any $X \in \mathcal{A}$ there exist $Y \in \mathcal{R}$ in which X can be included.

Example. Consider the left exact functor $\mathrm{Hom}(X, \cdot)$. Injective objects form an admissible class with respect to it.

Similarly we can define the same notion for right exact functors.

If $\mathcal{R}$ is an admissible class with respect to F, then RF exist and can be defined as follows: any object of $\mathcal{D}(\mathcal{A})$ is isomorphic to the image of some object $K \in \mathcal{C}^+(\mathcal{R})$. We set $RF(Q_{\mathcal{A}}(K)) = Q_{\mathcal{B}}(F(K))$.

Proposition 2.12. 1. *$H^n(RF(K^\bullet))$ is a subquotient of*

$$\bigoplus_{p+q=n} R^pF(H^q(K^\bullet)).$$

2. *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$. If there exist $\mathcal{R}_{\mathcal{A}}$ and $\mathcal{R}_{\mathcal{B}}$ – admissible classes for F and G such that $F(\mathcal{R}_{\mathcal{A}}) \subset \mathcal{R}_{\mathcal{B}}$, then $R(G \circ F) = RG \circ RF$.*

The same can be done for right exact functors.

Examples. 1. Let R be any commutative ring, $N \mapsto M \otimes_R N$ – right exact functor. The corresponding derived functor is $N \mapsto M \overset{L}{\otimes} N$, where $M \overset{L}{\otimes} N$ is in $\mathcal{D}^-(R-\text{modules})$.

2. $Y \mapsto \mathrm{RHom}(X, Y) \in \mathcal{D}^+(\mathcal{A}b)$.

Functor $M \overset{L}{\otimes} N$ does not depend on the variable, with respect to which we derive. If $\mathcal{A}$ has enough injectives and projectives, the same is true for RHom.

Let's go back to $\mathcal{D}$-modules. Define $D(\mathcal{D}_X) = \mathcal{D}^b(\mathcal{M}(\mathcal{D}_X))$.

Theorem 2.13. *If X is smooth, $\mathcal{D}^b(\mathcal{M}_{hol}(\mathcal{D}_X)) \simeq D_{hol}(\mathcal{D}_X)$, where the last category is the full subcategory of $D(\mathcal{D}_X)$ with holonomic cohomologies.*

This kind of statement is not true in general. For example $\mathcal{D}^b_{f.dim.}(\mathfrak{g}-\text{modules}) \not\simeq \mathcal{D}^b(\text{ finite dimensional } \mathfrak{g}-\text{modules})$, where $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$.

3. Lectures 13 and 16 (03/14/02 and 04/02/02)

Let us now go apply the machinery of derived categories to $\mathcal{D}$-modules. We shall denote by $D(\mathcal{D}_X)$ the bounded derived category of $\mathcal{D}_X$-modules. When we want to stress that we work with left (resp. right) $\mathcal{D}_X$ we shall write $D^l(\mathcal{D}_X)$ (resp. $D^r(\mathcal{D}_X)$).

3.1. Duality. Let's define duality $\mathbb{D} : D^l(\mathcal{D}_X) \to D^r(\mathcal{D}_X)$ by $M \mapsto \underline{\mathrm{RHom}}(M, \mathcal{D}_X)$ for any left $\mathcal{D}_X$-module. Here $\underline{\mathrm{Hom}}(M, \mathcal{D}_X)$ is a quasi coherent sheaf of $\mathcal{D}_X$-modules, whose sections on every open subset $U \subset X$ are $\mathrm{Hom}(\Gamma(U, M), \mathcal{D}_U)$.

Let's define the map $\mathbb{D} : D^l(\mathcal{D}_X) \to D^l(\mathcal{D}_X)$ by $\mathbb{D}(M) = \underline{\mathrm{RHom}}(M, \mathcal{D}_X \otimes \Omega_X^{-1})[n]$, where $n = \dim X$.

THEOREM 3.2. (1) $\mathbb{D}^2 \simeq \mathrm{Id}$.
(2) *Let* $M, N \in \mathcal{D}^b(\mathcal{M}_{coh}(\mathcal{D}_X))$. *Then*

$$\mathrm{Hom}_{D(\mathcal{D}_X)}(M, \mathbb{D}(N)) = \mathrm{Hom}_{D(\mathcal{D}_X)}(N, \mathbb{D}(M)).$$

PROOF. Let $\mathcal{R}$ be the class of locally free $\mathcal{D}_X$-modules. To check that $\mathbb{D}^2 = \mathrm{Id}$, it's enough to check it on complexes of locally free locally finitely generated $\mathcal{D}_X$-modules. For such modules $\underline{\mathrm{Hom}}\ \underline{\mathrm{Hom}}((M, \mathcal{D}_X), \mathcal{D}_X) \simeq M$. So we have a natural morphism $\mathrm{Id} \to \mathbb{D}^2$. Since it is an isomorphism for every object in $\mathcal{R}$, it is an isomorphism in general. □

THEOREM 3.3. $\mathrm{RHom}(M, N) \simeq \mathrm{RHom}(\mathbb{D}N, \mathbb{D}M)$ *for any left* $\mathcal{D}_X$*-modules* M *and* N.

PROOF. There exist a natural morphism $\mathrm{RHom}(M, N) \to \mathrm{RHom}(\mathbb{D}N, \mathbb{D}M)$. For locally free modules this is an isomorphism, so it is an isomorphism in general (we can take a resolution by locally free modules). □

3.4. Inverse image. Let $\pi : X \to Y$ be a morphism and $\pi^\bullet$ – sheaf theoretical inverse image. Define the inverse image functor $\pi^!$ by $\pi^!(M) := \mathcal{O}_X \overset{L}{\otimes}_{\pi^\bullet \mathcal{O}_Y} \pi^\bullet M[\dim X - \dim Y]$ for any $\mathcal{D}_Y$-module M. Here $\overset{L}{\otimes}$ means the derived functor.

If π is closed embedding, then $\pi^! : D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X)$ in derived category, is the left derived of the functor $\pi^! : \mathcal{M}(\mathcal{D}_Y) \to \mathcal{M}(\mathcal{D}_X)$ for modules.

As before define $\mathcal{D}_{X\to Y} = \pi^! \mathcal{D}_Y[\dim Y - \dim X] = \mathcal{O}_X \overset{L}{\otimes}_{\pi^\bullet \mathcal{O}_Y} \pi^\bullet \mathcal{D}_Y$.

3.5. Direct image. Let $\pi : X \to Y$ be any morphism. Let us define the direct image of a right $\mathcal{D}_X$-module $M \in D^b(\mathcal{D}_X)$ as follows

$$\pi_*(M) = R\pi_\bullet(M \overset{L}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}).$$

Here $\pi_\bullet$ is the usual sheaf theoretical direct image. By definition $\pi_*(M)$ is a complex of sheaves of $\mathcal{D}_Y$-modules. A priori it's not clear, why these sheaves are quasi coherent. One way to show it is to use the following theorem

THEOREM 3.6. *[Bernstein] Let* $\mathcal{A}$ *be a quasi coherent sheaf of associative algebras on* X. *Then*

$$\mathcal{D}^b_{q.coh.}(\mathcal{M}(\mathcal{A})) \simeq \mathcal{D}^b(\mathcal{M}_{q.coh}(\mathcal{A})),$$

where the first category is the full subcategory of $\mathcal{D}^b(\mathcal{M}(\mathcal{A}))$ *consisting of complexes with quasi coherent cohomologies.*

Using a decomposition of π into a locally closed embedding and projection, one can show that $\pi_*(M)$ has quasi coherent cohomologies.

We shall give another proof of this fact, based on the explicit construction of direct image. The main idea is the following: suppose there exists a complex of quasi coherent sheaves $K^\bullet = K^\bullet(M)$, quasi isomorphic to $M \overset{L}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$. Let's choose a cover $X = \bigcup_i U_i$ by affine open subsets. Consider the Cech complex $\check{C}(K^\bullet)$ with respect to this cover, i.e. the total complex of the following bicomplex

$$\bigoplus_\alpha (j_\alpha)_* K^\bullet|_{U_\alpha} \to \bigoplus_{\alpha_1,\alpha_2} (j_{\alpha_1,\alpha_2})_* K^\bullet|_{U_{\alpha_1}\cap U_{\alpha_2}\to\ldots},$$

where $j_{\alpha_1,\ldots,\alpha_k} : U_{\alpha_1} \cap \cdots \cap U_{\alpha_k} \to X$. $K^\bullet$ and $\check{C}(K^\bullet)$ are quasi isomorphic and moreover

$$\pi_\bullet \check{C}(K^\bullet) = R\pi_\bullet(K^\bullet)$$

Since $K^i(M)$ are quasi coherent, so are $\check{C}(K^\bullet)^i$ and hence $\pi_\bullet\check{C}(K^\bullet)$. So, in this case $\pi_*(M) = R\pi_\bullet(K^\bullet)$ consist of quasi coherent modules.

Now, let us find such a complex $K^\bullet(M)$ (complex of quasi coherent sheaves, quasi isomorphic to $M \overset{L}{\otimes} \mathcal{D}_{X\to Y}$). Let's consider Koszul complex $Kos(M)$ which is quasi isomorphic to M. This complex is defined as follows: we know that $dR(\mathcal{D}_X)$ is a locally free resolution of the sheaf K_X of the top forms on X. Then $dR(\mathcal{D}_X)\otimes K_X^{-1}$ is a locally free resolution of $\mathcal{O}_X$. Let $Kos(M) = M \otimes dR(\mathcal{D}_X)\otimes K_X^{-1}$, i.e. $Kos(M)^i = M\otimes\Omega_X^i\otimes\mathcal{D}_X\otimes K_X^{-1}$. This complex is obviously a resolution of M. It carries an $\mathcal{O}_X$ action from the left (acts on M and $\mathcal{D}_X$) and right $\mathcal{D}_X$ action (acts on $\mathcal{D}_X$). Since M might not be locally free over $\mathcal{D}_X$ it follows that $Kos(M)$ also doesn't have to be locally free. However, it is easy to see that $Kos(M)$ is still locally free in ∂-direction. (This means that if we choose coordinates $x_1,\ldots,x_n,\partial_1,\ldots,\partial_n$ $Kos(M)$ consists of free $k[\partial_1,\ldots,\partial_n]$-modules.)

Modules, which are locally free in ∂-direction form an admissible class with respect to $\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}$, since $\mathcal{D}_{X\to Y}$ is locally free over $\mathcal{O}_X$. This means that $Kos(M)\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}$ is quasi isomorphic to $M \overset{L}{\otimes} \mathcal{D}_{X\to Y}$. Since the $\mathcal{O}_X$-action and $\mathcal{D}_X$-action commutes, $K^\bullet(M) = Kos(M)\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}$ is a complex of quasi coherent $\mathcal{O}_X$-modules. Using the arguments above, we get that $\pi_\bullet(K^\bullet(M)) = \pi_*(M)$ is a complex of quasi coherent modules.

3.7. Some computations of direct images. Example. Let $\pi : X \to pt$. We claim that $\pi_* M = R\pi_\bullet(dR(M^l))$ where $M^l = M\otimes K_X^{-1}$. Indeed, if $Y = pt$ then $\mathcal{D}_{X\to Y} = \mathcal{O}_X$. As before $M \overset{L}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$ is quasi isomorphic to $Kos(M)\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}$. And $(Kos(M)\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y})^i = (M\otimes\Omega_X^i\otimes\mathcal{D}_X\otimes K_X^{-1})\otimes_{\mathcal{D}_X}\mathcal{O}_X = M\otimes\Omega_X^i\otimes K_X^{-1} = dR(M^l)^i$. We get $\pi_*M = R\pi_\bullet(dR(M^l))$.

We can also say that if is a left module then we just have $\pi_*(M)$ is just isomorphic to the hypercohomology of $dR(M)$.

Let $\pi : X \to Y$ be any morphism. It can be decomposed as a product of locally closed embedding and smooth morphism.

