NOTES ON FACTORIZABLE SHEAVES

This is a preliminary version. Imprecisions are likely.

1. From Hopf algebras to factorizable sheaves

1.1. **Configuration spaces.** Let Λ be a lattice and $\Lambda^{neg} \subset \Lambda$ a sub-semigroup, isomorphic to $(\mathbb{Z}^{\geq 0})^k$.

Let X be an algebraic curve (smooth, but not necessarily complete). For $\lambda \in \Lambda^{neg}$, we will denote by X^λ the algebraic variety classifying Λ-valued divisors $D := \Sigma\, \lambda_i \cdot x_i$ with $x_i \neq x_j$ and $\lambda_i \in \Lambda^{neg}$. By definition, $X^0 = \mathrm{pt}$.

For a marked point $x_0 \in X$ and an arbitrary $\lambda \in \Lambda$, let $X^\lambda_{x_0}$ be the ind-scheme classifying Λ-valued divisors $\Sigma\, \lambda_i \cdot x_i$ as above but with a weaker condition, namely, that $\lambda_i \in \Lambda^{neg}$ for $x_i \neq x_0$.

Let us denote by $add_{\lambda_1,\lambda_2}$ either of the maps

$$X^{\lambda_1} \times X^{\lambda_2} \to X^{\lambda_1+\lambda_2} \text{ and } X^{\lambda_1} \times X^{\lambda_2}_{x_0} \to X^{\lambda_1+\lambda_2}_{x_0}.$$

The map $add_{\lambda_1,\lambda_2}$ is finite. For a perverse sheaves $\mathcal{F}_1$ on X^{λ_1} and $\mathcal{F}_2$ on X^{λ_2} or $X^{\lambda_2}_{x_0}$, we shall denote by $\mathcal{F}_1 \star \mathcal{F}_2$ the perverse sheaf

$$(add_{\lambda_1,\lambda_2})_!(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \simeq (add_{\lambda_1,\lambda_2})_*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$$

on $X^{\lambda_1+\lambda_2}$ (or $X^{\lambda_1+\lambda_2}_{x_0}$).

Let

$$(X^{\lambda_1} \times X^{\lambda_2})_{disj} \subset X^{\lambda_1} \times X^{\lambda_2} \text{ and } (X^{\lambda_1} \times X^{\lambda_2}_{x_0})_{disj} \subset X^{\lambda_1} \times X^{\lambda_2}_{x_0}$$

the open subschemes, corresponding to pairs of divisors (D_1, D_2) with the condition that the support of D_1 does not intersect the support of D_2 in the former case, and is also disjoint from $\{x_0\}$ in the latter case.

1.2. **The construction.** Let A be a Λ^{neg}-graded Hopf algebra. For the following construction we will work over the ground field $\mathbb{C}$ and we take X to be the affine line $\mathbb{A}^1$. (For the construction to work for any X, we need that the antipode on A be involutive.) We will assume that $A^0 \simeq \mathbb{C}$ and that its graded components are finite-dimensional.

Theorem-Construction 1.3.
(1) *To A one can canonically attach a system Ω_A of perverse sheaves Ω^λ_A on X^λ endowed with factorization isomorphisms*

$$add^*_{\lambda_1,\lambda_2}(\Omega^{\lambda_1+\lambda_2}_A)|_{(X^{\lambda_1}\times X^{\lambda_2})_{disj}} \simeq \Omega^{\lambda_1}_A \boxtimes \Omega^{\lambda_2}_A|_{(X^{\lambda_1}\times X^{\lambda_2})_{disj}}. \tag{1.1}$$

Moreover, the isomorphisms (1.1) *are associative in a natural sense.*
(2) *The *-stalk of Ω^λ_A at a point $\Sigma\, \lambda_i \cdot x_i \in X^\lambda$ is quasi-isomorphic to*

$$\bigotimes_i \left(\mathrm{Tor}^\bullet_A(\mathbb{C},\mathbb{C})\right)^{\lambda_i},$$

where the super-script refers to the corresponding graded component.

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1.4. **Verdier duality.** For a perverse sheaf $\mathcal{F}$ on X^λ let us view its Verdier dual $\mathbb{D}(\mathcal{F})$ as living over $X^{-\lambda}$ (i.e., we change our semi-group to $\Lambda^{pos} := -\Lambda_{pos}$).

For a Λ^{neg}-graded Hopf algebra A as above, consider its graded linear dual $A^\vee$ as a Λ^{pos}-graded Hopf algebra.

Proposition 1.5. *There exists a natural isomorphism of perverse sheaves over $X^{-\lambda}$.*

$$\mathbb{D}(\Omega^\lambda_A) \simeq \Omega^\lambda_{A^{\vee,oc}},$$

where the superscript "oc" denotes the Hopf algebra with reversed co-multiplication (and the old multiplication).

The above isomorphisms for all $\lambda \in \Lambda^{pos}$ are compatible with the factorization isomorphisms (1.1).

Note that $\mathbb{D}(\mathbb{D}(\Omega^\lambda_A)) \simeq \Omega_{A^{om,oc}}$, where "om" denotes the Hopf algebra with reversed multiplication. However, $A^{om,oc} \simeq A$, by means of the antipode.

As a corollary, we obtain the following description of the !-stalk of Ω^λ_A at a point $\Sigma\, \lambda_i \cdot x_i \in X^\lambda$. Namely, it is quasi-isomorphic to

$$\bigotimes_i \left(\mathrm{Ext}^\bullet_{A^\vee}(\mathbb{C},\mathbb{C})\right)^{\lambda_i}.$$

1.6. **Modules.** We will now extend the construction of Sect. 1.2 to the case of modules. Let $\mathbf{Dr}(A)$ be the Drinfeld double of A. We will consider the category $\mathbf{Dr}(A)$-mod of Λ-graded modules M over $\mathbf{Dr}(A)$, with finite-dimensional graded components and such that the set $\{\lambda \,|\, M^\lambda \neq 0\}$ is bounded from above (i.e., is contained in a set of the form $\lambda' + \Lambda^{neg}$ for some $\lambda' \in \Lambda$).

Theorem-Construction 1.7.

(1) *To an object $M \in \mathbf{Dr}(A)$-mod one can canonically associate system $\Omega_{A,M}$ of perverse sheaves $\Omega^\lambda_{A,M}$ on $X^\lambda_{x_0}$ for $\lambda \in \Lambda$ endowed with following system of factorization isomorphisms with respect to Ω_A:*

$$add^*_{\lambda_1,\lambda_2}(\Omega^{\lambda_1+\lambda_2}_{A,M})|_{(X^{\lambda_1}\times X^{\lambda_2}_{x_0})_{disj}} \simeq \Omega^{\lambda_1}_A \boxtimes \Omega^{\lambda_2}_{A,M}|_{(X^{\lambda_1}\times X^{\lambda_2}_{x_0})_{disj}}. \tag{1.2}$$

The isomorphisms (1.2) *are associative in a natural sense with respect to* (1.1).

(2) *The *-stalk of $\Omega^\lambda_{A,M}$ at a point $\Sigma\, \lambda_i \cdot x_i \in X^\lambda + \lambda_0 \cdot x_0$ is quasi-isomorphic to*

$$\bigotimes_i \left(\mathrm{Tor}^\bullet_A(\mathbb{C},\mathbb{C})\right)^{\lambda_i} \otimes \left(\mathrm{Tor}^\bullet_A(\mathbb{C},M)\right)^{\lambda_0}.$$

Note that linear duality $M \mapsto M^\vee$ defines a contravariant equivalence between $\mathbf{Dr}(A)$-mod and $\mathbf{Dr}(A^{\vee,oc})$-mod, (reversing the braiding).

Proposition 1.8. *There is a natural isomorphism*

$$\mathbb{D}(\Omega^\lambda_{A,M}) \simeq \Omega^{-\lambda}_{A^{\vee,oc},M^\vee},$$

compatible with the isomorphisms (1.2).

The proposition implies the following description of the !-stalks of $\Omega^\lambda_{A,M}$. At a point $\Sigma\, \lambda_i \cdot x_i \in X^\lambda + \lambda_0 \cdot x_0$, the !-stalk is isomorphic to

$$\bigotimes_i \left(\mathrm{Ext}^\bullet_{A^\vee}(\mathbb{C},\mathbb{C})\right)^{\lambda_i} \otimes \left(\mathrm{Ext}^\bullet_{A^\vee}(\mathbb{C},M)\right)^{\lambda_0}.$$

1.9. **Factorizable sheaves.**

Definition 1.10. *A factorizable sheaf with respect to Ω_A is a system $\mathcal{F}$ of perverse sheaves $\mathcal{F}^\lambda$ on $X^\lambda_{x_0}$ equipped with an associative system of isomorphism*

$$add^*_{\lambda_1,\lambda_2}(\mathcal{F}^{\lambda_1+\lambda_2})|_{(X^{\lambda_1}\times X^{\lambda_2}_{x_0})_{disj}} \simeq \Omega_A^{\lambda_1}\boxtimes\mathcal{F}^{\lambda_2}|_{(X^{\lambda_1}\times X^{\lambda_2}_{x_0})_{disj}}, \tag{1.3}$$

and such that the set $\{\lambda \,|\, \mathcal{F}^\lambda\neq 0\}$ is bounded from above.

Factorizable sheaves with respect to Ω_A naturally form a category that we shall denote $\mathrm{FS}(\Omega_A)$.

Theorem 1.11. *The assignment*

$$M\mapsto \Omega_{A,M}$$

is an equivalence between the category $\mathbf{Dr}(A)$-mod and $\mathrm{FS}(\Omega_A)$.

2. The commutative case

2.1. We begin with the following observation:

Lemma 2.2. *A Hopf agebra A is co-commutative (resp., commutative) if and only if the isomorphism* (1.1) *extends to a map*

$$add^*_{\lambda_1,\lambda_2}(\Omega_A^{\lambda_1+\lambda_2})\to\Omega_A^{\lambda_1}\boxtimes\Omega_A^{\lambda_2} \text{ or } \Omega_A^{\lambda_1}\boxtimes\Omega_A^{\lambda_2}\to add^!_{\lambda_1,\lambda_2}(\Omega_A^{\lambda_1+\lambda_2}),$$

respectively.