(1) If $\pi : X \to Y$ is an open embedding direct image for modules coincides with the usual direct images of $\mathcal{O}_X$-modules.
(2) Let $\pi : X \to Y$ be closed embedding. In this case we can write $\pi_*(M) = \pi_\bullet(M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$.
(3) Suppose $\pi : X \to Y$ is smooth and let M be a left $\mathcal{D}_X$-module. In this case $\pi_*(M) = R\pi_\bullet(dR_{X/Y}(M))$, where $dR_{X/Y}(M)$ is a relative de Rham complex

$$0 \to M \to M \otimes \Omega^1_{X/Y} \to \cdots \to M \otimes \Omega^{\dim X - \dim Y}_{X/Y} \to 0.$$

The relative 1-forms are defined from the following exact sequence

$$0 \to \pi^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0.$$

and $\Omega^i_{X/Y} = \Lambda^i \Omega^1_{X/Y}$. Proof of this statement is analogous to the case of $Y = pt$, considered in example above.

3.8. Base change property. For any diagram

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\tilde{\tau}} & X \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ S & \xrightarrow[\tau]{} & Y \end{array}$$

we have a natural isomorphism of functors $\tau^!\pi_*\mathcal{F} = \tilde{\pi}_*\tilde{\tau}^!\mathcal{F}$, where $\mathcal{F} \in D^b(\mathcal{D}_X)$.

Proof. Let us decompose τ as a locally closed embedding and projection.

(1) If τ is an open embedding, the statement is obvious, since direct image is compatible with restriction to open subsets.
(2) Let $\tau : X \to Y$ be projection. Let $S = Y \times Z$, then $X \times_Y S = X \times Z$. We have the following diagram

$$\begin{array}{ccc} X \times Z & \xrightarrow{\tilde{\tau}} & X \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y \times Z & \xrightarrow[\tau]{} & Y \end{array}$$

For any $\mathcal{D}_X$-module $\mathcal{F}$ $\quad \tilde{\tau}^!\mathcal{F} = \mathcal{F} \boxtimes \mathcal{O}_Z[\dim Z]$ and
$\tilde{\pi}_*\tilde{\tau}^!(\mathcal{F}) = \pi_*\mathcal{F} \boxtimes \mathcal{O}_Z[\dim Z] = \tau^!\pi_*\mathcal{F}$.

(3) Suppose that τ is a closed embedding. In this case we are going to call it i (this is our stadard notation for a closed embedding). Consider the following diagram

$$\begin{array}{ccccc} W & \xhookrightarrow{\tilde{i}} & X & \xhookleftarrow[\tilde{j}]{} & U \\ \downarrow & & \downarrow{\scriptstyle\pi} & & \downarrow \\ S & \xhookrightarrow{i} & Y & \xhookleftarrow[j]{} & V \end{array}$$

Here S is closed in Y and V is the open complement to S. For each $W \overset{\tilde{i}}{\hookrightarrow} X \underset{\tilde{j}}{\hookleftarrow} U$ we have an exact triangle $\tilde{i}_*\tilde{i}^!\mathcal{F} \to \mathcal{F} \to \tilde{j}_*\mathcal{F}|_U$, which get rise to the following diagram

$$\begin{array}{ccccc} \tilde{i}_*\tilde{i}^!\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \tilde{j}_*\mathcal{F}|_U \\ & & \| & & \| \\ i_*i^!\pi_*\mathcal{F} & \longrightarrow & \pi_*\mathcal{F} & \longrightarrow & j_*\pi_*\mathcal{F}|_U \end{array}$$

Here the last terms are isomorphic by 1.

We need to construct an isomorphism between the two left terms in this diagram. Note that if we lived in an abelian category and the rows of the above diagram were short exact sequences we would get such an isomorphism automatically. However, here we are dealing with derived categories and we cannot derive the existence of such an isomorphism from general reasons because cone in the derived category is not canonical. However, it follows from the 5-lemma that if we construct any morphism between the two left terms of the above diagram which makes the whole diagram commutative then this morphism will necessarily be an isomorphism. This can be done in the following way.

Let $R \in D_S(\mathcal{D}_Y)$ and $T \in D(\mathcal{D}_V)$. Then

$$\mathrm{Hom}(R, j_*T) = \mathrm{Hom}(R|_V, T) = 0.$$

Let $R = \pi_*\tilde{i}_*\tilde{i}^!\mathcal{F}$. In this case we have

$$0 = \mathrm{Hom}(R, j_*\pi^*\mathcal{F}|_U[-1]) \to \mathrm{Hom}(R, i_*i^!\pi_*\mathcal{F}) \to$$
$$\to \mathrm{Hom}(R, \pi_*\mathcal{F}) \to \mathrm{Hom}(R, j_*\pi^*\mathcal{F}) = 0.$$

So $\mathrm{Hom}(R, i_*i^!\pi_*\mathcal{F}) \simeq \mathrm{Hom}(R, \pi_*\mathcal{F})$. Since we are given canonical element in $\mathrm{Hom}(R, \pi_*\mathcal{F})$ we also get an element in $\mathrm{Hom}(R, i_*i^!\pi_*\mathcal{F})$. The fact it makes the whole diagram commutative is clear.

Let us now explain why the isomorphism of functors constructed above doesn't depend on the decomposition of of τ as a product of a closed embedding and a smooth morphism. □

Example. If $\pi : X \to Y$ is a smooth map, then $\pi^!$ maps coherent modules to coherent (in this case $\pi^!$ is exact up to a shift). If $\pi : X \to Y$ is projective, then π_* maps coherent modules to coherent. To show the last statement, it's enough to assume, that $\pi : X = \mathbb{P}^N \times Y \to Y$, where Y is affine. In the category of coherent modules $\mathcal{D}_X$ is a projective generator (we showed this in the proof of the fact that $\mathbb{P}^N$ is $\mathcal{D}$-affine), so it's enough to prove the statement for $\mathcal{D}_X$. We have $\mathcal{D}_{X\to Y} = \mathcal{O}_{\mathbb{P}^N} \boxtimes \mathcal{D}_Y$. Thus

$$(K_X \otimes \mathcal{D}_X) \otimes \mathcal{D}_{X\to Y} = ((\Omega_{\mathbb{P}^N} \boxtimes \Omega_Y) \otimes_{\mathcal{O}_X} (\mathcal{D}_{\mathbb{P}^N} \boxtimes \mathcal{D}_Y)) \otimes_{\mathcal{D}_X} (\mathcal{O}_{\mathbb{P}^N} \boxtimes \mathcal{D}_Y) = K_{\mathbb{P}^N} \boxtimes (\Omega_Y \otimes \mathcal{D}_Y).$$

So $\pi_*\mathcal{D}_X = \mathcal{D}_Y[-N]$.

THEOREM 3.9. *Let $\pi : X \to Y$ be projective. Then*

(1) *π_* is left adjoint to $\pi^!$.*
(2) *$\mathbb{D}\pi_* = \pi_*\mathbb{D}$.*

PROOF. As before, it is enough to assume that $X = \mathbb{P}^N \times Y$, where Y is affine.

Let us prove 1. Let $\mathcal{F} \in D(\mathcal{D}_X)$ and $\mathcal{G} \in D(\mathcal{D}_Y)$. We want to prove that $\mathrm{RHom}(\pi_*\mathcal{F}, \mathcal{G}) \simeq \mathrm{RHom}(\mathcal{F}, \pi^!\mathcal{G})$. It is enough to consider $\mathcal{F} = \mathcal{D}_X$, since it's a projective generator and $D(\mathcal{D}_X)$ is equivalent to the homotopic category of free complexes.

As we have computed, $\pi_*\mathcal{D}_X = \mathcal{D}_Y[-N]$. So

$$\mathrm{RHom}(\pi_*\mathcal{D}_X, \mathcal{G}) = \mathrm{RHom}(\mathcal{D}_Y[-N], \mathcal{G}) = R\Gamma(Y, \mathcal{G})[N]$$

On the other hand $\pi^!\mathcal{G} = \mathcal{O}_{\mathbb{P}^N} \boxtimes \mathcal{G}[N]$, and hence

$$\mathrm{RHom}(\mathcal{D}_X, \pi^!\mathcal{G}) = \mathrm{R}\Gamma(\pi^!\mathcal{G}) = \Gamma(Y, \mathcal{G})[N].$$

To prove 2 it is again enough to construct the above isomorphism for $\mathcal{D}_X$ which is done by means of a similiar calculation. □

THEOREM 3.10. *Let $\pi : X \to Y$ be smooth. Then*

(1) $\mathbb{D}\pi^![\dim Y - \dim X] = \pi^!\mathbb{D}[\dim X - \dim Y]$.
(2) $\pi^![2(\dim Y - \dim X)]$ *is left adjoint to π_*.*

CHAPTER 3

The derived category of holonomic $\mathcal{D}$-modules

1. Lecture 17

So far we have discussed the general properties of the derived category of holonomic D-modules. However, it turns out that many interesting things can only be done for holonomic D-modules. So, let us turn to a more detailed study of the derived category of holonomic D-modules.

Let us first summarize (and reformulate a little) what we already know about the direct and inverse image functors. Let $\pi : X \to Y$. There are two functors associated with π: π_* and $\pi_!$ (the last one is not always defined; recall that $\pi_! M$ exists if $\pi_* \mathbb{D} M$ is coherent). If $\pi_!$ is defined, there exists a canonical morphism $\pi_! \to \pi_*$, defined as follows. Firs of all let us decompose π as a product of an open embedding and projective morphism (as usual we shall leave the verification of the fact that the resulting morphism does not depend on the choice of this decomposition to the reader). Recall that if π is projective then $\pi_!$ and π_* are the same. Thus it is enough to construct our morphism for an open embedding. If we have an open embedding $j : U \hookrightarrow X$ then for any module M on U we have $\operatorname{Hom}(N, j_* M) = \operatorname{Hom}(N|_U, M)$. For $N = j_! M$ we have $N|_U = M$ and hence $\operatorname{Hom}(j_! M, j_* M) = \operatorname{Hom}(M, M)$. We let the morphism $j_I M \to j_* M$ be the image of the identity in $\operatorname{Hom}(M, M)$.

Similarly we have the functors $\pi^!$ and π^* (where the latter is only partially defined).

Theorem 1.1. *Let $\pi : X \to Y$ be smooth morphism. Then*
(1) *$\pi^!$ maps $D_{coh}(\mathcal{D}_Y)$ to $D_{coh}(\mathcal{D}_X)$.*
(2) *$\pi^* = \mathbb{D}\pi^!\mathbb{D} = \pi^![2(\dim Y - \dim X)]$.*
(3) *π^* is left adjoint to π_*.*

In case when π is open embedding, $\pi^! = \pi^*$.

Of course it would be very useful to have a situation in which we are guaranteed that π^* and $\pi_!$ exist. It turns out that this is always the case for holonomic modules. Moreover, we have the following theorem.

Theorem 1.2. *$D_{hol}(\mathcal{D})$ is stable under π_*, $\pi^!$, $\mathbb{D}$ and $\boxtimes$.*

Corollary 1.3. *On $D_{hol}(\mathcal{D}_X)$ we can always define $\pi_! = \mathbb{D}\pi_*\mathbb{D}$ and $\pi^* = \mathbb{D}\pi^!\mathbb{D}$.*

Defined in this way, $\pi_!$ is left adjoint to $\pi^!$ and $\pi_!$ is right adjoint to π^*.
?? and Corollary 1.3 above give us an example of a so called Grothendieck's formalism of six functors. We do not give a precise definition; let us just note that it follows

that we have 6 kinds of functors acting between the categories $D_{hol}(\mathcal{D}_X)$ - namely, $\pi^*, \pi_*, \pi^!, \pi_!$ (for a morphism $\pi : X \to Y$), $\mathbb{D}$ and $\boxtimes$.

Before proving this theorem, let us give an example of its application. Namely let us discuss the structure of holonomic modules and classify irreducible ones.

Lemma 1.4. *Let M be a holonomic module on X with supp$M = X$. Then there exists a smooth open $U \subset X$ such that $j^! M$ is $\mathcal{O}_U$-coherent on U. (Here $j : U \hookrightarrow X$.)*

Recall that an $\mathcal{O}_U$-coherent $\mathcal{D}_U$-modules may be thought of as a vector bundle on U endowed with a flat connection.

Proof. We know that $j^! M$ is $\mathcal{O}_U$-coherent iff $s.s.(M) = \{$ zero-section in $T^*U\}$.
Let us consider the following diagram

$$\begin{array}{ccc} s.s.(M) & \hookrightarrow & T^*X \\ & \searrow & \downarrow \\ & & X \end{array}$$

Since $s.s.(M)$ is conic it follows that any fiber of the map $s.s.(M) \to X$ is either 0 or of dimension greater or equal to 1. Since $\dim s.s.(M) = \dim X$ the fiber over generic point cannot be of dimension > 0. So, over the generic point the diagonal arrow is an isomorphism. Hence there exists U as above , such that $j^! M$ is $\mathcal{O}$-coherent. □

It is clear that we may choose the open subset U above to be affine.

Let us now look at the case when M is irreducible. In this case let us try to restore M from $j^! M$. Here $j : U \to X$ is any open embedding. First of all, if M is irreducible, so is $j^! M$. Indeed, suppose we have a short exact sequence $0 \to K \to j^! M \to N \to 0$ with $K, N \neq 0$. Then we have the map $M \to j_* N$ which in fact factorizes as $M \to H^0(j_* N) \to j_* N$ (since $j_* N$ lives in degress ≥ 0. Let $\widetilde{K}$ denote the kernel of this map. Then $j^! \widetilde{K} = K$ which shows that $\widetilde{K}$ is non-zero but different from M. This contradicts irreducibility of M.

Let us now set $N = j^! M$ and pretend that we are only given N nut not M and we want to effectively construct M. In fact it turns out that we can do more: to every holonomic $\mathcal{D}_U$-module we are going to associate (canonically) a newholonomic $calD_X$-module $j_{!*} N$ called the *intermediate* or *minimal* (or Deligne-Goresky-MacPherson) extension of N to X which in particular will solve our problem in the case when N is irreducible. This extension will in fact be uniquely characterized in the following way.

Theorem 1.5. *Let X be an irreducible variety and let $U \subset X$ be an open subset. For every holonomic $\mathcal{D}_U$-module N there exists unique $\mathcal{D}_X$-module M satisfying the following properties:*

(1) $j^!(j_{!*}(N)) = N$;

(2) $j_{!*}$ *has no submodules or quotients concentrated on* $X\backslash U$.

Moreover, we claim that if N is irreducible then $j_{!*}(N)$ (defined by (1) and (2) above) is also irreducible. Indeed, suppose there is a submodule $M \hookrightarrow j_{!*}(N)$. Applying the functor $j^!$ we get $j^!M \to N$. Since N is irreducible, this map is either an isomorphism or it is equal to 0. Assume first that this map is 0; in this case M is concentrated on $X\backslash U$ which contradicts property (2). In the case when this map is an isomorphism it follows $j_{!*}(N)/M$ is concentrated on $X\backslash U$ which again contradicts property (2).

In particular, it follows that if N is an irreducible $\mathcal{D}_U$-module then N has unique irreducible extension to X - namely the intermediate extension $j_{!*}N$.

1.6. Construction of intermediate extension. Let us now construct the extension $j_{!*}N$ satisfying properties (1) and (2) of Theorem 1.5. As is suggested by both the name and the notation it should somehow be constructed out of $j_!N$ and j_*N.

General remark: Let $\mathcal{A}$ and $\mathcal{B}$ two be abelian categories. Let us introduce the following notations: $D^{\geq 0} = \{K^\bullet \mid H^i(K^\bullet) = 0 \text{ for } i < 0\}$ and $D^{\leq 0} = \{K^\bullet \mid H^i(K^\bullet) = 0 \text{ for } i > 0\}$. Then for any left exact functor $F : \mathcal{A} \to \mathcal{B}$ we have $RF : D^{\geq 0}(\mathcal{A}) \to D^{\geq 0}(\mathcal{B})$ and for any right exact functor $G : \mathcal{A} \to \mathcal{B}$ we have $LG : D^{\leq 0}(\mathcal{A}) \to D^{\leq 0}(\mathcal{B})$.

In particular, $j_* : D^{\geq 0}_{hol}(\mathcal{D}_U) \to D^{\geq 0}_{hol}(\mathcal{D}_X)$ and $j_! : D^{\leq 0}_{hol}(\mathcal{D}_U) \to D^{\leq 0}_{hol}(\mathcal{D}_X)$.

Consider the map $j_!N \to j_*N$. By the remark above $j_!N \in D^{\leq 0}_{hol}$ and $j_*N \in D^{\geq 0}_{hol}$. Hence this map factorizes through the chain

$$j_!N \to H^0(j_!N) \to H^0(j_*N) \to j_*N. \tag{1.1}$$

Let $j_{!*}$ be the image of this map. Note that is j is an affine.embedding (we can in fact always reduce the situation to this case) then $j_{!*}N$ is simply the image of $j_!N$ in j_*N.

We claim that $j_{!*}N$ defined in this way satisfies properties (1) and (2) of Theorem 1.5.

First, all the maps in (1.1) become isomorphisms after restricting to U. Thus the restriction of $j_{!*}N$ to U is equal to N. Hence property 1 is satisfied.

Let us now show property 2. We have the maps $j_!N \to j_{!*}N \to j_*N$. It is easy to see from the definition that for any non-zero $\mathcal{D}_X$-module L the induced maps $\mathrm{Hom}(j_{!*}N, L) \to \mathrm{Hom}(j_!N, L)$ and $\mathrm{Hom}(L, j_{!*}N) \to \mathrm{Hom}(L, j_*N)$ are injective.

Let L be any $\mathcal{D}_X$-module concentrated on $X\backslash U$. Then $\mathrm{Hom}(j_!N, L) = \mathrm{Hom}(N, j^!L) = 0$ since $j^!L = 0$. Similarly, $\mathrm{Hom}(L, j_*N) = \mathrm{Hom}(j^*L, N) = \mathrm{Hom}(j^!L, N) = 0$. This means that j_*N has no submodules concentrated on $X\backslash U$ and $j_!N$ has no quotients concentrated on $X\backslash U$. Hence by the above observation it follows that $\mathrm{Hom}(j_{!*}N, L) = \mathrm{Hom}(L, j_{!*}N) = 0$.

Warning. We have shown above that $j_{!*}N$ has neither quotients nor submodules concentrated on $X\backslash U$. However, we do not claim (and in general it is not true) that $j_{!*}N$ has no *subquotients* concentrated on $X\backslash U$.

Let us now show that the properties (1) and (2) of Theorem 1.5 define $j_{!*}N$ uniquely.

Let M be $\mathcal{D}_X$ module satisfying these properties. In particular, $j^!(M) = j^*M = N$. Thus by adjointness the map $j_!N \to j_*N$ factorizes through the sequence

$$j_!N \to M \to j_*N.$$

Since M is concentrated in cohomological degree 0 it follows that this sequence in fact factorizes through

$$j_!N \to H^0(j_!N) \xrightarrow{\alpha} M \xrightarrow{\beta} H^0(j_*N) \to j_*N.$$

Also all of these maps become isomorphisms when restricted to U. Thus the cokernel of α is a quotient module of M which is concentrated on $X\backslash U$ and therefore it is 0. Similarly the kernel of β is a submodule of M concentrated on $X\backslash U$ and hence it is 0. In other words, α is surjective and β is injective. This means that M is the image of the map $H^0(j_!N) \to H^0(j_*N)$ which finishes the proof.

Example. Let $X = \mathbb{A}^1, \quad U = \mathbb{A}^1\backslash\{0\}$ and $N = \mathcal{O}_U$. In this case $j_!\mathcal{O}_U = \mathbb{D}j_*\mathcal{O}_U$ and there are two exact sequences $0 \to \mathcal{O}_X \to j_*\mathcal{O}_U \to \delta_0 \to 0$ and $0 \to \delta_0 \to j_!\mathcal{O}_U \to \mathcal{O}_X \to 0$, dual to each other. So we have a sequence $j_!\mathcal{O}_U \twoheadrightarrow \mathcal{O}_X \hookrightarrow j_*\mathcal{O}_U$, which means that $j_{!*}\mathcal{O}_U = \mathcal{O}_X$.

If N is $\mathcal{D}_U$-module generated by x^λ, where $\lambda \neq \mathbb{Z}$, then $j_!N \to j_*N$ is an isomorphism, since both modules are irreducible.

Note that in fact $j_{!*}$ is defined not only for an open embedding j but for any locally closed embedding.

Let us point out several properties of $j_{!*}$.

Lemma 1.7. (1) *$j_{!*}N$ is functorial in N.*
(2) *If N is irreducible then so is $j_{!*}N$.*
(3) *$j_{!*}(\mathbb{D}N) = \mathbb{D}(j_{!*}N)$.*

Proof. (1) follows immediately from the fact that $j_!$ and j_* are functors. Note, however, that $j_{!*}$ is neither left exact nor right exact. As a result the functor $j_{!*}$ makes sense only for modules and it doesn't make sense for general objects of the derived category.

(2) we have already discussed above.

To prove (3) let us check that $\mathbb{D}(j_{!*}N)$ satisfies properties (1) and (2) of Theorem 1.5. Property (1) is in fact obvious. To prove (2) let us remark that $\mathbb{D}$ is a contravariant equivalence of categories which does not change the support of a module. Hence it transforms submodules concentrated on $X\backslash U$ to quotient modules concentrated on $X\backslash U$ and vice versa. □

Here is some interesting application of the notion of intermediate extension (probably one of the most important ones). Let X be singular variety. Let $U \subset X$ be smooth dense open subset. Set

$$\mathrm{IC}_X = j_{!*}O_U \in \mathcal{M}_{hol}(\mathcal{D}_X)$$

We shall call IC_X the *intersection cohomology $\mathcal{D}$-module of X*.

If $\pi : X \to pt$, then $\pi_*(IC_X)$ is called the *intersection cohomology of X*. In the case when X is smooth one clearly has $\mathrm{IC}_X = \mathcal{O}_X$ since $\mathcal{O}_X$ is irreducible. Thus by Theorem ?? in the case when X is smooth its intersection cohomology is naturally isomorphic to the ordinary cohomology (with complex coefficients) of the underlying analytic space.

The intersection cohomology $\mathcal{D}$-module of X has many nice properties. For example since $\mathcal{O}_U$ is self-dual then so is IC_X. Assume for example X is proper, then $\mathbb{D}\pi_* = \pi_*\mathbb{D}$. Hence

$$\mathbb{D}\pi_*(IC_X) = \pi_*\mathbb{D}(IC_X) = \pi_*(IC_X)$$

i.e. the intersection cohomology of an proper variety satisfies Poincare duality.

1.8. Description of irreducible holonomic modules. It is clear from the above discussion are essentially classified by pairs (Z, N), where $Z \subset X$ is an irreducible smooth locally closed and N is an irreducible $\mathcal{O}$-coherent module on Z. The corresponding module is then $j_{!*}N$ where $j : Z \to X$ is the natural embedding. Indeed, if M is an irreducible $\mathcal{D}_X$-module, let $Y \subset X$ be the support of M. This is an irreducible closed subset of X and by Kashiwara's theorem M corresponds to some irreducible holonomic $\mathcal{D}_Y$-module M_Y. We now choose Z to be any smooth open subset of Y such that $M_Y|_Z$ is $\mathcal{O}$=coherent.

Moreover, if $Z' \subset Z$ is open, then $(Z', M\mid_{Z'})$ and (Z, M) correspond to the same holonomic module on X. In this case we say that (Z, M) and (Z', M') are equivalent. Let us generate by this an equivalence relation on the pairs (Z, M) as above. Then irreducible holonomic $\mathcal{D}_X$-modules are in one-to-one correspondence with equivalence classes of pairs (Z, M).

Exercise: Assume that an algebraic group G acts on an algebraic variety X with finitely many orbits. Let $x_1, \dots x_k$ be representatives of orbits and $Z_i = \mathrm{Stab}_G(x_i)$ be stabilizators of x_i in G.

1. Define the notion of G-equivariant modules.
2. Prove, that irreducible G-equivariant modules are in one to one correspondence with set of pairs $\{i$, irreducible representation of $Z_i/Z_i^0\}$.

2. Lecture 18: Proof of Theorem 1.2

We now turn to the proof of Theorem 1.2. It is easy to see that the theorem is true for the functor $\boxtimes$. Also, we already know that it is true for the functor $\mathbb{D}$. It remains to prove it for inverse and direct images.

Let us first prove this Theorem 1.2 for $\pi^!$. Any morphism can be decomposed as a product of closed embedding and projection. Let $\pi : X = Y \times Z \to Y$ be projection.

For any $\mathcal{F} \in D_{hol}(\mathcal{D}_Y)$ $\quad \pi^! \mathcal{F} = \mathcal{F} \boxtimes \mathcal{O}_Z[\dim Z]$ and hence is holonomic. The case when π is closed embedding, we've already proved.