Thus, for A which is co-commutative (resp., commutative) we have the maps

$$\Omega_A^{\lambda_1+\lambda_2}\to\Omega_A^{\lambda_1}\star\Omega_A^{\lambda_2} \text{ and } \Omega_A^{\lambda_1}\star\Omega_A^{\lambda_2}\to\Omega_A^{\lambda_1+\lambda_2}, \tag{2.1}$$

that we shall refer to as co-multiplication and multiplication, respectively.

2.3. The Bar-complex. Again, for A co-commutative (resp., commutative) we can form a complex of perverse sheaves on X^λ's, denoted $\mathrm{Bar}(\Omega_A)$ by taking

$$\bigoplus_{n\geq 0}\underbrace{A[\pm 1]\star ...\star A[\pm 1]}_{n},$$

with the appropriate Λ^{neg}-grading and with $\pm1=-1$ in the co-commutative and $\pm1=1$ in the commutative case.

The system $\mathrm{Bar}(\Omega_A)$ itself is endowed with factorization isomorphisms, and, moreover, an associative algebra (resp., co-associative co-algebra) structure with respect to $\star$. For a point $x\in X$, the direct sums

$$\bigoplus_{\lambda\in\Lambda^{neg}}\iota^*_{\lambda\cdot x}(\mathrm{Bar}(\Omega_A)) \text{ and } \bigoplus_{\lambda\in\Lambda^{neg}}\iota^!_{\lambda\cdot x}(\mathrm{Bar}(\Omega_A))$$

thus acquire structures of a co-associative co-algebra and associative algebra, respectively (here $\iota_{\lambda\cdot x}$ denotes the embedding of the corresponding point into X^λ).

Lemma 2.4. *We have canonical quasi-isomorphisms:*

$$\bigoplus_{\lambda\in\Lambda^{neg}}\iota^*_{\lambda\cdot x}(\mathrm{Bar}(\Omega_A))\simeq A, \textit{ as associative algebras for } A \textit{ co-commutative,}$$

$$\bigoplus_{\lambda\in\Lambda^{neg}}\iota^!_{\lambda\cdot x}(\mathrm{Bar}(\Omega_A))\simeq A, \textit{ as co-associative co-algebras for } A \textit{ commutative.}$$

2.5. Modules. Let A be co-commutative (resp., commutative). Then we can speak about systems $\mathcal{F}$ of perverse sheaves $\mathcal{F}^\lambda$ on X^λ (or $X^\lambda_{x_0}$) that are co-modules (resp., modules) with respect to Ω_A and the $\star$ operation. I.e., we have the maps

$$\mathcal{F}^{\lambda_1+\lambda_2} \to \Omega_A^{\lambda_1} \star \mathcal{F}^{\lambda_2} \text{ or } \Omega_A^{\lambda_1} \star \mathcal{F}^{\lambda_2} \to \mathcal{F}^{\lambda_1+\lambda_2},$$

respectively, that are associative with respect to the maps (2.1).

For a co-module (resp., module) $\mathcal{F}$ with respect to Ω_A, the convolution

$$\mathrm{Bar}(\Omega_A, \mathcal{F}) := \mathrm{Bar}(\Omega_A) \star \mathcal{F} \tag{2.2}$$

acquires a natural differential, and as such is a module (resp., co-module) with respect to $\mathrm{Bar}(\Omega_A)$.

Moreover, the assignment (2.2) establishes an equivalence of suitably defined derived categories. The functor in the opposite direction sends a system of complexes $\mathcal{F}'$ with an action (resp., co-action) of $\mathrm{Bar}(\Omega_A)$ to

$$\text{Un-Bar}(\mathcal{F}') := \Omega_A \star \mathcal{F}'$$

with a canonical differential.

2.6. Modules vs. factorizable sheaves. Assume that A is co-commutative (resp., commutative).

Definition 2.7. *A factorizable sheaf $\mathcal{F}$ with respect to Ω_A is said to be co-commutative (resp., commutative) if the factorization isomorphisms* (1.3) *extend to maps*

$$add^*_{\lambda_1,\lambda_2}(\mathcal{F}^{\lambda_1+\lambda_2}) \to \Omega_A^{\lambda_1} \boxtimes \mathcal{F}^{\lambda_2} \text{ and } \Omega_A^{\lambda_1} \boxtimes \mathcal{F}^{\lambda_2} \to add^!_{\lambda_1,\lambda_2}(\mathcal{F}^{\lambda_1+\lambda_2}),$$

respectively.

We note that in the derived category being co-commutative (resp., commutative) is not a property but an additional structure.

Let $\mathcal{F}$ be a co-commutative (resp., commutative) factorizable sheaf with respect to Ω_A. By adjunction, we obtain a co-associative co-action (resp., associative action)

$$\mathcal{F} \to \Omega_A \star \mathcal{F} \text{ or } \Omega_A \star \mathcal{F} \to \mathcal{F},$$

respectively.

Thus, we obtain a functor from the category of co-commutative (resp., commutative) factorizable sheaves with respect to Ω_A to that of Ω_A-comodules (resp., modules).

Proposition 2.8.

(1) *The above functor is a fully faithful embedding.*

(2) *An Ω_A-comodule (resp., module) $\mathcal{F}$ is a factorizable sheaf if and only if* $\mathrm{Bar}(\Omega_A, \mathcal{F})$ *is supported on*

$$\underset{\lambda\in\Lambda}{\cup} \lambda \cdot x_0 \subset \bigcup_{\lambda\in\Lambda} X^\lambda_{x_0}.$$

2.9. Let us see how the notion of (co)commutative factorizable sheaf couples with Theorem-Construction 1.7.

Let A be co-commutative (resp., commutative). Let A-comod (resp., A-mod) denote the category of graded A-comodules (resp. modules) N with finite-dimensional graded components and such that the set $\{\lambda \,|\, N^\lambda \neq 0\}$ is bounded from above.

We have a natural functor from A-comod (resp., A-mod) to $\mathbf{Dr}(A)$-mod, which makes A act (resp., co-act) via the augmentation.

Proposition 2.10.
(1) *For $N \in A$-comod (resp., $N \in A$-mod) the corresponding object $N \in \mathbf{Dr}(A)$-mod has the property that $\Omega_{A,N}$ is co-commutative (resp., commutative) with respect to Ω_A.*
(2) *The assignment $N \mapsto \Omega_{A,N}$ establishes an equivalence between the category of A-comodules (resp., A-modules) and that of co-commutative (resp., commutative) factorizable sheaves with respect to Ω_A.*
(3) *For $N \in A$-comod (resp., $N \in A$-mod),*

$$(\mathrm{Bar}(\Omega_A, \Omega_{A,N}))^\lambda \simeq (\iota_{\lambda\cdot x_0})_*(N^\lambda).$$

3. The twisted case: preliminaries

3.1. Categories over a braided monoidal category. Let $\mathcal{C}$ be a braided monoidal category. We denote by S_{c_1,c_2} the braiding isomorphism $c_1 \otimes c_2 \to c_2 \otimes c_1$. Let $\overline{\mathcal{C}}$ denote the braided monoidal category, which equals $\mathcal{C}$ as a monoidal category, but with the reversed braiding $\overline{S}_{c_1,c_2} := S^{-1}_{c_2,c_1}$.

Let $\mathbf{O}$ be a monoidal category equipped with a *central* functor $\phi : \mathcal{C} \to \mathbf{O}$, by which we mean a monoidal functor endowed with functorial isomorphisms $S_{c,M} : \phi(c) \otimes M \to M \otimes \phi(c)$ for $c \in \mathcal{C}$ and $M \in \mathbf{O}$, compatible with tensor products and such that for c_1, c_2, $S_{c_1,\phi(c_2)} = \phi(S_{c_1,c_2})$. In this case we will say that $\mathbf{O}$ is a monoidal category *over* $\mathcal{C}$.

Note that the category $\mathbf{O}^{op}$, obtained from $\mathbf{O}$ by reversing the monoidal structure naturally receives a central functor from $\overline{\mathcal{C}}$.

3.2. The Drinfeld double. For $(\mathcal{C}, \mathbf{O}, \phi)$ as above, we can consider the category $Z_{\mathcal{C}}(\mathbf{O})$ whose objects are $V \in \mathcal{C}$, equipped with a system of isomorphisms

$$R_{M,V} : M \otimes V \to V \otimes M, \ \forall M \in \mathbf{O},$$

compatible with tensor products of M's and such that for $M = \phi(c), c \in \mathcal{C}$, $R_{\phi(c),V} = S_{c,V}$.

The tensor product in $\mathbf{O}$ equips $Z_{\mathcal{C}}(\mathbf{O})$ with a monoidal structure. Moreover, $Z_{\mathcal{C}}(\mathbf{O})$ is naturally braided by means of $S_{V_1,V_2} := R_{V_1,V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$.

Note, however, that unless $\mathcal{C}$ is symmetric, we do *not* have any naturally defined functor $\mathcal{C} \to Z_{\mathcal{C}}(\mathbf{O})$. In other words, $Z_{\mathcal{C}}(\mathbf{O})$ is *not* a category over $\mathcal{C}$ (unless $\mathcal{C}$ is symmetric).

Note that we have a naturally defined monoidal functor $Z_{\mathcal{C}}(\mathbf{O}^{op}) \to (Z_{\mathcal{C}}(\mathbf{O}))^{op}$, which reverses the braiding, and makes the following diagram commute:

$$\begin{array}{ccc} Z_{\mathcal{C}}(\mathbf{O}^{op}) & \longrightarrow & (Z_{\mathcal{C}}(\mathbf{O}))^{op} \\ \big\downarrow & & \big\downarrow \\ \mathbf{O}^{op} & \xrightarrow{\mathrm{Id}} & \mathbf{O}^{op}. \end{array}$$

3.3. Hopf algebras. Assume that we also have a faithful monoidal functor $F : \mathbf{O} \to \mathcal{C}$ such that $F \circ \phi = \mathrm{Id}_{\mathcal{C}}$, as monoidal functors, and such that $F(S_{c,M}) = S_{c,F(M)}$. In this case, we will say that $\mathbf{O}$ is a monoidal category *in* $\mathcal{C}$. Modulo representability, a datum of a monoidal category in $\mathcal{C}$ is equivalent to that of a Hopf algebra A in $\mathcal{C}$.

Note that $\mathbf{O}^{op}$ is then naturally a category over $\overline{\mathcal{C}}$. The corresponding Hopf algebra may be thought of as A^{oc} (which it literally is, if $\mathcal{C}$ is symmetic).

As was mentioned above, the category $Z_{\mathcal{C}}(\mathbf{O})$, although endowed with a monoidal functor to $\mathcal{C}$, does not correspond to a *Hopf* algebra in $\mathcal{C}$. However, it corresponds to an algebra in this category, denoted $\mathbf{Dr}(A)$. By definition, $\mathbf{Dr}(A)$-mod $:= Z_{\mathcal{C}}(\mathbf{O})$. The forgetful functor $Z_{\mathcal{C}}(\mathbf{O}) \to \mathbf{O}$ defines a homomorphism of algebras $A \to \mathbf{Dr}(A)$.

3.4. The Koszul dual category. For $(\mathbf{O}, \mathcal{C}, \phi, F)$ as above consider the category $\mathbf{O}' = \mathrm{Funct}_{\mathbf{O}}(\mathcal{C}, \mathcal{C})$ of functors $\mathcal{C} \to \mathcal{C}$ that commute in a natural sense with the action of $\mathbf{O}$, the latter being

$$M \in \mathbf{O}, c \in \mathcal{C} \mapsto F(M) \otimes c.$$

Composition makes $\mathbf{O}'$ into a monoidal category. We have a natural monoidal functor

$$F' : \mathbf{O}' \to \mathrm{Funct}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) \simeq \mathcal{C}^{op} \simeq \mathcal{C},$$

where the last arrow comes from the braiding.

In addition, since $\mathcal{C}$ maps to $Z(\mathbf{O})$ by means of ϕ, we have a natural monoidal functor $\phi' : \overline{\mathcal{C}} \to Z(\mathbf{O}')$. I.e., we obtain that $\mathbf{O}'$ is naturally a monoidal category over $\overline{\mathcal{C}}$. The functor F' makes $\mathbf{O}'$ into a monoidal category in $\overline{\mathcal{C}}$.

If $\mathbf{O}$ corresponds to a Hopf A in $\mathcal{C}$, and assume that A is dualizable as an object of $\mathcal{C}$. Then $\mathbf{O}'$ corresponds to the Hopf algebra $A^{\vee,oc}$ in $\overline{\mathcal{C}}$.

3.5. Relationship between Drinfeld doubles. Note that we have a natural functor

$$Z_{\mathcal{C}}(\mathbf{O}) \to Z_{\mathcal{C}}(\mathbf{O}')^{op},$$

which reverses the braiding and makes the following diagram commute

$$\begin{array}{ccc} Z_{\mathcal{C}}(\mathbf{O}) & \longrightarrow & Z_{\mathcal{C}}(\mathbf{O}') \\ \big\downarrow & & \big\downarrow \\ \mathcal{C} & \longrightarrow & \overline{\mathcal{C}}, \end{array}$$

where the bottom horizontal line is the identity functor.

In the case when $\mathbf{O}$ corresponds to a Hopf algebra A in $\mathcal{C}$, we obtain an isomorphism of algebras

$$\mathbf{Dr}(A) \simeq \mathbf{Dr}(A^{\vee,oc});$$

in particular, $\mathbf{Dr}(A)$ receives a homomorphism from $A^{\vee,oc}$.

4. The twisted case: construction

4.1. Gerbes. Recall that on a topological space, a data of a $\mathbb{C}^*$-gerbe allows to twist the notion of a sheaf. Given a $\mathbb{C}^*$-gerbe $\mathcal{P}$ over $\mathcal{Y}$, we will refer to the corresponding category as $\mathcal{P}$-twisted sheaves on $\mathcal{Y}$. If $\mathcal{Y}$ is stratified, we can talk about $\mathcal{P}$-twisted constructible (or perverse, if we are given a perversity function) sheaves on $\mathcal{Y}$.

Let us recall the following constriction. Let $\mathcal{L}$ be a complex line bundle over $\mathcal{Y}$, and $q' \in \mathbb{C}^*$. We define the gerbe $\mathcal{L}^{\otimes log(q')}$ over $\mathcal{Y}$ as follows: its sections (over an open subset of $\mathcal{Y}$) is the category of 1-dimensional local systems on the corresponding circle bundle (over the given open subset) with monodromy q' along the fiber.

The corresponding category of $\mathcal{L}^{\otimes log(q')}$-twisted sheaves identifies with that of sheaves on the circle bundle, which are monodromic with monodromy q' along the fiber.

4.2. Gerbes attaches to a pairing. Let q be a symmetric pairing $\Lambda \otimes \Lambda \to \mathbb{C}^*$.

The following mimics [CHA], Sect. 3.10.3. Assume that for every $\lambda \in \Lambda$ we are given a $\mathbb{C}^*$-gerbe $\mathcal{P}^{\lambda}_{X,q}$ over X plus the following data: for every λ_1, λ_2 an identification

$$c^{\lambda_1,\lambda_2} : \mathcal{P}^{\lambda_1}_{X,q} \otimes \mathcal{P}^{\lambda_2}_{X,q} \simeq \mathcal{P}^{\lambda_1+\lambda_2}_{X,q} \otimes \omega_X^{2log(q(\lambda_1,\lambda_2))},$$

where ω_X is the canonical line bundle on X. We need the data of c^{λ_1,λ_2} to be symmetric in a natural sense.

For three elements $\lambda_1, \lambda_2, \lambda_3$ we need a data that identifies the three arising isomorphisms of gerbes,

$$\mathcal{P}^{\lambda_1}_{X,q} \otimes \mathcal{P}^{\lambda_2}_{X,q} \otimes \mathcal{P}^{\lambda_3}_{X,q} \simeq \mathcal{P}^{\lambda_1+\lambda_2+\lambda_3}_{X,q} \otimes \omega_X^{2log(q(\lambda_1,\lambda_2))} \otimes \omega_X^{2log(q(\lambda_2,\lambda_3))} \otimes \omega_X^{2log(q(\lambda_3,\lambda_1))},$$

such that the corresponding identity holds for 4-tuples of λ's.

Given a pairing q, it is easy to see that the data of $(\mathcal{P}^{\lambda}_{X,q}, c^{\lambda_1,\lambda_2})$ exists (see below for a construction). Having two pieces of data like this, their ratio is canonically a $\check{T}$-gerbe over X, where $\check{T}$ is the torus $\mathbb{C}^* \underset{\mathbb{Z}}{\otimes} \Lambda$.

Given q, we will refer to a data $\mathcal{P}_q := \{(\mathcal{P}^{\lambda}_{X,q}, c^{\lambda_1,\lambda_2})\}$ as above as a q-twisted $\check{T}$-gerbe on X.

4.3. The canonical gerbe attached to a pairing. We define a q-twisted $\check{T}$-gerbe on X by setting:

$$\mathcal{P}^{\lambda}_{can,X,q} := (\omega_X)^{log(q(\lambda,\lambda))},$$

and c^{λ_1,λ_2} to be the natural identification

$$(\omega)^{log(q(\lambda_1+\lambda_2,\lambda_1+\lambda_2))} \simeq (\omega_X)^{log(q(\lambda_1,\lambda_1))} \otimes (\omega_X)^{log(q(\lambda_2,\lambda_2))} \otimes (\omega_X)^{2\cdot log(q(\lambda_1,\lambda_2))}.$$

In what follows we will refer to this choice as a canonical q-twisted $\check{T}$-gerbe on X, and denote it $\mathcal{P}_{q,can}$.

4.4. Gerbes over configuration spaces. A datum of a q-twisted $\check{T}$-gerbe on X gives rise to the following construction:

For every n-tuple $\lambda_1, ..., \lambda_n$ of elements of Λ we obtain a $\mathbb{C}^*$-gerbe $\mathcal{P}^{\lambda_1,...,\lambda_n}_{X^n,q}$ over X^n, whose restriction to the diagonal

$$\Delta_\phi : X^m \hookrightarrow X^n$$

corresponding to a surjection $\phi : \{1, ..., n\} \twoheadrightarrow \{1, ..., m\}$ identifies with $\mathcal{P}^{\mu_1,...,\mu_m}_{X^m,q}$, where

$$\mu_j = \sum_{i,\phi(i)=j} \lambda_i.$$

Namely, we set

$$\mathcal{P}^{\lambda_1,...,\lambda_n}_{X^n,q} := \mathcal{P}^{\lambda_1}_{X,q} \boxtimes ... \boxtimes \mathcal{P}^{\lambda_n}_{X,q} \boxtimes \left(\underset{i\neq j}{\boxtimes} \mathcal{O}(-\Delta_{i,j})^{2log(q(\lambda_i,\lambda_j))} \right).$$

For $n = n_1 + n_2$ and $\{\lambda_1, ..., \lambda_n\} = \{\lambda^1_1, ..., \lambda^1_{n_1}\} \cup \{\lambda^2_1, ..., \lambda^2_{n_2}\}$ we have an identification of gerbes

$$\mathcal{P}^{\lambda_1,...,\lambda_n}_{X^n,q}|_{(X^{n_1}\times X^{n_2})_{disj}} \simeq \mathcal{P}^{\lambda^1_1,...,\lambda^1_{n_1}}_{X^{n_1},q} \boxtimes \mathcal{P}^{\lambda^2_1,...,\lambda^2_{n_2}}_{X^{n_2},q}|_{(X^{n_1}\times X^{n_2})_{disj}}.$$

In particular, for a semi-group $\Lambda^{neg} \subset \Lambda$, the above datum gives to a $\mathbb{C}^*$-gerbe denoted $\mathcal{P}_{X^\lambda,q}$ over X^λ and a $\mathbb{C}^*$-gerbe denoted $\mathcal{P}_{X^\lambda_{x_0},q}$ over $X^\lambda_{x_0}$ endowed with factorization isomorphisms

$$\mathcal{P}_{X^{\lambda_1+\lambda_2},q}|_{(X^{\lambda_1}\times X^{\lambda_2})_{disj}} \simeq \mathcal{P}_{X^{\lambda_1},q} \boxtimes \mathcal{P}_{X^{\lambda_2},q}|_{(X^{\lambda_1}\times X^{\lambda_2})_{disj}} \tag{4.1}$$

and

$$\mathcal{P}_{X^{\lambda_1+\lambda_2}_{x_0},q}|_{(X^{\lambda_1}\times X^{\lambda_2}_{x_0})_{disj}} \simeq \mathcal{P}_{X^{\lambda_1},q} \boxtimes \mathcal{P}_{X^{\lambda_2}_{x_0},q}|_{(X^{\lambda_1}\times X^{\lambda_2}_{x_0})_{disj}}, \tag{4.2}$$

which are associative in the natural sense.