Now let us consider the case of π_*. Any arbitrary morphism $\pi : X \to Y$ can be decomposed as a product of a closed embedding and projective morphism. We have already proved the theorem for closed embedding. The case of projective morphism follows from the **main step**: the case of an open embedding.

First way of proving this statement for projective morphisms (without using the main step).
Let $\mathcal{F}$ be any holonomic $\mathcal{D}_Y$-module. Then $\pi_*\mathcal{F}$ is coherent and has finite dimensional fibers. This means that for any $i : y \hookrightarrow Y$ and $\tilde{i} : \pi^{-1}(y) \hookrightarrow Y$ $\quad i^! \pi_* \mathcal{F} = \pi_* \tilde{i}^! \mathcal{F}$ is finite dimensional (this is coherent module on a point and hence a vector space). Theorem follows from the following proposition.

PROPOSITION 2.1. *[Bernstein] Let $\mathcal{F} \in D_{coh}(\mathcal{D}_X)$. Then $\mathcal{F}$ is holonomic iff it has finite dimensional fibers.*

Second way of proving theorem in case of projective morphism.
We have to prove that for $\pi : Y \times \mathbb{P}^N \to Y$ $\quad \pi_* : D_{hol}(\mathcal{D}_X) \to D_{hol}(\mathcal{D}_Y)$. We have already proved the analogous claim for $\pi : \mathbb{A}^n \to \mathbb{A}^m$. Using the same method one can prove it for $\pi : Y \times \mathbb{A}^N \to Y$ (namely reducing to the case of affine Y and $N = 1$, where $\pi_* M = \{M \xrightarrow{\frac{\partial}{\partial x}} M\}$).

Now let us consider $\pi : Y \times \mathbb{P}^N \to Y$. We'll do induction on N. Let $j : U = Y \times \mathbb{A}^N \hookrightarrow X = Y \times \mathbb{P}^N$. For any sheaf $\mathcal{F}$ on X, there is the following exact triangle $\mathcal{F} \to j_* \mathcal{F} \mid_U \to K$. Here K is supported on $Y \times \mathbb{P}^{N-1}$.

General remark: Let $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$ be an exact triangle. Then if two of the terms are holonomic, so is the third one. To see this one should write the corresponding long exact sequence of cohomologies.

In our case $\mathcal{F}$ is holonomic by assumption and $j_*\mathcal{F} \mid_U$ is holonomic by main step (which we have not proved yet). Hence K is also holonomic. Applying π_* to that triangle we get

$$\pi_* \mathcal{F} \to \pi_* j_* \mathcal{F} \mid_U \to \pi_* K.$$

Let us denote $\tilde{\pi} : Y \times \mathbb{A}^N \to Y$. Then $\pi_* j_* = \tilde{\pi}_*$ and hence $\pi_* j_* \mathcal{F} \mid_U$ is holonomic by the previous step. By induction hypothesis $\pi_* K$ is also holonomic and hence so is $\pi_* \mathcal{F}$.

Proof of the main step:

Let $j : U \hookrightarrow X$ be open embedding. Since holonomicity is a local property, we can assume that X is affine. We can also assume that U is affine and moreover U is a zero locus of $p \in \mathcal{O}_X$. (If not, we can cover U with finitely many affine open subsets $\{U_\alpha\}$ and consider the Check complex $\check{C}(M)$ with respect to this cover. This is a complex $\mathcal{D}_X$-modules $j_{\alpha_1,\dots,\alpha_k *}(M \mid_{U_{\alpha_1} \cap \dots \cap U_{\alpha_k}})$ where $j_{\alpha_1,\dots,\alpha_k} : U_{\alpha_1} \cap \dots \cap U_{\alpha_k} \hookrightarrow X$ and $\check{C}(M) = j_*(M) \in D(\mathcal{D}_X)$. Hence $j_* M$ is holonomic when $j_{\alpha_1,\dots,\alpha_k *}(M \mid_{U_{\alpha_1} \cap \dots \cap U_{\alpha_k}})$ are holonomic.)

Example. Let $X = \mathbb{A}^n$, $p \in \mathcal{O}_X$ and $\lambda \in \Bbbk$. Consider $\mathcal{D}_U$-module $M(p^\lambda)$ (here U is the zero locus of p). At the begining of the course we've proved that j_*M is holonomic.

One way of proving the main step is to reduce it to the case $X = \mathbb{A}^n$, embedding X into an affine space. Consider U – zero locus of p and generalize the proof for $M(p^\lambda)$ to any module M.

We'll give another proof of this fact. Here is the sketch of it.

Step 1. We can assume that M is generated by some element $u \in M$. Let $N = j_*M$. Consider a submodule $N_k \subset N$ generated by up^k. Then for $k \gg 0$ N_k is holonomic and $N_k = N_{k+1}$.

Step 2. Lemma on b-functions.
Let ${}^\lambda X = X \otimes \Bbbk(\lambda)$ be the same variety as X considered over a field $\Bbbk(\lambda)$. Let ${}^\lambda M = M \otimes p^\lambda$. Then there exist $d \in \mathcal{D}_{{}^\lambda X}$ such that $d(up^\lambda) = up^{\lambda-1}$. (This is equivalent to say, that there exist $d \in \mathcal{D}_X[\lambda]$ and $b(\lambda) \in \Bbbk[\lambda]$ such that $d(up^\lambda) = b(\lambda)up^{\lambda-1}$.)

If lemma on b-functions holds, then ${}^\lambda M$ is generated by $up^{\lambda+k}$. And for $k \gg 0$ it is holonomic by step 1.

Step 3. Let $\widetilde{M}$ be $\mathcal{O}$-$\mathcal{D}$-module on $\mathbb{A}^1 \times X$ (i.e. an $\mathcal{O}$-module on $\mathbb{A}^1$ and $\mathcal{D}$-module on X), defined as follows:

$$\widetilde{M} = span\{q(\lambda)mp^{\lambda+i} \mid i \in \mathbb{Z}, m \in M, q(\lambda) \in \Bbbk[\lambda]\}.$$

For any $\alpha \in \mathbb{A}^1$ let $M_\alpha = \widetilde{M}/(\lambda - \alpha)\widetilde{M}$. We claim that for generic α this module is holonomic. Indeed, let Z be singular support in $\mathcal{D}_X$-direction of $\widetilde{M}$ in $T^*X \times \mathbb{A}^1$. After passing to a generic point of $\mathbb{A}^1$ we'll get the singular support of ${}^\lambda M$, which is holonomic by step 2. This means that $\dim(Z \otimes \Bbbk(\lambda)) = \dim X$.

So, we have a map $Z \to \mathbb{A}^1$ with fibers $Z_\alpha = s.s.M_\alpha$, $M_0 = M$ (and in fact $M_i = M$ for any $i \in \mathbb{Z}$). The general fiber of this map has dimension $\dim X$. Hence there exist an integer i, such that $\dim Z_i = \dim X$ and this means that M is holonomic.

Now let us deduce step 1 from the following lemma

Lemma 2.2. *Consider an inclusion $I : \mathcal{M}_{hol}(\mathcal{D}_X) \to \mathcal{M}_{coh}(\mathcal{D}_X)$. It has a right adjoint functor $G : \mathcal{M}_{coh}(\mathcal{D}_X) \to \mathcal{M}_{hol}(\mathcal{D}_X)$ defined by $G(N) = \{$ the maximal holonomic submodule of $N\}$. This functor commutes with restriction to open subsets.*

Let $N = j_*M$. By the lemma $G(N)\mid_U = M$. For $k \gg 0$ we have $up^k \in G(N)$ and hence $N_k \subset G(N)$. In particular N_k is holonomic. Since $G(N)$ is holonomic, it has finite length and this means taht $N_k = N_{k+1}$ for k large enough.

Now let us deduce step 2. Consider ${}^\lambda N = j_*({}^\lambda M)$ and apply step 1 to it. We'll get ${}^\lambda N_k = {}^\lambda N_{k+1}$ for $k \gg 0$. And this gives us the lemma on b-functions.

For any k ${}^\lambda N$ is generated by $up^{\lambda+k}$, so ${}^\lambda N = {}^\lambda N_k$ and hence is holonomic.

Proof of Lemma 2.2: Let us give an another construction of G. Given N we can constuct $\mathbb{D}(N)$, which is a complex, living in negative degrees. Let $G(N) =$

$\mathbb{D}H^0(\mathbb{D}(N))$. Defined in such a way, G obviously commutes with restriction to an open sets. To prove, that it coincides with G, let's prove that it's right adjoint to I. Let $K \in \mathcal{M}_{hol}(\mathcal{D}_X)$ be holonomic. Then

$$\mathrm{Hom}(K, \mathbb{D}H^0(\mathbb{D}N)) = \mathrm{Hom}(H^0(\mathbb{D}N), \mathbb{D}K) = \mathrm{Hom}(\mathbb{D}N, \mathbb{D}K)$$

since $\mathbb{D}N \in D^{\leq 0}(\mathcal{D}_X)$, and $\mathbb{D}K$ is a module, since K is holonomic. So, $\mathrm{Hom}(K, \mathbb{D}H^0(\mathbb{D}N)) = \mathrm{Hom}(N, K)$ and our functor coincedes with G. □

3. Lecture 18 (04/09/02)

Let $\pi : X \to Y$ be any morphism. By the theorem we proved last time, the functor $\pi^* = \mathbb{D}\pi^!\mathbb{D} : D_{hol}(\mathcal{D}_Y) \to D_{hol}(\mathcal{D}_X)$ is well defined.

Proposition 3.1. (1) *π^* is left adjoint to π_*.*
(2) *If π is smooth $\pi^! = \pi^*[2(\dim Y - \dim X)]$.*

Proof. (1) Let's decompose π as a product of an open embedding and projective morphism.

If $\pi : U \hookrightarrow X$ is an open embedding, then $\pi^* = \pi^!$ and is simply the restriction to U. We've already discussed that in this case π^* is left adjoint to π_*.

If π is projective, then $\pi_! = \pi_*$ and

$$\mathrm{Hom}(\pi^*M, N) = \mathrm{Hom}(\mathbb{D}\pi^!\mathbb{D}M, N) = \mathrm{Hom}(\mathbb{D}N, \pi^!\mathbb{D}M) =$$
$$= \mathrm{Hom}(\pi_!\mathbb{D}N, \mathbb{D}M) = \mathrm{Hom}(\pi_*\mathbb{D}N, \mathbb{D}M) = \mathrm{Hom}(M, \pi_*N).$$

(2) Let $\pi : Z \times Y \to Y$, where Z is smooth. Then $\pi^!M = \mathcal{O}_Z \boxtimes M[\dim Z]$ and $\pi^*M = \mathbb{D}\pi^!\mathbb{D}M = \mathbb{D}(\mathcal{O}_Z \boxtimes \mathbb{D}M[\dim Z]) = \mathcal{O}_Z \boxtimes M[-\dim Z] = \pi^!M[-2\dim Z]$.

Any smooth morphism $\pi : X \to Y$ is formally a projection, i.e. for any $x \in X$ and $y = \pi(x)$ there exist a formal neighborhood of x $\hat{X}_x = \hat{Y}_y \times Z$ (here $\hat{Y}_y$ is a formal neighborhood of y and Z is smooth) such that $\pi|_{\hat{X}_x} : \hat{Y}_y \times Z \to \hat{Y}_y$. This proves proposition.

□

3.2. Elementary comlexes. Let Z be a smooth variety and $j : Z \hookrightarrow X$ be locally closed embedding.

Definition. If M is an $\mathcal{O}$-coherent $\mathcal{D}_Z$-module, then j_*M is called an elementary complex.

Let $\mathcal{A}$ be an abelian category.

Definition. An object $K \in D(\mathcal{A})$ is glued from L_1 and L_2, if there exist an exact triangle $L_1 \to L_2 \to K \to L_1[1]$.

Definition. Let $\mathcal{L}$ be a class of objects of $D(\mathcal{A})$ invariant under shift functor. Then $\mathcal{K}$ is called class of objects glued from $\mathcal{L}$, if it satisfies the following properties

(1) $\mathcal{L} \subset \mathcal{K}$
(2) If $L_1, L_2 \in \mathcal{K}$, then K, glued from L_1 and L_2 is also in $\mathcal{K}$.

LEMMA 3.3. *Every object of $D_{hol}(\mathcal{D}_X)$ is glued from elementary objects.*

PROOF. Let us assume, that $M \in D_{hol}(\mathcal{D}_X)$ is a module.

We'll prove this lemma by induction on $\dim(suppM)$. Any module on a point is by definition an elementary complex. By induction hypothesis we know this lemma for any M, such that $suppM = Z \subset X$ is a proper subvariety. Suppose $suppM = X$. Then there exist an open subset $U \subset X$, such that $M\mid_U$ is $\mathcal{O}$-coherent. Consider the following exact sequence

$$M \to j_*(M\mid_U) \to K,$$

where $j : U \hookrightarrow X$. Here K is supported on $Z = X\backslash U$ and hence by induction hypothesis is glued from elementary complexes. Since $j_*(M\mid_U)$ is an elementary complex by definition, M is also glued from elementary complexes. □

Example. (from problem set)
Let $X = \mathbb{A}^1$ and $U = \mathbb{A}^1\backslash 0 \subset U$. For any $n \in \mathbb{Z}$ let us introduce $\mathcal{D}_U$-module $\mathcal{E}_n = M(x^\lambda)/\lambda^n$.

Topological interpretation. For every n $\mathcal{E}_n$ has regular singularities and hence corresponds to some representation of $\pi_1(U) = \mathbb{Z}$. Namely, $\mathcal{E}_n$ corresponds to the representation $\mathbb{Z} \ni 1 \mapsto$ Jordan block of size n.

Consider $\mathcal{D}_U$-module $\mathcal{E}_2$. Let us describe $j_{!*}\mathcal{E}_2$ as a module, glued from elementary complexes. (Here $j : U \hookrightarrow X$.) For $\mathcal{E}_2$ we have the following exact sequence $0 \to \mathcal{O}_U \to \mathcal{E}_2 \to \mathcal{O}_U \to 0$. This gives us two other exact sequences

$$0 \to j_*\mathcal{O}_U \to j_*\mathcal{E}_2 \to j_*\mathcal{O}_U \to 0$$

and

$$0 \to j_!\mathcal{O}_U \to j_!\mathcal{E}_2 \to j_!\mathcal{O}_U \to 0.$$

Hence the following two sequences are exact

$$0 \to j_*\mathcal{O}_U \to j_{!*}\mathcal{E}_2 \to \mathcal{O}_X \to 0;$$

$$0 \to \mathcal{O}_X \to j_{!*}\mathcal{E}_2 \to j_!\mathcal{O}_U \to 0.$$

So $j_{!*}\mathcal{E}_2$ is glued from $\mathcal{O}_X$ and $j_*\mathcal{O}_U$ or from $j_!\mathcal{O}_U$ and $\mathcal{O}_X$.

Because of the following exact sequences

$$0 \to \mathcal{O}_X \to j_*\mathcal{O}_U \to \delta \to 0,$$

$$0 \to \delta \to j_!\mathcal{O}_U \to \mathcal{O}_X \to 0,$$

we can say, that $j_{!*}\mathcal{E}_2$ has form $\mathcal{O}_X)\delta)\mathcal{O}_X$ (i.e. $j_{!*}\mathcal{E}_2$ is glued from δ and two copies of $\mathcal{O}_X$ in the indicated order).

More generally, $j_{!*}\mathcal{E}_n$ has a form $\mathcal{O}_X)\delta)\mathcal{O}_X)\dots)\delta)\mathcal{O}_X$.

3.4. A little bit more about intersection cohomology module. In the examples below we'll use the following notations

$$H^i(M) = H^i(\pi_* M); \quad H^i_c(M) = H^i(\pi_! M).$$

Examples (Poincare duality). Let X be any variety, $M \in D_{hol}(\mathcal{D}_X)$ and $\pi : X \to pt$.

(1) If X is smooth then $\pi_* M = H^*_{dR}(M)$. On the other hand

$$\pi_* M = \mathbb{D}\pi_! \mathbb{D} M = (\pi_! \mathbb{D} M)\check{}.$$

So, $H^i(\pi_* M) = H^{-i}(\pi_! \mathbb{D} M)\check{}$.

(2) Suppose X is smooth and proper and $M = \mathcal{O}_X$. Then $\mathbb{D}\mathcal{O}_X = \mathcal{O}_X$ and $\pi_* = \pi_!$. Hence $H^i(\mathcal{O}_X)\check{} = H^{-i}(\mathcal{O}_X)$.

(3) Let X be projective, but not smooth. Let $j : U \hookrightarrow X$ be a smooth open subset of X. We've defined the intersection cohomology module $IC_X = j_{!*}\mathcal{O}_U = \mathrm{Image}(H^0 j_! \mathcal{O}_U \to H^0 j_* \mathcal{O}_U)$.

For any module M on U the intermediate extension $j_{!*}M$ has the following properties:

(1) $j_{!*}M \mid_U = M$,
(2) $j_{!*}M$ has no submodules or quotients, supported on $X \backslash U$,
(3) $\mathbb{D}(j_{!*}M) = j_{!*}(\mathbb{D}M)$. In our case $\mathbb{D}(IC_X) = IC_X$, since $\mathbb{D}\mathcal{O}_U = \mathcal{O}_U$.
So, $H^i(IC_X)\check{} = H^{-i}(IC_X)$.
(4) If X is any variety, then $H^i(IC_X) = H^{-i}_c(IC_X)\check{}$.

3.5. Examples.

(1) Let $X \subset \mathbb{A}^2$ be given by equation $xy = 0$. Then $X = X_1 \sqcup X_2$, where $X_1 = \{x = 0\}$ and $X_2 = \{y = 0\}$. It's easy to see, that $IC_X = i_{1_*}\mathcal{O}_{X_1} \oplus i_{2_*}\mathcal{O}_{X_2}$ (conditions (a) and (b) are satisfied, and those two determine the intermediate extension uniquely). Here $i_1 : X_1 \to X$ and $i_2 : X_2 \to X$. For cohomology we have

$$\dim H^i(IC_X) = \begin{cases} 2 & \text{if } i = -1; \\ 0 & \text{otherwise.} \end{cases}$$

(2) Let X be a smooth variety and Γ – a finite group, acting on X freely at the general point. Let $Y = X/\Gamma$ and $\pi : X \to Y$ be a natural morphism. Then Γ acts on $\pi_*\mathcal{O}_X$ as follows: let $\Gamma \ni \gamma : X \to X$. By definition $\pi\gamma = \pi$ and $\gamma_*\mathcal{O}_X \cong \mathcal{O}_X$. Consider the following composition

$$\pi_*\mathcal{O}_X \to (\pi \circ \gamma)_*\mathcal{O}_X = \pi_*(\gamma_*\mathcal{O}_X) \cong \pi_*\mathcal{O}_X$$

It gives us a map $\pi_*\mathcal{O}_X \xrightarrow{\sim} \pi_*\mathcal{O}_X$, which depends on γ.

Lemma 3.6. *In the situation above, let $j : V \hookrightarrow Y$, where V is smooth. Then*

$$j_{!*}((\pi_*\mathcal{O}_X) \mid_V) = \pi_*\mathcal{O}_X.$$

Proof. Let $\tilde{j} : U \hookrightarrow X$, where $U = \pi^{-1}V$. Since Γ acts freely at the general point of X, π is proper. Hence

$$j_!((\pi_*\mathcal{O}_X) \mid_V) = \pi_*(\tilde{j}_!\mathcal{O} \mid_U) \text{ and } j_*((\pi_*\mathcal{O}_X) \mid_V) = \pi_*(\tilde{j}_*\mathcal{O} \mid_U).$$

So

$$\operatorname{Im}(j_!(\pi_*\mathcal{O}_X \mid_V) \to j_*(\pi_*\mathcal{O}_X \mid_V)) = \pi_*(\operatorname{Im}(\tilde{j}_!\mathcal{O}_U \to \tilde{j}_*\mathcal{O}_U)) = \pi_*\mathcal{O}_X,$$

since X is smooth. And this means that $j_{!*}((\pi_*\mathcal{O}_X) \mid_V) = \pi_*\mathcal{O}_X$. □

Corollary 3.7. $IC_Y = (\pi_*\mathcal{O}_X)^\Gamma$.

Corollary 3.8. $H^i(IC_Y) = H^{n+i}(Y^{an}, \mathbb{C}) = H^{n+i}(X^{an}, \mathbb{C})^\Gamma$, *where* $n = \dim X$.

3.9. Example. Let Y be a quadratic cone in $\mathbb{C}^3$. Then $H^*(IC_Y) = H^*(Y^{an}, \mathbb{C})[2]$. This follows from Corollary 3.8 together with the fact that $Y \simeq \mathbb{C}^2/ZZ_2$.

CHAPTER 4

$\mathcal{D}$-modules with regular singularities

1. Lectures 14 and 15 (by Pavel Etingof): Regular singularities and the Riemann-Hilbert correspondence for curves

Let X be a C^∞-manifold. Recall that a local system on X consists of the following data:

1) A vector space V_x for every point $x \in X$

2) An isomorphism $\alpha_\gamma : V_{x_1} \to V_{x_2}$ for every C^∞-path γ starting at x_1 and ending at x_2.

This data should depend only on the homotopy class of γ and should be compatible with composition of paths.

If X is connected, then by choosing a point $x \in X$ we may identify the category of local systems on X with the category of representations of the fundamental group $\pi_1(X, x)$ (we shall often omit the point x in the notations).

Let X be a complex manifold. Then we have equivalance of categories "holomorphic vector bundles on X with connection=representations of $\pi_1(X)$ (=local systems on X)".

This is not true in the algebraic setting. For example let $X = \mathbb{A}^1$ and and consider the connection on the trivial vector bundle on X given by the formula:

$$\nabla(f(x)) = (f'(x) - f(x))dx.$$

Then this connection has a nowhere vanishing flat section given by the function e^x. Hence the corresponding local system is trivial. On the other hand it is clear (for example, bacause e^x is not a polynomial function) that over $\mathbb{C}[x]$ the above connection is not isomorphic to the trivial one. Thus we cannot hope to have an equivalence between the category of algebraic vector bundles with connection with the category of local systems.

It turns out that we can single out some nice subcategory of $\mathcal{O}$-coherent D-modules on smooth algebraic variety X (called D-modules with regular singularities) for which the above equivalence is still valid. Today we are going to do it for curves.

1.1. Regular connection on a disc. First of all let us develop some analytic theory and then we'll apply it to the algebraic setting. Let D denote the complex disc $\{x \in \mathbb{C} | |x| < r\}$ and let D^* denote the punctured disc. Let $\mathcal{O}_D$ denote the algebra of holomorphic functions on D and let $\mathcal{O}_D[x^{-1}]$ be the algebra of meromorphic functions. Also we denote by $\Omega_D[x^{-1}]$ the space of meromorphic one-forms on D which are holomorphic outside 0. Let also Ω_{log} denote the space of 1-forms with pole of order at most 1 at 0.

By a meromorphic connection on D we mean a vector bundle M with a connection $\nabla : M \to M \otimes \Omega_D[x^{-1}]$.

By a morphism of meromorphic connections we mean a *meromorphic* map $\alpha : M_1 \to M_2$ which is holomorphic outside 0 and which is compatible with the connections. Thus meromorphic connections form a category (note that there is no functor from this category to the category of vector bundles on D).

If we choose a meromorphic trivialization of M then ∇ is given by a matrix A of meromorphic one forms. A is defined uniquely up to gauge trasformations $A \mapsto gAg^{-1} + g^{-1}dg$ where g is a holomorphic function on D^* taking values in the vactor space $Mat(n)$ of $n \times n$-matrices which is meromorphic at 0.

DEFINITION 1.2. *We say that ∇ has regular singularities if there exists a trivialization a as above such that all the entries of A have a pole of order at most one 1. More invariantly (M, ∇) is regular if it is isomorphic to some (M', ∇') where $\nabla' : M' \to M' \otimes \Omega_{log}$.*

In other words, M' above stable under the algebra generated by $\mathcal{O}_D$ and the vector field $x\frac{d}{dx}$. In a coordinate-free way it means that $\nabla(M') \subset M' \otimes \Omega_{log}$.

Examples. Let us give some examples of regular and non-regular connections. First of all any connection with connection matrix having poles of order ≤ 1 is regular. We claim that the converse is true if $\text{rank}(M) = 1$. Indeed, in this case the connection matrix A is just a differential one-form which is meromorphic at 0. The RS-condition says that there exists a meromorphic function g such that $A + g^{-1}dg$ has pole of order ≤ 1 at 0. But $g^{-1}dg$ also has a pole of order ≤ 1 at 0. Hence the same is true for A.