4.5. The braided category attached to q. Let q be as above. We consider a new braiding on the category of Λ-graded vector spaces. Namely, for two such vector spaces M_1 and M_2 and elements $m_i \in M_i$ od degrees λ_i, respectively, we set

$$m_1 \otimes m_2 \mapsto q(\lambda_1, \lambda_2) \cdot m_2 \otimes m_1.$$

We denote this category by Vect^Λ_q.

Let now A be a Hopf algebra in Vect^Λ_q. Assume that as a Λ-graded vector space, A satisfies the same assumptions as in the non-twisted case.

We take our curve X to be $\mathbb{A}^1$ with a fixed coordinate. Let $\mathcal{P}_q$ be $\mathcal{P}_{q,can}$ as defined in Sect. 4.3. (Note that the choice of a coordinate trivializes the gerbes $\mathcal{P}^\lambda_{q,can,X}$ for every λ, however, the gerbe $\mathcal{P}_{X^\lambda,q,can}$ over X^λ is non-trivial [1].)

The following generalizes Theorem-Construction 1.3 to the present context:

Theorem-Construction 4.6.

(1) *To A one can associate a system Ω_A of $\mathcal{P}_{X^\lambda,q}$-twisted perverse sheaves Ω^λ on X^λ, endowed with factorization isomorphisms as in* (1.1), (*the latter make sense in view of* (4.1)), *which are associative in the natural sense.*

(2) *The *- and !-stalks of Ω_A are calculated by the same formula as in the non-twisted case.* (*This makes sense, since according to our choice of the q-twisted $\check{T}$-gerbe, its restriction to the strata of the diagonal stratification of X^λ is trivial, so *- and !-fibers of a twisted sheaf are (non-twisted) vector spaces*).

[1]The choice of the coordinate allows to trivialize each individual $\mathcal{P}_{X^\lambda,q,can}$ as well, which is how it is done in [BFS], but this will not be compatible with the factorization isomorphisms.

4.7. Factorizable sheaves in the twisted situation. Due to (4.2), the definition of factorizable sheaf given in Sect. 1.9 transfers to the twisted context, where the $\mathcal{F}^\lambda$'s are now $\mathcal{P}_{X^\lambda_{x_0},q}$-twisted perverse sheaves. Thus, for A as above, we have a well-defined category $\mathrm{FS}(\Omega_A)$.

Recall now the category (which is in fact braided monoidal) $\mathbf{Dr}(A)$-mod. Generalizing Theorem-Construction 1.7 and Theorem 1.11 we have:

Theorem-Construction 4.8.

(1) *To an object $M \in \mathbf{Dr}(A)$-mod one can canonically associate an object $\Omega_{A,M}$ in $\mathrm{FS}(\Omega_A)$. Moreover, the assignment $M \to \Omega_{A,M}$ is an equivalence of categories.*

(2) *The *- and !-stalks of $\Omega_{A,M}$ are calculated by the same formula as in the non-twisted case.*

4.9. Verdier duality in the twisted case. Given a $\mathbb{C}^*$-gerbe $\mathcal{P}$ over $\mathcal{Y}$ we have the Verdier duality functor that maps the (derived category of) $\mathcal{P}$-twisted sheaves to that of $\mathcal{P}^{-1}$-twisted sheaves.

Recall that we can think of $A^{\vee,oc}$ as a Hopf algebra in $\mathrm{Vect}^\Lambda_{q^{-1}}$, and there is an isomorphism

$$\mathbf{Dr}(A)\text{-mod} \to \mathbf{Dr}(A^{\vee,oc})\text{-mod},$$

as monoidal categories that reverses the braiding. The rigidity functor on $\mathbf{Dr}(A)$-mod can be thought of as a contravariant functor

$$\mathbf{Dr}(A)\text{-mod} \to \mathbf{Dr}(A^{\vee,oc})\text{-mod},$$

which underlies the dualization functor

$$M \mapsto M^\vee : \mathrm{Vect}^\Lambda_q \to \mathrm{Vect}^\Lambda_{q^{-1}}.$$

We have:

Proposition 4.10. $\mathbb{D}(\Omega_A) \simeq \Omega_{A^{\vee,oc}}$ *and* $\mathbb{D}(\Omega_{A,M}) \simeq \Omega_{A^{\vee,oc},M^\vee}$.

5. The quantum group

5.1. Lusztig's algebra. Let Λ be the weight lattice of a reductive group G. We denote by I the Dynkin diagram, and for $i \in I$ by α_i the corresponding root. Let Λ^{pos} be the positive span of the α_i's. Let q be as above.

Our primary interest is the category of modules over Lusztig's quantum group ${}^L\mathbf{U}_q$ (with divided powers). We will denote by ${}^L\mathbf{U}_q$-mod the category of Λ-graded modules with weights bounded from above. This quantum R-matrix makes ${}^L\mathbf{U}_q$-mod into a braided monoidal category.

Along with ${}^L\mathbf{U}_q$ we have the Kac-De Concini version of the quantum group, denoted ${}^{KD}\mathbf{U}_q$, and the corresponding category of modules ${}^{KD}\mathbf{U}_q$-mod. We have a homomorphism of Hopf algebras

$$\phi : {}^{KD}\mathbf{U}_q \to {}^L\mathbf{U}_q,$$

compatible with the the fiber functor to Vect. The morphism ϕ is known to be an isomorphism "away from roots of unity", i.e., when $q(\alpha_i, \alpha_i)$ is not a root of unity for any $i \in I$, and at $q = 1$.

We remark, however, that ${}^{KD}\mathbf{U}_q$-mod fails to be a braided category at roots of unity.

5.2. Positive and negative parts. Let $\mathbf{U}_q[B^+]$ be either Lusztig's are the Kac-De Concini sub-Hopf algebra of $\mathbf{U}_q$, corresponding to the quantum Borel.

The projection on the torus and the forgetful functor to Vect^Λ define monoidal functors

$$\mathrm{Vect}^\Lambda \leftrightarrows \mathbf{U}_q[B^+]\text{-mod},$$

and it is easy to see that $\mathbf{U}_q[B^+]$-mod becomes a monoidal category in $\mathrm{Vect}^\Lambda_{q^{-1}}$. We shall denote by ${}^L\mathbf{U}_q^+$ and ${}^{KD}\mathbf{U}_q^+$, respectively, the resulting Hopf algebras in $\mathrm{Vect}^\Lambda_{q^{-1}}$. As associative algebras, they coincide with the same-named sub-algebras of ${}^L\mathbf{U}_q$ and ${}^{KD}\mathbf{U}_q$, respectively.

Similarly, we consider the Hopf algebra $\mathbf{U}_q[B^-]$, and the category $\mathbf{U}_q[B^-]$-mod with the corresponding functors $\mathrm{Vect}^\Lambda \leftrightarrows \mathbf{U}_q[B^-]$-mod. We change the monoidal structure on the above functors by multiplying it by $q(\lambda,\mu)$ on the (λ,μ) graded component for the $\leftarrow$ functor, and by its inverse for the $\rightarrow$ functor. This makes $\mathbf{U}_q[B^-]$-mod into a monoidal category over Vect^Λ. We shall denote by ${}^L\mathbf{U}_q^-$ and ${}^{KD}\mathbf{U}_q^-$, respectively, the resulting Hopf algebras in Vect^Λ_q. As associative algebras, they coincide with the same-named sub-algebras of ${}^L\mathbf{U}_q$ and ${}^{KD}\mathbf{U}_q$, respectively.

The Drinfeld doubles

$$\mathbf{Dr}({}^L\mathbf{U}_q^-),\ \mathbf{Dr}({}^{KD}\mathbf{U}_q^-),\ \mathbf{Dr}({}^L\mathbf{U}_q^+) \text{ and } \mathbf{Dr}({}^{KD}\mathbf{U}_q^+)$$

make sense as Hopf algebras in Vect.

The quantum R-martix on $\mathbf{U}_q$ has the property that the monoidal restriction functors ${}^L\mathbf{U}_q$-mod $\to {}^L\mathbf{U}_q[B^-]$-mod $\to {}^{KD}\mathbf{U}_q[B^-]$ naturally lift to a braided monoidal functors

$$ {}^L\mathbf{U}_q\text{-mod} \to Z_{\mathrm{Vect}^\Lambda_q}({}^L\mathbf{U}_q[B^-]\text{-mod}) \text{ and } {}^L\mathbf{U}_q\text{-mod} \to Z_{\mathrm{Vect}^\Lambda_q}({}^{KD}\mathbf{U}_q[B^-]\text{-mod}) \tag{5.1}$$

Moreover, the above functors are equivalences away from roots of unity.

Similarly, when we consider ${}^L\mathbf{U}_q$-mod with the inverted braiding, we have the functors

$$ {}^L\mathbf{U}_q\text{-mod} \to Z_{\mathrm{Vect}^\Lambda_q}({}^L\mathbf{U}_q[B^+]\text{-mod}) \text{ and } {}^L\mathbf{U}_q\text{-mod} \to Z_{\mathrm{Vect}^\Lambda_q}({}^{KD}\mathbf{U}_q[B^+]\text{-mod}), \tag{5.2}$$

which are equivalences away from roots of unity.

We have the corresponding homomorphisms of Hopf algebras

$$\mathbf{Dr}({}^L\mathbf{U}_q^-) \to {}^L\mathbf{U}_q,\ \mathbf{Dr}({}^{KD}\mathbf{U}_q^-) \to {}^L\mathbf{U}_q,\ \mathbf{Dr}({}^L\mathbf{U}_q^+) \to {}^L\mathbf{U}_q \text{ and } \mathbf{Dr}({}^{KD}\mathbf{U}_q^+) \to {}^L\mathbf{U}_q. \tag{5.3}$$

The categories

$$\begin{aligned} &Z_{\mathrm{Vect}^\Lambda_q}({}^L\mathbf{U}_q[B^-]\text{-mod}),\ Z_{\mathrm{Vect}^\Lambda_q}({}^{KD}\mathbf{U}_q[B^-]\text{-mod}), \\ &\qquad\qquad Z_{\mathrm{Vect}^\Lambda_q}({}^L\mathbf{U}_q[B^+]\text{-mod}),\ Z_{\mathrm{Vect}^\Lambda_q}({}^{KD}\mathbf{U}_q[B^+]\text{-mod}) \end{aligned} \tag{5.4}$$

will serve as approximations for ${}^L\mathbf{U}_q$-mod. In the sequel we will make an assumption that $q(\alpha_i,\alpha_i)^2 \neq 1$ for any $i \in I$, and will define categories that approximate ${}^L\mathbf{U}_q$-mod even better.

5.3. The free algebras. We consider the braided monoidal category Vect_q^Λ, and we let ${}^{fr}\mathbf{U}_q^-$ be the free associative algebra in it, generated by the elements F_i, $i \in I$, where $\deg(F_i) = -\alpha_i$. We define a Hopf algebra structure on it by $F_i \mapsto F_i \otimes 1 + 1 \otimes F_i$.

Similarly, we consider the braided monoidal category $\mathrm{Vect}_{q^{-1}}^\Lambda$, and we let ${}^{fr}\mathbf{U}_q^+$ be the free associative algebra in it, generated by the elements E_i, $i \in I$, where $\deg(E_i) = \alpha_i$. We define a Hopf algebra structure on ${}^{fr}\mathbf{U}_q^+$ by $E_i \mapsto E_i \otimes 1 + 1 \otimes E_i$.

We let ${}^{cofr}\mathbf{U}_q^-$ be $({}^{fr}\mathbf{U}_q^+)^{\vee,oc}$, which is a Hopf algebra in Vect_q^Λ, and we let ${}^{cofr}\mathbf{U}_q^+$ be $({}^{fr}\mathbf{U}_q^-)^{\vee,oc}$, which is a Hopf algebra in $\mathrm{Vect}_{q^{-1}}^\Lambda$.

From now on, we will assume that $q(\alpha_i,\alpha_i)^2 \neq 1$ for any $i \in I$. Then the formulas $(F_i, E_i) = \frac{1}{1-q(\alpha_i,\alpha_i)^{-2}}$ define canonical maps of Hopf algebras

$$ {}^{fr}\mathbf{U}_q^- \to {}^{cofr}\mathbf{U}_q^- \text{ and } {}^{fr}\mathbf{U}_q^+ \to {}^{cofr}\mathbf{U}_q^+. \tag{5.5}$$

5.4. Lusztig's and Kac-De Concini algebras. By definition, the Kac-De Concini version ${}^{KD}\mathbf{U}_q^-$ of $\mathbf{U}_q^-$ is a quotient Hopf algebra of ${}^{fr}\mathbf{U}_q^-$ given by the quantum Serre relations, and similarly for ${}^{KD}\mathbf{U}_q^+$.

Under the above assumption on q, Lusztig's algebra ${}^{L}\mathbf{U}_q^-$ is a Hopf sub-algebra of ${}^{cofr}\mathbf{U}_q^-$, and similarly for ${}^{L}\mathbf{U}_q^+$, such that the isomorphisms

$$ {}^{cofr}\mathbf{U}_q^- \simeq ({}^{fr}\mathbf{U}_q^+)^{\vee,oc} \text{ and } {}^{cofr}\mathbf{U}_q^+ \simeq ({}^{fr}\mathbf{U}_q^-)^{\vee,oc} $$

induce isomorphisms

$$ {}^{L}\mathbf{U}_q^- \simeq ({}^{KD}\mathbf{U}_q^+)^{\vee,oc} \text{ and } {}^{L}\mathbf{U}_q^+ \simeq ({}^{KD}\mathbf{U}_q^-)^{\vee,oc}. $$

The maps

$$ {}^{KD}\mathbf{U}_q^- \to {}^{L}\mathbf{U}_q^- \text{ and } {}^{KD}\mathbf{U}_q^+ \to {}^{L}\mathbf{U}_q^+, $$

induced by (5.5) coincide with those induced by the map $\phi : {}^{KD}\mathbf{U}_q \to {}^{L}\mathbf{U}_q$.

In particular, we have:

$$ Z_{\mathrm{Vect}_q^\Lambda}({}^{L}\mathbf{U}_q[B^-]\text{-mod}) \simeq Z_{\mathrm{Vect}_{q^{-1}}^\Lambda}({}^{KD}\mathbf{U}_q[B^+]\text{-mod}) $$

and

$$ Z_{\mathrm{Vect}_q^\Lambda}({}^{KD}\mathbf{U}_q[B^-]\text{-mod}) \simeq Z_{\mathrm{Vect}_{q^{-1}}^\Lambda}({}^{L}\mathbf{U}_q[B^+]\text{-mod}), $$

i.e.,

$$ \mathbf{Dr}({}^{L}\mathbf{U}_q^-) \simeq \mathbf{Dr}({}^{KD}\mathbf{U}_q^+) \text{ and } \mathbf{Dr}({}^{L}\mathbf{U}_q^+) \simeq \mathbf{Dr}({}^{KD}\mathbf{U}_q^-), $$

reversing the braiding, and compatible with the homomorphisms (5.3).

5.5. The small quantum group. Let $\mathfrak{u}_q^-$ (resp., $\mathfrak{u}_q^+$) be the images of the maps

$$ \phi^- : {}^{KD}\mathbf{U}_q^- \to {}^{L}\mathbf{U}_q^- \text{ and } \phi^- : {}^{KD}\mathbf{U}_q^+ \to {}^{L}\mathbf{U}_q^+, $$

respectively. These are Hopf algebras in Vect_q^Λ and $\mathrm{Vect}_{q^{-1}}^\Lambda$, respectively.

By the above,

$$ \mathfrak{u}_q^+ \simeq (\mathfrak{u}_q^-)^{\vee,oc}. $$

Let $\mathbf{U}_q^{-,\frac{\pm}{2}} \subset {}^{L}\mathbf{U}_q$ and, $\mathbf{U}_q^{+,\frac{=}{2}} \subset {}^{L}\mathbf{U}_q$ be the sub-Hopf algebras, equal to the images of

$$ \mathbf{Dr}({}^{L}\mathbf{U}_q^-) \to {}^{L}\mathbf{U}_q \text{ and } \mathbf{Dr}({}^{L}\mathbf{U}_q^+) \to {}^{L}\mathbf{U}_q, $$

respectively. They are generated as algebras by $({}^L\mathbf{U}_q^-, \mathfrak{u}_q^+)$ and $(\mathfrak{u}_q^-, {}^L\mathbf{U}_q^+)$. I.e., the corresponding categories of modules, denoted $\mathbf{U}_q^{-,\frac{\pm}{2}}$-mod and $\mathbf{U}_q^{,\frac{\mp}{2}}$-mod are monoidal, and braided since the subalgebras $\mathbf{U}_q^{-,\frac{\pm}{2}} \subset {}^L\mathbf{U}_q$ and, $\mathbf{U}_q^{+,\frac{\mp}{2}} \subset {}^L\mathbf{U}_q$ contain the R-matrix of ${}^L\mathbf{U}_q$.

We have a commutative diagram of homomorphisms of Hopf algebras:

$$\begin{tikzcd}
\mathbf{Dr}({}^L\mathbf{U}_q^-) \simeq \mathbf{Dr}({}^D\mathbf{U}_q^+) \arrow[r] & \mathbf{U}_q^{-,\frac{\pm}{2}} \arrow[r] & {}^L\mathbf{U}_q \\
& \mathbf{Dr}(\mathfrak{u}_q^-) \simeq \mathbf{Dr}(\mathfrak{u}_q^+) \arrow[u] \arrow[r] & \mathbf{U}_q^{+,\frac{\mp}{2}} \arrow[u] \\
& & \mathbf{Dr}({}^{KD}\mathbf{U}_q^-) \simeq \mathbf{Dr}({}^L\mathbf{U}_q^+). \arrow[u]
\end{tikzcd}$$

The above diagram is compatible with the R-matrices of

$$\mathbf{Dr}({}^L\mathbf{U}_q^-),\ \mathbf{Dr}(\mathfrak{u}_q^-),\ \mathbf{Dr}({}^{KD}\mathbf{U}_q^-) \text{ and } {}^L\mathbf{U}_q,$$

respectively.

The intersection

$$\mathfrak{u}_q := \mathbf{U}_q^{-,\frac{\pm}{2}} \cap \mathbf{U}_q^{+,\frac{\mp}{2}} \subset {}^L\mathbf{U}_q$$

is the small quantum group. The above diagram shows that $\mathfrak{u}_q$ identifies both with $\mathbf{Dr}(\mathfrak{u}_q^-)$ (as a Hopf algebra with an R-matrix), and with the image of $\phi : {}^{KD}\mathbf{U}_q \to {}^L\mathbf{U}_q$.

5.6. The quantum Frobenius. Let $\Lambda_{cl} \subset \Lambda$ be the sub-lattice consisting of elements $\{\lambda \,|\, q(\lambda,\mu)^2 = 1, \forall \mu \in \Lambda\}$. Note that the corresponding braided monoidal subcategory $\mathrm{Vect}_q^{\Lambda_{cl}} \subset \mathrm{Vect}_q^{\Lambda}$ is symmetric. We denote the corresponding category simply by $\mathrm{Vect}^{\Lambda_{cl}}$.

For $i \in I$ let ℓ_i be the minimal integer such that $q(\alpha_i,\alpha_i)^{\ell_i} = 1$. Then, it is easy to see that $\alpha_{i,cl} := \ell_i \cdot \alpha_i \in \Lambda_{cl}$, and $\check{\alpha}_{i,cl} := \frac{\check{\alpha}_i}{\ell_i} \in (\Lambda_{cl})^\vee$.