Here is an example of a connection matrix of rank 2 which has poles of order > 1 but still defines a maromorphic connection with regular singularities. Namely let β an arbitrary meromorphic function on D and consider the connection ∇_β whose connection matrix is

$$A_\beta = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

We claim that for every *beta* the connection ∇_β has regular singularities at 0. Indeed, it is easy to see that there exists a meromorphic function u on D such that $-u' + \beta$ has pole of order ≤ 1. Define

$$g = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

Then it is easy to see that the matrix $gA_\beta g^{-1} + g^{-1}dg$ has poles of order ≤ 1 at 0. Hence ∇_β has regular singularities.

Here are some first properties of RS.

Given two meromorphic connections (M_1, ∇_1) and (M_2, ∇_2) w may consider their tensor product $\nabla_1 \otimes \nabla_2$ (this is a connection on $M_1 \underset{\mathcal{O}_{D_2}}{M}$). It is easy to see that if both ∇_1 and ∇_2 have RS then their tensor product has that property too.

Also given a meromorphic connection (M, ∇) we may define the dual meromorphic coneection $\nabla^\vee$ on $M^\vee = \underline{\mathrm{Hom}}(M, \mathcal{O}_D)$. If ∇ has RS then $\nabla^\vee$ has RS too.

Given two meromorphic connections as above we can define their inner Hom by

$$\underline{\mathrm{Hom}}(M_1, M_2) = M_1^\vee \otimes M_2.$$

It follows from the above that if ∇_1 and ∇_2 have RS then the same is true for the corresponding connection on $\underline{\mathrm{Hom}}(M_1, M_2)$.

Here is an analytic characterisation of regular singularities (RS) on D.

Definition 1.3. *Let f be a vector-valued function defined in some sector*

$$\{z = \rho e^{i\theta} |\ 0 < |\rho| < r,\ \alpha < \theta < \beta\}.$$

We say that f has moderate growth if there exist some constants C and γ such that

$$||f(\rho e^{i\theta})|| \leq C\rho^{-\gamma}.$$

Theorem 1.4. *A meromorphic connection ∇ regular if and only if for every sector $\alpha < arg(x) < \beta$ the horizontal sections for ∇ on this sector have moderate growth.*

Proof. Suppose first that ∇ is regular. Thus we are looking for the asymptotics of solutions of the equation

$$\frac{dF}{dz} = A(z)F$$

where A is an $n \times n$-matrix of meromorphic functions having poles of order ≤ 1 at 0 and F is a function of z with values in $\mathbb{C}^n$. Let $\widetilde{A}(z) = zA(z)$. Then $\widetilde{A}$ is regular at 0. So, we have the equation

$$\rho\frac{d}{d\rho} = \widetilde{A}(\rho e^{i\theta})F.$$

Lemma 1.5. *Let $f, B : [0, L] \to Mat_n(\mathbb{C})$ be two C^1-functions such that*

$$f'(t) = B(t)f(t).$$

Then we have

$$||f(L)|| \leq ||f(0)||e^{L\max||B||}. \tag{1.1}$$

Proof. We have

$$f(L) = \lim_{n\to\infty}\prod_{j=0}^{n-1} e^{\frac{L}{n}B(\frac{j}{n})}f(0).$$

Let us set $M = \max||B||$. Then for $\varepsilon >?$ we have

$$||f(L)|| \leq \limsup_{n\to\infty}\prod_{j=0}^{n-1}||e^{\frac{L}{n}B(\frac{j}{n})||\cdot||f(0)||} \leq \prod_{j=0}^{n-1}(1 + \frac{L}{n}(1+\varepsilon)M)||f(0)|| \leq e^{L(1+\varepsilon)M}||f(0)||.$$

As $n \to \infty$ we set that the above inequality becomes true for all $= \varepsilon > 0$. $\square$

$\square$

THEOREM 1.6. *Restriction to D^* is an equivalence of categories "meromorphic connections with regular singularities= connections on D^*".*

PROOF. The restriction functor is clearly exact and faithful. Also we claim that it is surjective on objects. Indeed, since $\pi_1(D^*) = \mathbb{Z}$ it follows that every connection on D^* of rank n is isomorphic to a connection on the trivial rank n vector bundle on D^* with connection form equal to $A\frac{dz}{z}$ where A is some constant $n \times n$-matrix. Extending this bundle in the trivial way to D we get a connection with regular singularities.

Let us denote the restriction functor by R. To prove Theorem 1.6 it is enough now to show that for two meromorphic connections (M_1, ∇_1) and (M_2, ∇_2) with regular singularities we have

$$\mathrm{Hom}(M_1, M_2) = \mathrm{Hom}(R(M_1), R(M_2))$$

(here in the left hand side we look at the morphisms in the category of meromorphic connections on D and in the right hand side we deal with morphisms in the category of connections (local systems) on D^*).

Let $\phi \in \mathrm{Hom}(R(M_1), R(M_2))$. We can regard ϕ as a flat section of $\underline{\mathrm{Hom}}(M_1, M_2)$ on D^*. By Theorem 1.4 ϕ has moderate growth. Also ϕ is holomorphic on D^*. Such a section is automatically holomorphic on D. Hence ϕ comes from an element of $\mathrm{Hom}(M_1, M_2)$. $\square$

1.7. Regular connections on an arbitrary curve. Let X be a smooth projective curve, $j : Y \hookrightarrow X$ an open subset, S – the complement of Y (finite set). Let $\mathcal{D}_X^S$ denote the subsheaf of algebras of the sheaf $\mathcal{D}_X$ generated (locally) by $\mathcal{O}_X$ and vector field which vanish on S. We say that an $\mathcal{O}_Y$-coherent $\mathcal{D}_Y$-module N has regular singularities if there exists an $\mathcal{O}_X$-coherent (=vector bundle on X) submodule M of $j_*(N)$ which is stable under $\mathcal{D}_X^S$.

It is clear that the category of $\mathcal{O}_Y$-coherent RS $\mathcal{D}_Y$-modules is closed under subquotients.

This definition of regular singularities is connected with what we studied before in the following way.

Let M be as above. Let $s \in S$ and let D be a small disc around s (not containing any other point from S). Then the restriction of M to D acquires a meromorphic connection with regular singularities.

THEOREM 1.8. *The natural functor "$\mathcal{O}$-coherent $\mathcal{D}$-modules on Y with regular singularities" $\to$ "connection on Y^{an} (i.e. Y considered as a complex analytic manifold)" is an equivalence of categories. In particular, we have an equivalence "$\mathcal{O}$-coherent $\mathcal{D}$-modules on Y with regular singularities" $\simeq$ "representations of $\pi_1(Y)$.*

Proof. Let us denote the functor in question by $N \mapsto N^{\mathrm{an}}$ (we shall call the analytification functor). This functor is clearly exact and faithful. Hence to show that it is an equivalence of categories we must show the following two statements:

1. The analytification functor is surjective on objects.
2. For every two $\mathcal{O}_Y$-cohere $\mathcal{D}_Y$-modueles N_1, N_2 we have

$$\mathrm{Hom}(N_1, N_2) = \mathrm{Hom}(N_1^{\mathrm{an}}, N_2^{\mathrm{an}}).$$

Let us prove 1. Let $(N^{\mathrm{an}}, \nabla^{\mathrm{an}})$ be a holomorphic vector bundle with a connection on Y. We must show that there exists an algebraic vector bundle M on X with a connection ∇ with poles of first order along S such that $(M, \nabla)|_Y^{\mathrm{an}} = (N^{\mathrm{an}}, \nabla^{\mathrm{an}})$. By Theorem 1.6 such an M exists locally in the analytic topology. Thus globally we get a holomorphic vector bundle M^{an} on X^{an} with a meromorphic connection ∇^{an} having poles of first order along S. By GAGA M^{an} ha unique algebraic structure and thus gives rise to an algebraic vector bundle M on X. Thus ∇^{an} becomes a holomorphic section of some algebraic vector bundle on X and therefore is also algebraic by GAGA. □

Theorem 1.9. *The notion of RS-modules is stable under extensions.*

Enough to prove this for the disc – there we have to make an explicit calculation.

In general we say that a holonomic $\mathcal{D}$-module on a (not necessarily projective) curve X is RS if at the generic point it is an $\mathcal{O}$-coherent module which is RS.

As an example let us consider $X = \mathbb{C}$. Consider $\mathcal{D}$-modules on $\mathbb{C}$ which have RS and which are $\mathcal{O}$-coherent on $\mathbb{C}^*$. We can completely describe this category. Namely let $\tau : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ be any lift of the natural projection (we shall assume that $\tau(0) = 0$. We claim that this category is equivalent to the following one:

"pairs of finite-dim. vector spaces E, F with morphisms $u : E \to F$ and $v : F \to E$ such that the eigenvalues of vu lie in the image of τ (it is then automatically true also for uv)". The functor in one direction is described in the following way. If M is a module as above then let M^α (for $\alpha \in \mathbb{C}$) denote the generalized α-eigenspace of $x\frac{d}{dx}$. Then we define

$$F = \oplus_{\alpha \in Im(\tau)} M^{\alpha-1}, \quad E = \oplus_{\alpha \in Im(\tau)} M^{\alpha}$$

We also let v be multiplication by x and let u be $\frac{d}{dx}$.

The functor in the opposite direction is: given (E, F, u, v) define

$$M = \mathbb{C}[x] \otimes E \oplus \mathbb{C}[\frac{d}{dx}] \otimes F$$

Also define the action of x and $\frac{d}{dx}$ by

$$\frac{d}{dx}(1 \otimes e) = 1 \otimes u(e) \quad x(1 \otimes f) = 1 \otimes v(f)$$

(the action of x on $\mathbb{C}[x] \otimes E$ and the action of $\frac{d}{dx}$ on $\mathbb{C}[\frac{d}{dx}] \otimes F$ are assumed to be the natural ones).

It is easy to see that these two functors are mutually inverse.

CHAPTER 5

The Riemann-Hilbert correspondence and perverse sheaves

1. Riemann-Hilbert correspondence

1.1. Constructible sheaves and complexes. Let X be a complex algebraic variety. We denote by X^{an} the correspondent analytic variety, considered in classical topology.

Let $\mathbb{C}_X$ be the constant sheaf of complex numbers on X^{an}. We denote by $\mathrm{Sh}(X^{\mathrm{an}})$ the category of sheaves of $\mathbb{C}_X$-modules, i.e., the category of sheaves of $\mathbb{C}$-vector spaces. Derived category of bounded complexes of sheaves will be denoted by $D(X^{an})$. We shall call sheaves $\mathcal{F} \in Sh(X^{an})$ $\mathbb{C}_X$-*modules* and complexes $\mathcal{F}^{\cdot} \in D(X^{an})$ $\mathbb{C}_X$-*complexes*. We shall usually omit the $^{\cdot}$ superscript when it does not lead to a confusion.

We shall call a $\mathbb{C}_X$-module $\mathcal{F}$ *constructible* if there exists a stratification $X = \cup X_i$ of X by locally closed *algebraic* subvarieties X_i, such that $F|_{X_i^{\mathrm{an}}}$ is a locally constant (in classical topology) sheaf of finite-dimensional vector spaces. We shall call a $\mathbb{C}_X$-complex $\mathcal{F}^{\cdot}$ constructible if all its cohomology sheaves are constructible $\mathbb{C}_X$-modules. The full subcategory of $D(X^{\mathrm{an}})$ consisting of constructible complexes will be denoted by $D_{\mathrm{con}}(X^{\mathrm{an}})$.

Any morphism $\pi : Y \to X$ of algebraic varieties induces the continuous map $\pi^{\mathrm{an}} : Y^{\mathrm{an}} \to X^{\mathrm{an}}$ and we can consider functors

$$\pi_!, \pi_* : D(Y^{\mathrm{an}}) \longrightarrow D(X^{\mathrm{an}})$$
$$\pi^*, \pi^! : D(X^{\mathrm{an}}) \longrightarrow D(Y^{\mathrm{an}}).$$

Also we shall consider the Verdier duality functor

$$\mathbb{D} : D(X^{\mathrm{an}}) \longrightarrow D(X^{\mathrm{an}}).$$

THEOREM 1.2. *The functors $\pi_*, \pi_!, \pi^*, \pi^!$ and $\mathbb{D}$ preserve subcategories $D_{con}(\quad)$. On this categories $\mathbb{D} \circ \mathbb{D} \simeq Id$ and*

$$\mathbb{D}\pi^* D = \pi^!, \qquad \mathbb{D}\pi_*\mathbb{D} = \pi_!.$$

1.3. De Rham functor. Denote by $\mathcal{O}_X^{\mathrm{an}}$ the structure sheaf of the analytic variety X^{an}. We will assign to each $\mathcal{O}_X$-module $\mathcal{F}$ corresponding "analytic" sheaf of $\mathcal{O}_X^{an}$-modules M^{an}, which is locally given by

$$M^{an} = \mathcal{O}_{X^{\mathrm{an}}} \underset{\mathcal{O}_X}{\otimes} M.$$

This defines an exact functor

$$\text{an} : \mathcal{M}(\mathcal{O}_X) \longrightarrow \mathcal{M}(\mathcal{O}_{X^{an}}).$$

In particular, the sheaf $\mathcal{D}_X^{an}$ is the sheaf of analytic (holomorphic) differential operators on X^{an} and we have an exact functor

$$\text{an} : \mathcal{M}(\mathcal{D}_X) \ \longrightarrow \ \mathcal{M}(\mathcal{D}_X^{\text{an}}).$$

Since this functor is exact it induces a functor

$$\text{an} : D(\mathcal{D}_X) \ \longrightarrow \ D(\mathcal{D}_X^{\text{an}}).$$

Definition 1.4. *Define the De Rham functor $DR : D(\mathcal{D}_X) \to D(X^{an}) = D(Sh(X^{an}))$ by*

$$DR(M^{\cdot}) = \Omega_X^{an} \underset{\mathcal{D}_X^{an}}{\otimes} (M^{\cdot})^{an}.$$

Remarks. 1. We know that the complex $dR(\mathcal{D}_X)$ is a locally projective resolution of the right $\mathcal{D}_X$-module Ω_X. Hence

$$DR(M^{\cdot}) = dR(\mathcal{D}_X^{\text{an}}) \underset{\mathcal{D}_X^{\text{an}}}{\otimes} (\mathcal{F}^{\cdot})^{\text{an}}[n] = dR((M^{\cdot})^{an})[n],$$

where $n = \dim X$.

In particular, if M is an $\mathcal{O}$-coherent $\mathcal{D}_X$-module corresponding to some vector bundle with a flat connection and $\mathcal{L} = M^{\text{flat}}$ is the local system of flat sections of $\mathcal{F}$ (in classical topology), then by Poincaré lemma

$$\text{DR}(M) = \mathcal{L}[n].$$

2. Kashiwara usually uses a slightly different functor $\text{Sol} : D_{\text{coh}}(\mathcal{D}_X)^o \to D(X^{\text{an}})$ defined by

$$\text{Sol}(M^{\cdot}) = \underline{\text{RHom}}_{D_X^{\text{an}}}(M^{\text{an}}, \mathcal{O}_X^{\text{an}}).$$

We claim that $\text{Sol}(\mathcal{M}^{\cdot}) = DR(\mathbb{D}M^{\cdot})[-n]$. This follows from the following formula. Let P be any locally projective $\mathcal{D}_X$-module and let $P^{\vee} = \text{Hom}_{\mathcal{D}_X}(P, \mathcal{D}_X^{\Omega})$. Then

$$\text{Hom}_{\mathcal{D}_X}(P, \mathcal{O}_X) = \Omega_X \underset{\mathcal{D}_X}{\otimes} (P^{\vee}),$$

Here is the main result about the relation between $\mathcal{D}$-modules and constructible sheaves.

Theorem 1.5. *a) $DR(D_{hol}(\mathcal{D}_X)) \subset D_{con}(X^{an})$. Also on $D_{hol}(\mathcal{D}_X)$ we have*

$$\mathbb{D} \circ DR = DR \circ \mathbb{D}.$$

(note that in the laft hand side $\mathbb{D}$ means the Verdier duality and in the right hand side $\mathbb{D}$ stands for the duality of $\mathcal{D}$-modules).

For $M^{\cdot} \in D_{hol}(\mathcal{D}_X)$ and $N^{\cdot} \in D(\mathcal{D}_Y)$ we have

$$DR(M^{\cdot} \boxtimes N^{\cdot}) \approx DR(M^{\cdot}) \boxtimes DR(N^{\cdot}).$$

b) On the subcategory D_{rs} the functor DR commutes with $\mathbb{D}, \pi_*, \pi^!, \pi_!, \pi^*$ and $\boxtimes$.
c) $DR : D_{rs}(\mathcal{D}_X) \to D_{con}(X^{an})$ is an equivalence of categories.

1.6. Simple statements. First let us consider some basic properties of the functor DR.
(i) DR commutes with restriction to an open subset. For an étale covering $\pi : Y \to X$ the functor DR commutes with π_* and $\pi^!$.

(ii) For a morphism $\pi : Y \to X$ there exists a natural morphism of functors $\alpha : \mathrm{DR}\pi_* \to \pi_* \circ \mathrm{DR}$ which is an isomorphism for proper π.

In order to prove this let us consider the functor
$\pi_*^{an} : D(\mathcal{D}_Y^{\mathrm{an}}) \to D(\mathcal{D}_X^{\mathrm{an}})$ on the categories of $\mathcal{D}^{\mathrm{an}}$-complexes, which is given by

$$\pi_*^{an}(M^\cdot) = R\pi_\bullet^{\mathrm{an}}(\mathcal{D}_{X\leftarrow Y}^{an} \underset{\mathcal{D}_Y^{an}}{\otimes} M^\cdot).$$

We claim that $DR \circ \pi_*^{\mathrm{an}} = \pi_* \circ \mathrm{DR}$. Indeed,

$$\mathrm{DR}(\pi_*^{\mathrm{an}}(M^\cdot)) = \Omega_X^{\mathrm{an}} \overset{L}{\underset{\mathcal{D}_X^{an}}{\otimes}} R\pi_\bullet^{\mathrm{an}}(\mathcal{D}_{X\leftarrow Y}^{\mathrm{an}} \overset{L}{\underset{\mathcal{D}_Y^{an}}{\otimes}} M^\cdot) =$$

$$R\pi_\bullet^{\mathrm{an}}(\pi^\bullet(\Omega_X^{\mathrm{an}}) \overset{L}{\underset{\pi^\bullet D_X^{an}}{\otimes}} \mathcal{D}_{X\leftarrow Y}^{\mathrm{an}} \overset{L}{\underset{\mathcal{D}_Y^{an}}{\otimes}} M^\cdot) = R\pi_\bullet^{an}(\Omega_Y^{an} \overset{L}{\underset{D_Y^{an}}{\otimes}} M^\cdot),$$

since $\pi^\bullet\Omega_X \underset{\pi^\cdot\mathcal{D}_X}{\otimes} \mathcal{D}_{X\leftarrow Y} \approx \Omega_Y$ as a $\mathcal{D}_Y$-module.

Now there exists in general the natural morphism of functors

$$\mathrm{an} \circ R\pi_\bullet(M^\cdot) \;\longrightarrow\; R\pi_\bullet^{\mathrm{an}}(M^\cdot)^{\mathrm{an}}.$$

This functor is not an isomorphism in general, since direct image on the left and on the right are taken in different topologies. But according to Serre's "GAGA" theorem it is an isomorphism for proper π. Combining these 2 observations we obtain (ii).

(iii) On the category of coherent $\mathcal{D}_X$-complexes there exists a natural morphism of functors

$$\beta : \mathrm{DR} \circ \mathbb{D}(M^\cdot) \to \mathbb{D} \circ \mathrm{DR}(M^\cdot)$$

which is an isomorphism for $\mathcal{O}$-coherent $M^\cdot$ and which is compatible with the isomorphism $\pi_* \circ \mathrm{DR} = \mathrm{DR} \circ \pi_*$ for proper π, described in (ii).

By definition of the duality functor $\mathbb{D}$ in the category $D(X^{\mathrm{an}})$ we have

$$\mathbb{D}(\mathcal{F}^\cdot) = \underline{\mathrm{RHom}}_{\mathbb{C}_X}(\mathcal{F}^\cdot, \mathbb{C}_X[2\dim X]).$$

(Note that $\mathbb{C}_X[2\dim X]$ is the dualizing sheaf of X^{an}). Hence in order to construct β it is sufficient to construct a morphism

$$\beta' : \mathrm{DR} \circ \mathbb{D}(M) \otimes_{\mathbb{C}_X} \mathrm{DR}(M) \longrightarrow \mathbb{C}_X[2\dim X].$$

As we have seen above $\mathrm{DR} \circ \mathbb{D}(M)$ is naturally isomorphic to

$$\mathrm{Sol}(M)[\dim X] = \underline{\mathrm{RHom}}_{\mathcal{D}_X^{\mathrm{an}}}(M^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}})[\dim X].$$

Let us realize $\mathrm{DR}(M)$ as $dR(M^{\mathrm{an}})$ and $\mathrm{DR}\circ\mathbb{D}(M)$ as $\underline{\mathrm{RHom}}_{\mathcal{D}_X^{\mathrm{an}}}(M^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}}[\dim X])$. Hence we get a morphism

$$\beta'' : M^{\mathrm{an}} \otimes_{\mathbb{C}_X} \mathrm{DR}(M) \longrightarrow \mathcal{O}_X^{\mathrm{an}}[\dim X]).$$

Since $dR(\mathcal{O}_X^{\mathrm{an}}) = \mathbb{C}_X[2\dim X]$ by applying dR to both sides we get β'.

It is easy to check that the corresponding β is an isomorphism for $\mathcal{O}$-coherent M.

It remains to check that β is compatible with (ii). By it is enough to do it for projections $Z\times X \to X$ where Z is smooth for which the statement is straightforward.

(iv) There is a natural morphism of functors

$$\gamma : \mathrm{DR}(M \boxtimes N) \longrightarrow \mathrm{DR}(M) \boxtimes \mathrm{DR}(N)$$

which is an isomorphism for $\mathcal{O}$-coherent M.

The morphism γ is defined by the natural imbedding $\Omega_X^{\mathrm{an}} \boxtimes_{\mathbb{C}} \Omega_Y^{\mathrm{an}} \longrightarrow \Omega_{X\times Y}^{\mathrm{an}}$. If M is O-coherent and N is locally projectively then γ is an isomorphism by partial Poincaré lemma. This implies the general statement.

(v) There is a natural morphism of functors

$$\delta : \mathrm{DR}\circ \pi^!(M) \to \pi^! DR(M)$$

which is an isomorphism for smooth π.

It is enough to construct δ in the following two cases:

1. π is an open embedding.
2. π is a smooth projection.
3. π is a closed embedding.

In the first case the construction of δ is obvious (it is also clear that in this case δ is an isomorphism). Consider the second case. In this case the isomorphism δ can be constructed on generators – locally projective modules. Indeed, let $\pi : Y = T\times X \to X$ be the projection, then $\pi^!(M) = \mathcal{O}_T \boxtimes M[\dim T]$ and $\pi^!\mathrm{DR}(M) = \mathbb{C}_T \boxtimes \mathrm{DR}(M)[2\dim T] = \mathrm{DR}(\mathcal{O}_T)\boxtimes \mathrm{DR}(M)[\dim T]$.

Consider now the case of a closed embedding $i : Y \to X$. Using i_*, which commutes with DR, we will identify sheaves on Y with sheaves on X, supported on Y. Then $i_*i^!M = R\Gamma_Y M$ in both categories, which gives the natural morphism

$$\delta : \mathrm{DR}\circ i_*i^!(M) = \mathrm{DR}(R\Gamma_Y M) \longrightarrow R\Gamma_Y \mathrm{DR}(M) = i_*i^!\mathrm{DR}(M).$$

1.7. Proof of Theorem 1.5 a). Let M be a holonomic $\mathcal{D}_X$-complex. Consider the maximal Zariski open subset $U\subset X$ such that $DR(M)|_U$ is constructible. Since M is $\mathcal{O}$-coherent almost everywhere it follows that U is dense in X.

Let W be an irreducible component of $X\setminus U$. We want to show that $\mathrm{DR}(M)$ is locally constant on some Zariski dense open subset $W_0 \subset W$.

Proposition 1.8. *We can assume that*

$$X = \mathbb{P} \times W, \qquad W = p \times W, \qquad \textit{where } p \in \mathbb{P}$$

and that $V = U \cup W$ is open in X. Here $\mathbb{P}$ denotes some projective space.