According to [Lu], $(\Lambda_{cl}, (\Lambda_{cl})^\vee, I, \alpha_{i,cl}, \check{\alpha}_{i,cl})$ form a root datum. We shall denote by $\mathbf{U}_{cl}$ the resulting quantum enveloping algebra. The braided monoidal category $\mathbf{U}_{cl}$-mod is in fact symmetric. Let G_{cl} denote the reductive group, corresponding to the above root datum. By *loc. cit.*, the category $\mathbf{U}_{cl}$-mod is equivalent, as a braided monoidal category, to $\mathcal{O}_{cl}$–the category $\mathcal{O}$ corresponding to G_{cl}.

In addition, we have a homomorphism of Hopf algebras (the quantum Frobenius):

$$\mathrm{Frob}_q : {}^L\mathbf{U}_q \to \mathbf{U}_{cl},$$

and the corresponding braided monoidal functor

$$\mathrm{Frob}_q^* : \mathbf{U}_{cl}\text{-mod} \to {}^L\mathbf{U}_q\text{-mod}.$$

The composition

$$\mathbf{U}_{cl}\text{-mod} \to {}^L\mathbf{U}_q\text{-mod} \to \mathfrak{u}_q\text{-mod}$$

factors as

$$\mathbf{U}_{cl}\text{-mod} \to \mathrm{Vect}^{\Lambda_{cl}} \to \mathfrak{u}_q\text{-mod},$$

where elements of $\mathrm{Vect}^{\Lambda_{cl}}$ are considered as 1-dimensional representations of $\mathfrak{u}_q$-mod with the E_i's and F_i's acting trivially.

Moreover, we have an exact sequence of Hopf algebras:

$$1 \to \mathfrak{u}_q^- \to {}^L\mathbf{U}_q^- \to \mathbf{U}_{cl}^- \to 1 \tag{5.6}$$

(see, [Lu], Theorem 35.4.2 or [AG]).

6. Factorizable sheaves attached to the quantum group

6.1. The algebras. Denote

$${}^L\Omega_q^- := \Omega_{{}^L\mathbf{U}_q^-},\ {}^L\Omega_q^+ := \Omega_{{}^L\mathbf{U}_q^+},\ {}^{KD}\Omega_q^- := \Omega_{{}^{KD}\mathbf{U}_q^-} \text{ and } {}^{KD}\Omega_q^+ := \Omega_{{}^{KD}\mathbf{U}_q^+} \tag{6.1}$$

and also

$$\Omega_{q,small}^- := \Omega_{\mathfrak{u}_q^-},\ \Omega_{q,small}^+ := \Omega_{\mathfrak{u}_q^+}. \tag{6.2}$$

We have canonical maps between factorization algebras:

$${}^L\Omega_q^- \to {}^{KD}\Omega_q^- \text{ and } {}^L\Omega_q^+ \to {}^{KD}\Omega_q^+.$$

By the previous section, the categories in (5.4) are equivalent to

$$\mathrm{FS}({}^L\Omega_q^-),\ \mathrm{FS}({}^{KD}\Omega_q^-),\ \mathrm{FS}({}^L\Omega_q^+) \text{ and } \mathrm{FS}({}^{KD}\Omega_q^+),$$

respectively, and

$$\mathrm{FS}(\Omega_{q,small}^-) \simeq \mathfrak{u}_q\text{-mod}. \tag{6.3}$$

6.2. Free factorization algebras. Assume now that $q(\alpha_i,\alpha_i)^2 \neq 1$ for any $i \in I$. Denote

$${}^{fr}\Omega_q^- := \Omega_{{}^{fr}\mathbf{U}_q^-},\ {}^{fr}\Omega_q^+ := \Omega_{{}^{fr}\mathbf{U}_q^+},\ {}^{cofr}\Omega_q^- := \Omega_{{}^{cofr}\mathbf{U}_q^-},\ {}^{cofr}\Omega_q^+ := \Omega_{{}^{cofr}\mathbf{U}_q^+}. \tag{6.4}$$

We obtain the following isomorphisms:

$${}^L\Omega_q^- \simeq \mathbb{D}({}^{KD}\Omega_q^+),\ {}^{KD}\Omega_q^- \simeq \mathbb{D}({}^L\Omega_q^+),\ {}^{fr}\Omega_q^- \simeq \mathbb{D}({}^{cofr}\Omega_q^+),\ {}^{cofr}\Omega_q^- \simeq \mathbb{D}({}^{fr}\Omega_q^+)$$

and

$$\Omega_{q,small}^- \simeq \mathbb{D}(\Omega_{q,small}^+).$$

We have canonical maps

$${}^{fr}\Omega_q^- \twoheadrightarrow {}^{KD}\Omega_q^- \twoheadrightarrow \Omega_{q,small}^- \hookrightarrow {}^L\Omega_q^- \hookrightarrow^{cofr} \Omega_q^-. \tag{6.5}$$

The Verdier dual of this sequence is

$${}^{fr}\Omega_q^+ \twoheadrightarrow {}^{KD}\Omega_q^+ \twoheadrightarrow \Omega_{q,small}^+ \hookrightarrow {}^L\Omega_q^+ \hookrightarrow^{cofr} \Omega_q^+.$$

In what follows we will describe the above factorization algebras more explicitly.

6.3. Change of gerbe. Recall that the factorization algebras of (6.5) were in the category of perverse sheaves twisted by the gerbes $\mathcal{P}_{q,can}$. Consider a different q-twisted $\check{T}$-gerbe $\mathcal{P}_{q,can'}$ on X, where we put

$$\mathcal{P}^{\lambda}_{X,q,can'} := \omega_X^{log(q(\lambda,\lambda))-log(q(\lambda,2\rho))},$$

where $2\rho \in \Lambda$ is the sum of positive roots. The data of c^{λ_1,λ_2} for $\mathcal{P}_{q,can'}$ follows from that of $\mathcal{P}_{q,can'}$ since the function $\lambda \mapsto q(\lambda, 2\rho) : \Lambda \to \mathbb{C}^*$ is linear.

Let us assume now that the $\mathbb{C}^*$-gerbes $\omega_X^{q(\lambda,2\rho)}$ have been trivialized in a way compatible with isomorphisms

$$\omega_X^{q(\lambda_1+\lambda_2,2\rho)} \simeq \omega_X^{q(\lambda_1,2\rho)} \otimes \omega_X^{q(\lambda_2,2\rho)}.$$

For example, such a trivialization arises from a trivialization of ω_X, and for $X = \mathbb{A}_1$ there is a canonical choice for one.

This data defines an equivalence of q-twisted $\check{T}$-gerbes $\mathcal{P}_{q,can'} = \mathcal{P}_{q,can}$. Thus from now on, we will think of the factorization algebras in (6.5) as being in $\mathcal{P}_{q,can'}$-twisted perverse sheaves.

Remark. The above replacement of the gerbe has no significance for $X = \mathbb{A}^1$, but would have one if we worked with arbitrary curves. The necessity of the replacement can be explained as follows:

In order for the construction $A \mapsto \Omega_A$ to be defined for any X, we need to consider $\mathrm{Vect}^{\Lambda_q}$ as a ribbon category, and we need that the square of the antipode on A be equal to the balancing on $\mathrm{Vect}^{\Lambda_q}$. The modification of the gerbe effects the required modification of the balancing on $\mathrm{Vect}^{\Lambda_q}$.

6.4. Description on the open part. A crucial property of $\mathcal{P}_{q,can'}$ is that the $\mathbb{C}^*$-gerbe $\mathcal{P}^{\lambda}_{X,q,can'}$ is canonically trivial for $\lambda = -\alpha_i$ for $i \in I$. Indeed, we have:

$$q(\lambda, \lambda) = q(\lambda, 2\rho) \text{ for } \lambda = w(\rho) - \rho,\ w \in W,$$

and $-\alpha_i = s_i(\rho) - \rho$.

For $\lambda \in \Lambda^{neg}$ let $\overset{\circ}{X}{}^{\lambda}$ be the open subset of X^{λ} corresponding to "multiplicity-free" divisors i.e., to points of the form $\Sigma\lambda_k \cdot x_k$ with $x_k \neq x_{k'}$ and each λ_k of the form $-\alpha_i$ for some $i \in I$.

By the above, the gerbe $\mathcal{P}_{X^{\lambda},q,can'}$, restricted to $\overset{\circ}{X}{}^{\lambda}$ is canonically trivial. Hence, we can think of $\mathcal{P}_{X^{\lambda},q,can'}$-twisted sheaves on $\overset{\circ}{X}{}^{\lambda}$ as ordinary sheaves.

Let ${}^{?}\overset{\circ}{\Omega}{}^{?}_{?}$ denote the restriction of any of the twisted perverse sheaves appearing in (6.5) to $\overset{\circ}{X}{}^{\lambda}$. By the above, we can think of it as ordinary perverse sheaves.

Proposition 6.5. *The maps*

$${}^{fr}\overset{\circ}{\Omega}{}^{-,\lambda}_{q} \twoheadrightarrow {}^{KD}\overset{\circ}{\Omega}{}^{-,\lambda}_{q} \twoheadrightarrow \overset{\circ}{\Omega}{}^{-,\lambda}_{q,small} \hookrightarrow {}^{L}\overset{\circ}{\Omega}{}^{-,\lambda}_{q} \hookrightarrow^{cofr} \overset{\circ}{\Omega}{}^{-,\lambda}_{q}$$

are isomorphisms. The corresponding perverse sheaf is canonically the sign local system on $\overset{\circ}{X}{}^{\lambda}$.

Proof. The assertion follows from the fact that for all of the algebras $\mathbf{U}^-$ involved $(\mathrm{Tor}_{\mathbf{U}^-}(\mathbb{C}, \mathbb{C}))^{-\alpha_i}$ is 1-dimensional and is concentrated in the cohomological degree -1. □

We will denote the resulting local system simply by $\overset{\circ}{\Omega}{}^{-,\lambda}_{q}$.

6.6. Description of extensions. Let j^λ denote the open embedding $\overset{\circ}{X}{}^\lambda \to X^\lambda$. We shall denote by $j^\lambda_?$, $? =!, *, !*$ the corresponding functors on the category of $\mathcal{P}_{X^\lambda,q,can'}$-twisted perverse sheaves. (Note that although the twisting was trivial on $\overset{\circ}{X}{}^\lambda$ and can be non-canonically trivialized over the entire X^λ, each of the above extension functors in the twisted and the non-twisted contexts is different.)