Indeed, consider an étale morphism of some open subset of W onto an open subset of an affine space $\mathbb{A}^k$ and extend it to an étale morphism of a neighbourhood of W onto an open subset of $\mathbb{A}^n \supset \mathbb{A}^k$. By changing base from $\mathbb{A}^k$ to W we can assume that $V = U \cup W$ is an open subset of $X' = \mathbb{P}^{n-k} \times W$. Then we can extend M to some holonomic $\mathcal{D}$-module on X'.

Now consider the projection $\mathrm{pr} : X = \mathbb{P} \times W \to W$. Since it is a proper morphism it follows that $\mathrm{DR}(\mathrm{pr}_*(M)) = \mathrm{pr}_* \mathrm{DR}(M)$. Since $\mathrm{pr}_*(M)$ is a holonomic $\mathcal{D}_W$-complex, it is $\mathcal{O}$-coherent almost everywhere. Hence $\mathrm{DR}(\mathrm{pr}_*(M))$ is locally constant almost everywhere.

Put $S = \mathrm{DR}(M) \subset D(X^{\mathrm{an}})$. Replacing W by an open subset, we can assume that $\mathrm{pr}_*(S) = \mathrm{DR}(\mathrm{pr}_*(M))$ is locally constant. We have an exact triangle

$$S_U \to S \to S_{X \setminus U}$$

where $S_U = (i_U)_! S|_U)$ and $i_U : U \to X$ is the natural embedding.

By the choice of U the complex $S|_U$ is constructible. Hence the complex S_U is constructible. Thus the complex $pr_*(S_{X \setminus U})$ is constructible. Replacing W once again by an open subset we can assume that it is locally constant.

Now $S_{X\setminus U}$ is a direct sum of 2 sheaves $(i_W)_! S|_W$ and something concentrated on $\{X \setminus U\} \setminus W$ (here we use the fact that V is open in X). This implies that $S|_W$ is a direct summand of the locally constant sheaf $\mathrm{pr}_*(S_{X\setminus U})$ and hence itself is locally constant. □

1.9. Proof of Theorem 1.5 b) for $\mathbb{D}$ and $\boxtimes$. Let us now show that DR commutes with $\mathbb{D}$ for holonomic complexes. Let $M \in D_{\mathrm{hol}}(\mathcal{D}_X)$. Put

$$\mathrm{Err}(M) = \mathrm{Cone}(\mathrm{DR} \circ \mathbb{D}(M) \to \mathbb{D} \circ \mathrm{DR}(M)).$$

This sheaf vanishes on a dense open subset where M is $\mathcal{O}$-coherent. Also we know that the functor Err commutes with direct image for proper morphisms. Repeating the above arguments we can show that $\mathrm{Err} = 0$, i.e., DR commutes with $\mathbb{D}$ on $D_{\mathrm{hol}}(\mathcal{D}_X)$.

Also the same arguments show that $\mathrm{DR}(M \boxtimes N) = DR(M) \boxtimes DR(N)$ for holonomic M.

Remark. Of course this proof is simply a variation of Deligne's proof of "Théorèmes de finitude" in SGA 4 1/2. Note the crucial role in the proof of both statements is played by the fact that we have a well-defined morphism between the two corresponding functors for *all* $\mathcal{D}$-modules. Then we use Deligne's trick to show that it is an isomorphism for holonomic ones.

1.10. Proof of Theorem 1.5 b) for direct image. Let us prove that the morphism

$$\mathrm{DR} \circ \pi_*(M) \to \pi_* \circ \mathrm{DR}(M)$$

is an isomorphism for $M \in D_{rs}(\mathcal{D}_Y)$.

Case 1. $\pi = j : Y \to X$ is a regular extension [1] and M is an $\mathcal{O}$-coherent D_Y-module with regular singularities. Blowing up Y and making use of (ii) above we see that it is enough to consider the case when Y has codimension 1 in X.

In this case the proof is straightforward, using the definition of RS (it was done by P. Deligne). Since M has RS there exists an $\mathcal{O}$-coherent submodule $M' \subset j_*M$ with respect to which our connection has a pole of order ≤ 1. It is clear that both $j_*(\mathrm{DR}(M)$ and $\mathrm{DR}(j_*(M)$ depend only on $(M')^{\mathrm{an}}$ (which is a meromorphic connection).

Now, locally in the neighbourhood of a point $x \in X \setminus Y$ we can choose coordinates $x_1, \ldots, x_n$ such that $X \setminus Y$ is given by the equation

$$x_1 = 0.$$

We may replace x by an *analytic* neighbourhood of x such that $\pi_1(X \setminus Y, x) = \mathbb{Z}$. Since the above fundamenta group is commutative, we can decompose M' into 1-dimensional subquotients. Using commutativity with $\boxtimes$ we can reduce to the case $\dim X = 1$. Hence as $\mathcal{O}_X$-module M' is generated by one element e, which satisfies the equation $x\partial(e) = \lambda e$. In this case our statement can be proved by a direct calculation.
Case 2. M is an $\mathcal{O}$-coherent D_Y-module with regular singularities.

In this case we decompose $\pi = \pi^+ \circ j$, where $j : Y \to Y^+$ is a regular extension and $\pi^+ : Y^+ \to X$ is a proper morphism. Then we know that DR commutes with j_* by Case 1 and with π^+_* by Section 1.6 (ii).

General Case. It is sufficient to check the statement on generators. Hence we can assume that $M = i_*(N)$, where $i : Z \to Y$ is a locally closed imbedding and N an $\mathcal{O}$-coherent D_Z-module with regular singularities. Then

$$\mathrm{DR}(\pi_*(M)) = \mathrm{DR}(\pi \circ i)_*(N) \overset{\text{Case 2}}{=} (\pi \circ i)_* \,\mathrm{DR}(N) =$$
$$\pi_*(i_*\mathrm{DR}(N)) \overset{\text{Case 2}}{=} \pi_*\mathrm{DR}(i_*(N)) = \pi_*\mathrm{DR}(N).$$

It follows that on $D_{rs}(\mathcal{D}_X)$ the functor DR also commutes with $\pi_!$ since $\pi_! = \mathbb{D} \circ \pi_* \circ \mathbb{D}$ and we have already checked that DR commutes with $\mathbb{D}$.

[1] by a regular extension we mean an open embedding $j : Y \to X$ such that the corresponding embedding $X \setminus Y \to X$ is regular

1.11. Proof of Theorem 1.5 b) for inverse image. It is enough to prove Theorem 1.5 for the functor $\pi^!$ (since $\pi^* = \mathbb{D} \circ \pi^! \circ \mathbb{D}$).

In Section 1.6 (v) we have constructed the morphism $\delta : \mathrm{DR} \circ \pi^! \to \pi^! \circ \mathrm{DR}$ which is an isomorphism for smooth π. Hence it is sufficient to check that for RS $\mathcal{D}_Y$-complexes the morphism δ is an isomorphism for the case of a closed embedding $\pi = i : Y \hookrightarrow X$. Denote by $j : U = X \setminus Y \to X$ the embedding of the complementary open set. Then we have the morphism of exact triangles

$$\begin{array}{ccccc} DR(i_*i^!M) & \longrightarrow & DR(M) & \longrightarrow & DR(j_*(M|_V)) \\ \Big\downarrow \delta & & \Big\downarrow id & & \Big\downarrow \alpha \\ i_*i^!DR(M) & \longrightarrow & DR(M) & \longrightarrow & j_*(DR(M)|_V). \end{array}$$

Since we already know that α is an isomorphism it follows that δ is an isomorphism.

1.12. Proof of Theorem 1.5 c). First of all, let us prove that DR gives an equivalence of $D_{rs}(\mathcal{D}_X)$ with a full subcategory of $D_{\mathrm{con}}(X^{an})$. We should prove that for $M, N \in D_{rs}(\mathcal{D}_X)$ the map

$$\mathrm{DR} : \mathrm{Hom}_{D_{rs}}(M, N) \longrightarrow \mathrm{Hom}_{D_{\mathrm{con}}}(\mathrm{DR}(M), \mathrm{DR}(N))$$

is an isomorphism.

It turns out that it is simpler to prove the isomorphism of RHom. Let π denote the morphism from X to pt. We know that

$$\mathrm{RHom}(M, N) = \pi_* \underline{\mathrm{RHom}}(M, N) = \pi_* \mathbb{D}M \otimes N.$$

Note that $\otimes$ in the sense of $\mathcal{D}$-modules is transformed to $\overset{!}{\otimes}$ in $D_{\mathrm{con}}(X^{\mathrm{an}})$. [2]

This implies that

$$\mathrm{DR}(\underline{\mathrm{RHom}}(M, N)) = \underline{\mathrm{RHom}}(\mathrm{DR}(M), \mathrm{DR}(N)).$$

This proves that DR gives an equivalence of the category $D_{rs}(\mathcal{D}_X)$ with a full subcategory of $D_{\mathrm{con}}(X^{\mathrm{an}})$.

Now let us prove that this subcategory contains all isomorphism classes of $D_{\mathrm{con}}(X^{\mathrm{an}})$. Since it is a full triangulated subcategory, it is sufficient to check that it contains some generators. As generators we can choose $\mathbb{C}_X$-complexes of the form $i_*(\mathcal{L})$ where $i : Y \to X$ is an imbedding and $\mathcal{L}$ is a local system on Y. We ca also assume that Y is smooth. Since DR commutes with direct images it is sufficient to check that there exists an $\mathcal{O}$-coherent $\mathcal{D}_Y$-module M such that $\mathrm{DR}(M) \simeq \mathcal{L}[\dim Y]$. This is a result by P. Deligne.

[2] By the definition $\mathcal{F} \overset{!}{\otimes} \mathcal{H} = \Delta^! \mathcal{F} \boxtimes \mathcal{H}$ where $\Delta : X \to X \times X$ is the diagonal embedding.

1.13. Perverse sheaves. Theorem 1.5 gives us a dictionary which allows to translate problems, statements and notions from $\mathcal{D}$-modules to constructible sheaves and back.

Consider one particular example. The category $D_{rs}(\mathcal{D}_X)$ of RS-complexes contains the natural full abelian subcategory RS-category of RS-modules.

How to translate it in the language of constructible sheaves?

From the description of the functor $i^!$ for locally closed imbedding one can immediately get the following

Criterion. Let M be a holonomic $\mathcal{D}_X$-complex. Then M is concentrated in non-negative degrees (i.e., $H^i(M) = 0$ for $i < 0$) if and only if it satisfies the following condition.
$(*)_{rs}$ For any locally closed embedding $i : Y \to X$ there exists an open dense subset $Y_0 \subset Y$ such that $i^! M|_{Y_0}$ is an $\mathcal{O}$-coherent $\mathcal{D}_{Y_0}$-complex, concentrated in degrees ≥ 0.
In terms of constructible complexes this condition can be written as
$(*)_{\text{con}}$ For any locally closed embedding $i : Y \to X$ there exists an open dense subset $Y_0 \subset Y$ such that $i^! S|_{Y_0}$ is locally constant and concentrated in degrees $\geq$ - dim Y.

Thus we have proved the following.

Criterion. A constructible complex S lies in the abelian subcategory

$$\mathrm{DR}(D_{rs}(\mathcal{D}_X)) \quad \text{iff} \quad S \quad \text{and} \quad \mathbb{D}S \quad \text{satisfy} \quad (*)_{con}.$$

Such a complex is called a *perverse sheaf* on X^{an}.

1.14. Analytic criterion of regularity. For any point $x \in X$ let us denote by $\mathcal{O}_x^{\mathrm{an}}$ (resp. $\mathcal{O}_x^{\mathrm{form}}$) the algebra of convergent (resp. formal) power series on X at the point x. For any $\mathcal{D}_X$-complex M the natural inclusion $\mathcal{O}_x^{\mathrm{an}} \to \mathcal{O}_x^{\mathrm{form}}$ induces a morphism

$$\nu_x : \mathrm{RHom}_{\mathcal{D}_X}(M, \mathcal{O}_x^{\mathrm{an}}) \longrightarrow \mathrm{RHom}_{\mathcal{D}_X}(M, \mathcal{O}_x^{\mathrm{form}}).$$

We say that M is good at x if ν_x is an isomorphism.

THEOREM 1.15. (1) *Let M be an RS $\mathcal{D}_X$-complex. Then M is good at all points.*
(2) *Assume that X is proper. Then M is good at all points of X if and only if M is RS.*

PROOF. FILL IN LATER □

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