Proposition 6.7. *We have:*

$$^{fr}\Omega_q^{-,\lambda} \simeq j^\lambda_!(\overset{\circ}{\Omega}{}_q^{-,\lambda}) \text{ and } {}^{cofr}\Omega_q^{-,\lambda} \simeq j^\lambda_*(\overset{\circ}{\Omega}{}_q^{-}),$$

compatibly with factorization isomorphisms. The map

$$^{fr}\Omega_q^{-,\lambda} \to {}^{cofr}\Omega_q^{-,\lambda}$$

corresponds to the canonical map $j^\lambda_! \to j^\lambda_*$.

Proof. It is enough to prove the first isomorphism, since the second one would follow by Verdier duality and replacing $-$ by $+$.

Our assertion is equivalent to

$$\left(\mathrm{Tor}_{^{fr}\mathbf{U}_q^-}(\mathbb{C},\mathbb{C})\right)^\lambda = 0$$

for λ being not one of the negative simple roots. However, the latter follows from the fact that $^{fr}\mathbf{U}_q^-$ is the free associative algebra with the generators in the specified degrees.

The assertion about factorization follows from the corresponding property over $\overset{\circ}{X}{}^\lambda$, and likewise the assertion about the map between Ω's.

□

Corollary 6.8. *We have:*

$$\Omega^{-,\lambda}_{q,small} \simeq j^\lambda_{!*}(\overset{\circ}{\Omega}{}_q^{-,\lambda})$$

compatibly with factorizations.

Proof. This follows by combining the previous proposition with (6.5). □

Combining the above corollary with (6.3), we reprove the main result of [BFS].

6.9. Description of $^L\Omega_q^-$ and $^{KD}\Omega_q^-$. By (6.5), as perverse sheaves, $^L\Omega_q^{-,\lambda}$ injects into $j^\lambda_*(\overset{\circ}{\Omega}{}_q^{-,\lambda})$ and $^{KD}\Omega_q^{-,\lambda}$ receives a surjective map from $j^\lambda_!(\overset{\circ}{\Omega}{}_q^{-,\lambda})$.

Let Δ^λ denote the main diagonal on X^λ. Let j'^λ denote the open embedding of its complement. By factorization and induction on $|\lambda|$, we can assume that we know the restrictions of $^L\Omega_q^{-,\lambda}$ and $^{KD}\Omega_q^{-,\lambda}$ to $X^\lambda - \Delta^\lambda$, and we can assume that $\lambda \neq -\alpha_i$, $i \in I$.

Proposition 6.10.
(1) *The perverse sheaf $^L\Omega_q^{-,\lambda}$ admits the following description in terms of its restriction to $X^\lambda - \Delta^\lambda$, i.e., $j'^*({}^L\Omega_q^{-,\lambda})$*

$$\begin{cases} \text{for } \lambda \neq w(\rho)-\rho, w \in W, & ^L\Omega_q^{-,\lambda} \simeq j'_*(j'^*({}^L\Omega_q^{-,\lambda})), \\ \text{for } \lambda = w(\rho)-\rho, \ell(w) \geq 3, & ^L\Omega_q^{-,\lambda} \simeq H^0\left(j'_*(j'^*({}^L\Omega_q^{-,\lambda}))\right), \\ \text{for } \lambda = w(\rho)-\rho, \ell(w) = 2, & ^L\Omega_q^{-,\lambda} \simeq j'_{!*}(j'^*({}^L\Omega_q^{-,\lambda})). \end{cases}$$

(2) *The perverse sheaf* ${}^{KD}\Omega_q^{-,\lambda}$ *admits the following description in terms of its restriction to* $X^\lambda - \Delta^\lambda$*, i.e.,* $j'^*({}^{KD}\Omega_q^{-,\lambda})$

$$\begin{cases} \text{for } \lambda \neq w(\rho)-\rho, w \in W, & {}^{KD}\Omega_q^{-,\lambda} \simeq j'_!(j'^*({}^{L}\Omega_q^{-,\lambda})), \\ \text{for } \lambda = w(\rho)-\rho, \ell(w) > 2, & {}^{KD}\Omega_q^{-,\lambda} \simeq H^0\left(j'_!(j'^*({}^{KD}\Omega_q^{-,\lambda}))\right), \\ \text{for } \lambda = w(\rho)-\rho, \ell(w) = 2, & {}^{KD}\Omega_q^{-,\lambda} \simeq j'_{!*}(j'^*({}^{KD}\Omega_q^{-,\lambda})). \end{cases}$$

Proof. Again, by Verdier duality, it suffices to prove statement (2). By Theorem-Construction 4.6(2), the *-restriction of ${}^{KD}\Omega_q^{-,\lambda}$ to Δ^λ is the constant sheaf tensored with $\mathrm{Tor}_{{}^{KD}\mathbf{U}_q^-}(\mathbb{C},\mathbb{C})$, and the latter satisfies

$$\begin{cases} 0 \text{ for } \lambda \neq w(\rho)-\rho, w \in W, \\ \text{is concentrated in cohomlogical degees } < -2 \text{ for } \lambda = w(\rho)-\rho, \ell(w) > 2, \\ \text{is concentrated in cohomlogical degee } -2 \text{ for } \lambda = w(\rho)-\rho, \ell(w) = 2, \end{cases}$$

since the Kac-De Concini algebra has the same homology as the classical $U(\mathfrak{n}^-)$.

This implies the assertion regarding ${}^{KD}\Omega_q^{-,\lambda}$ except in the case $\lambda = w(\rho)-\rho$, $\ell(w) = 2$. In the latter case, we have to show that the !-restriction of ${}^{KD}\Omega_q^{-,\lambda}$ to Δ^λ is concentrated in the perverse cohomological degrees ≥ 1. Again, by Verdier duality, it suffices to show that the *-restriction of ${}^{L}\Omega_q^{-,\lambda}$ to Δ^λ is concentrated in the perverse cohomological degrees ≤ -1, i.e., that $\left(\mathrm{Tor}_{{}^{KD}\mathbf{U}_q^-}(\mathbb{C},\mathbb{C})\right)^\lambda$ is concentrated in degrees ≤ -2.

However, $\mathrm{Tor}^1_{{}^{KD}\mathbf{U}_q^-}(\mathbb{C},\mathbb{C})$ is spanned by elements whose weights are proportional to the negative simple roots, implying our assertion.

□

7. Geometric incarnation of the quantum Frobenius

7.1. For $\lambda \in \Lambda_{cl}$ let us denote by $X^{\lambda_{cl}}$ the corresponding space taken with respect to the lattice Λ_{cl}, and by X^λ the corresponding space taken with respect to the lattice Λ. We have a natural closed embedding

$$\Delta^\lambda_{\mathrm{Frob}} : X^{\lambda_{cl}} \to X^\lambda.$$

Let

$$\Omega^-_{cl},\ \Omega^+_{cl},\ \Omega^{-,\vee}_{cl} \simeq \mathbb{D}(\Omega^-_{cl}),\ \Omega^{+,\vee}_{cl} \simeq \mathbb{D}(\Omega^+_{cl})$$

be the factorization algebras corresponding to the lattice Λ_{cl} and the "classical" quantum group $\mathbf{U}_{cl}$ with the sub-algebras $\mathbf{U}^-_{cl}$ and $\mathbf{U}^+_{cl}$. These are $\mathcal{P}_{X^{\lambda_{cl}},q_{cl},can}$-twisted perverse sheaves on the spaces $X^{\lambda_{cl}}$, $\lambda \in \Lambda_{cl}$.

Note that the corresponding q_{cl}-twisted $\check{T}_{cl}$-gerbe $\mathcal{P}_{q_{cl},can}$ has the property that the identifications (4.1) extend to identifications of gerbes over the entire $X^{\lambda_{1cl}} \times X^{\lambda_{2cl}}$. Thus, we can speak about factorization algebras that are commutative or co-commutative as in the non-twisted case.

Lemma 7.2. *The factorization algebras* Ω^-_{cl} *and* Ω^+_{cl} *are co-commutative and the factorization algebras* $\Omega^{-,\vee}_{cl}$ *and* $\Omega^{+,\vee}_{cl}$ *are commutative.*

7.3. Note that the $\mathbb{C}^*$-gerbe $\mathcal{P}_{X^{\lambda_{cl}},q_{cl},can}$ on $X^{\lambda_{cl}}$ identifies with the pull-back of $\mathcal{P}_{X^\lambda,q,can}$ by means of $\Delta^\lambda_{\mathrm{Frob}}$.

Note also that for $\lambda \in \Lambda_{cl}$ and $\mu \in \Lambda$ we have natural identifications of gerbes:

$$\mathcal{P}_{X^{\lambda+\mu},q,can}|_{X^{\lambda_{cl}}\times X^\mu} \simeq \mathcal{P}_{X^{\lambda_{cl}},q,can} \boxtimes \mathcal{P}_{X^\mu,q,can}$$

and also

$$\mathcal{P}_{X^{\lambda+\mu}_{x_0},q,can}|_{X^{\lambda_{cl}}\times X^\mu} \simeq \mathcal{P}_{X^{\lambda_{cl}},q,can} \boxtimes \mathcal{P}_{X^\mu_{x_0},q,can},$$

compatible with the factorization property of the $\mathcal{P}_{X^{\lambda_{cl}},q,can}$'s.

Therefore, given a co-commutative (resp., commutative) factorization algebra Ω_{cl} on the $X^{\lambda_{cl}}$, we can speak about co-modules (resp., modules) with respect to it on the X^μ's.

The maps of Hopf algebras

$${}^L\mathbf{U}_q^- \to \mathbf{U}_{cl}^- \text{ and } {}^L\mathbf{U}_q^+ \to \mathbf{U}_{cl}^+$$

define maps

$$\left(\Delta^\lambda_{\mathrm{Frob}}\right)^* ({}^L\Omega_q^{-,\lambda}) \to \Omega_{cl}^{-,\lambda} \text{ and } \left(\Delta^\lambda_{\mathrm{Frob}}\right)^* ({}^L\Omega_q^{+,\lambda}) \to \Omega_{cl}^{+,\lambda}, \tag{7.1}$$

and hence, by duality, a map

$$\Omega_{cl}^{+,\vee,\lambda} \to \left(\Delta^\lambda_{\mathrm{Frob}}\right)^! ({}^{KD}\Omega_q^{-,\lambda}).$$

Let $add_{\lambda_{cl},\mu}$ denote the corresponding maps

$$X^{\lambda_{cl}} \times X^\mu \to X^{\lambda+\mu} \text{ and } X^{\lambda_{cl}} \times X^\mu_{x_0} \to X^{\lambda+\mu}_{x_0},$$

and let

$$\left(X^{\lambda_{cl}} \times X^\mu\right)_{disj} \subset X^{\lambda_{cl}} \times X^\mu$$

denote the corresponding open subset.

Lemma 7.4. *The map*

$$add^*_{\lambda_{cl},\mu}({}^L\Omega_q^{-,\lambda+\mu})|_{\left(X^{\lambda_{cl}}\times X^\mu\right)_{disj}} \to \Omega_{cl}^{-,\lambda} \boxtimes {}^L\Omega_q^{-,\mu}|_{\left(X^{\lambda_{cl}}\times X^\mu\right)_{disj}},$$

which results from (7.1) *and* (1.1), *extends canonically onto the entire $X^{\lambda_{cl}} \times X^\mu$. Similarly, the map*

$$\Omega_{cl}^{+,\vee,\lambda} \boxtimes {}^{KD}\Omega_q^{-,\mu}|_{\left(X^{\lambda_{cl}}\times X^\mu\right)_{disj}} \to add^!_{\lambda_{cl},\mu}({}^{KD}\Omega_q^{-,\lambda+\mu})|_{\left(X^{\lambda_{cl}}\times X^\mu\right)_{disj}}$$

extends canonically onto the entire $X^{\lambda_{cl}} \times X^\mu$.

Thus, we obtain that ${}^L\Omega_q^-$ is naturally a co-module with respect to Ω_{cl}^-, and ${}^{KD}\Omega_q^-$ is naturally a module with respect to $\Omega_{cl}^{+,\vee}$.

7.5. Recall the natural maps

$${}^{KD}\Omega_q^- \to \Omega_{q,small}^- \text{ and } \Omega_{q,small}^- \to {}^L\Omega_q^-,$$

as well as the notation regarding the bar-construction.

Proposition 7.6. *The natural map*

$$\Omega_{q,small}^- \to \mathrm{Bar}(\Omega_{cl}^-) \star {}^L\Omega_q^-$$

is a quasi-isomorphism. Likewise, the map

$$\mathrm{Bar}(\Omega_{cl}^{+,\vee}) \star {}^{KD}\Omega_q^- \to \Omega_{q,small}^-$$

is a quasi-isomorphism.

Proof. The assertion of the proposition is the short exact sequence of Hopf algebras (5.6), i.e., that

$$\mathrm{Tor}_{{}^L\mathbf{U}_q^-}(\mathbb{C},\mathbb{C}) \simeq \mathrm{Tor}_{\mathbf{U}_{cl}^-}\left(\mathbb{C}, \mathrm{Tor}_{\mathfrak{u}_q^-}(\mathbb{C},\mathbb{C})\right).$$

□

Remark. From the above proposition, we obtain that $\Omega^-_{q,small}$, as an object of the derived category, carries an action of $\mathrm{Bar}(\Omega^-_{cl})$, which is an incarnation of $\mathbf{U}^-_{cl}$, and a co-action of $\mathrm{Bar}(\Omega^{+,\vee}_{cl})$, which incarnates an action of $\mathbf{U}^+_{cl}$. We would like to be able to phrase in what way these two structures are compatible, i.e., how to speak about an action on $\Omega^-_{q,small}$ of the entire $\mathbf{U}_{cl}$. Morally, this should correspond to the outer action of G_{cl} on $\mathfrak{u}_q$ coming from the short exact sequence of Hopf algebras

$$1 \to \mathfrak{u}_q \to {}^L\mathbf{U}_q \to \mathbf{U}_{cl} \to 1.$$

7.7. Let $\mathcal{F}$ be a factorizable sheaf with respect to ${}^L\Omega^-_q$. We shall say that it is co-commutative with respect to Ω^-_{cl} if the map

$$add^*_{\lambda_{cl},\mu}(\mathcal{F}^{\lambda+\mu})|_{\left(X^{\lambda_{cl}}\times X^\mu\right)_{disj}} \to \Omega^{-,\lambda}_{cl} \boxtimes \mathcal{F}^\mu|_{\left(X^{\lambda_{cl}}\times X^\mu\right)_{disj}}$$

(which follows from (7.1) and (1.3)) has been extended to a map

$$add^*_{\lambda_{cl},\mu}(\mathcal{F}^{\lambda+\mu}) \to \Omega^{-,\lambda}_{cl} \boxtimes \mathcal{F}^\mu$$

on the entire $X^{\lambda_{cl}} \times X^\mu$.

Similarly, if $\mathcal{F}$ is a factorizable sheaf with respect to ${}^{KD}\Omega^+_q$, we shall say that it is commutative with respect to $\Omega^{+,\vee}_{cl}$ if the map

$$\Omega^{+,\vee,\lambda}_{cl} \boxtimes \mathcal{F}^\mu|_{\left(X^{\lambda_{cl}}\times X^\mu\right)_{disj}} \to add^!_{\lambda_{cl},\mu}(\mathcal{F}^{\lambda+\mu})|_{\left(X^{\lambda_{cl}}\times X^\mu\right)_{disj}}$$

has been extended to a map

$$\Omega^{+,\vee,\lambda}_{cl} \boxtimes \mathcal{F}^\mu \to add^!_{\lambda_{cl},\mu}(\mathcal{F}^{\lambda+\mu})$$

on the entire $X^{\lambda_{cl}} \times X^\mu$.

We remark that in both cases above, if $\mathcal{F}$ consists of twisted perverse sheaves, than an extension of the required map, if exists, is unique. This is no longer the case in the derived category, where such an extension is an additional structure.

Proposition 7.8.

(1) *There exists an equivalence between the category of factorizable sheaves with respect to* ${}^L\mathbf{U}^-_q$ *co-commutative with respect to* Ω^-_{cl} *and* $\mathbf{U}_q^{-,\frac{\pm}{2}}$*-mod, such that we have a commutative diagram of functors*

$$\begin{array}{ccc} \mathbf{U}_q^{-,\frac{\pm}{2}}\textit{-mod} & \longrightarrow & \mathbf{Dr}({}^L\mathbf{U}^-_q)\textit{-mod} \\ \sim\big\downarrow & & \sim\big\downarrow \\ \mathrm{FS}({}^L\Omega^-_q)_{\Omega^-_{cl}-\mathrm{cocom}} & \longrightarrow & \mathrm{FS}({}^L\Omega^-_q). \end{array}$$

(2) *The assignment* $\mathcal{F} \mapsto \mathrm{Bar}(\Omega^-_{cl}) \star \mathcal{F}$ *defines a functor*

$$\mathrm{FS}({}^L\Omega^-_q)_{\Omega^-_{cl}-\mathrm{cocom}} \to \mathrm{FS}(\Omega^-_{q,small})$$

that makes the following diagram of categories commutative:

$$\begin{array}{ccc} \mathbf{U}_q^{-,\frac{\pm}{2}}\text{-}mod & \longrightarrow & \mathfrak{u}_q\text{-}mod \\ \sim\big\downarrow & & \sim\big\downarrow \\ \mathrm{FS}({}^L\Omega_q^-)_{\Omega_{cl}^- -\mathrm{co-com}} & \longrightarrow & \mathrm{FS}(\Omega_{q,small}^-). \end{array}$$

Similarly, we have:

Proposition 7.9.
(1) *There exists an equivalence between the category of factorizable sheaves with respect to* ${}^{KD}\Omega_q^-$ *commutative with respect to* $\Omega_{cl}^{+,\vee}$ *and* $\mathbf{U}_q^{+,\frac{=}{2}}$*-mod, such that we have a commutative diagram of functors*

$$\begin{array}{ccc} \mathbf{U}_q^{+,\frac{=}{2}}\text{-}mod & \longrightarrow & \mathbf{Dr}({}^{KD}\mathbf{U}_q^-)\text{-}mod \\ \sim\big\downarrow & & \sim\big\downarrow \\ \mathrm{FS}({}^{KD}\Omega_q^-)_{\Omega_{cl}^{+,\vee} -\mathrm{com}} & \longrightarrow & \mathrm{FS}({}^{KD}\Omega_q^-). \end{array}$$

(2) *The assignment* $\mathcal{F} \mapsto \mathrm{Bar}(\Omega_{cl}^{+,\vee}) \star \mathcal{F}$ *defines a functor*

$$\mathrm{FS}({}^{KD}\Omega_q^-)_{\Omega_{cl}^{+,\vee} -\mathrm{com}} \to \mathrm{FS}(\Omega_{q,small}^-)$$

that makes the following diagram of categories commutative:

$$\begin{array}{ccc} \mathbf{U}_q^{+,\frac{=}{2}}\text{-}mod & \longrightarrow & \mathfrak{u}_q\text{-}mod \\ \sim\big\downarrow & & \sim\big\downarrow \\ \mathrm{FS}({}^{KD}\Omega_q^-)_{\Omega_{cl}^{+,\vee} -\mathrm{co-com}} & \longrightarrow & \mathrm{FS}(\Omega_{q,small}^-). \end{array}$$

Remark. Let $\mathcal{M}$ be a module over ${}^L\mathbf{U}_q$. Consider its restriction to $\mathfrak{u}_q$ and let $\mathcal{F}$ be the corresponding factorizable sheaf with respect to $\Omega_{q,small}^-$. We obtain that it carries an action of $\mathrm{Bar}(\Omega_{cl}^-)$ and a co-action of $\mathrm{Bar}(\Omega_{cl}^{+,\vee})$. We would like to be able to spell out the compatibility of these two structures, which would determine which factorizable sheaves with respect to $\Omega_{q,small}^-$ correspond to modules that came by restriction from ${}^L\mathbf{U}_q$.

References

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