NOTES ON REPRESENTATIONS OF REAL REDUCTIVE GROUPS
MATH 224, SPRING 2017

DENNIS GAITSGORY

1. Week 1, Day 1 (Tue, Jan 24)

1.1. **Action of distributions.**

1.1.1. Let K be a compact topological group. Let $C(K)$ denote the space of continuous complex-valued functions on K. Unless specified otherwise, integration over K will be with respect to the Haar measure μ_{Haar} on K, normalized so that the volume of K equals 1.

Let $\text{Meas}(K)$ be the topological dual of $C(K)$ (it is known that $\text{Meas}(K)$ can be described as *Radon measures*). Unless specified otherwise, we will regard it as equipped with the *weak* topology.

For $k \in K$ we let $\delta_k \in \text{Meas}(K)$ denote the corresponding δ-function, i.e., $\langle \delta_k, f \rangle = f(k)$.

Note that we have a canonical embedding

$$L_1(K) \hookrightarrow \text{Meas}(K), \quad f \mapsto f \cdot \mu_{\text{Haar}} \tag{1.1}$$

with dense image (with respect to the weak topology on $\text{Meas}(K)$).

Remark 1.1.2. Note that, being the topological dual of a Banach space, $\text{Meas}(K)$ has a natural norm, isn which it is a Banach space. The corresponding topology (called the norm topology) is stronger than the weak topology. The map (1.1) is an isometric embedding with respect to the corresponding norms (in particular, the image of $L_1(K)$ in $\text{Meas}(K)$ is *not* dense in the norm topology).

The above discussion is applicable to (K, μ_{Haar}) replaced by an arbitrary compact topological set equipped with a Borel measure.

1.1.3. Pushforward of measures along the multiplication map $K \times K \to K$ (i.e., the operator dual to that of pullback of functions) makes $\text{Meas}(K)$ into an associative algebra. We denote the corresponding operation by

$$f_1, f_2 \mapsto f_1 \star f_2$$

and refer to it as the convolution product. The unit for this algebra is δ_1. We have

$$\delta_{k_1} \cdot \delta_{k_2} = \delta_{k_1 \cdot k_2}.$$

This algebra structure is compatible with the embedding (1.1), i.e., it induces a structure of associative algebra on $L_1(K)$. We have

$$\mu_{\text{Haar}} \star \mu_{\text{Haar}} = \mu_{\text{Haar}}.$$

Date: September 22, 2021.

1.1.4. Let V be a finite-dimensional continuous representation of K. For $k \in K$ we let T_k the corresponding element of $\mathrm{End}(V)$.

Since for every $v \in V, v^* \in V^*$, the matrix coefficient function

$$k \mapsto \langle v^*, T_k(v)\rangle$$

is continuous, we obtain that the assignment $k \mapsto T_k$ uniquely extends to a continuous linear map

$$\mathrm{Meas}(K) \to \mathrm{End}(V), \quad f \mapsto T_f, \tag{1.2}$$

determined by the condition that $T_{\delta_k} = T_k$.

The map (1.2) is a homomorphism of associative algebras.

1.2. **Matrix coefficients.**

1.2.1. We view $C(K)$ as a representation of $K \times K$ by

$$((k_1, k_2) \cdot f)(k) = f(k_1^{-1} \cdot k \cdot k_2).$$

By adjunction, we obtain action of $K \times K$ on $\mathrm{Meas}(K)$. We have

$$(k_1, k_2) \cdot \delta_k = \delta_{k_1 \cdot k \cdot k_2^{-1}}.$$

1.2.2. For a finite-dimensional continuous representation V of K, we view $\mathrm{End}(V)$ as a representation of $K \times K$ via

$$(k_1, k_2) \cdot S = T_{k_1} \circ S \circ T_{k_2^{-1}}.$$

The action map

$$\mathrm{Act}_V : \mathrm{Meas}(K) \to \mathrm{End}(V), \quad f \mapsto T_f \tag{1.3}$$

and the matrix coefficient map

$$\mathrm{MC}_V : \mathrm{End}(V) \to C(K), \quad \mathrm{MC}_{S,V}(k) = \mathrm{Tr}(S \circ T_{k^{-1}}, V) \tag{1.4}$$

are maps of $K \times K$-representations.

1.2.3. Note that we can canonically identify $\mathrm{End}(V)$ with the dual of $\mathrm{End}(V^*)$ as $K \times K$-representations by setting

$$\langle S_1, S_2\rangle = \mathrm{Tr}(S_1 \circ S_2^*, V), \quad S_1 \in \mathrm{End}(V), \quad S_2 \in \mathrm{End}(V^*),$$

where $S \mapsto S^*$ denotes the operation of taking the dual linear operator.

In terms of this identification, the maps (1.3) and (1.4) are each other's duals.

Remark 1.2.4. Note that the map $S \mapsto S^*$ defines an isomorphism of vector spaces

$$\mathrm{End}(V) \simeq \mathrm{End}(V^*).$$

This isomorphism is compatible with the $K \times K$-actions *up to the swap of factors* in $K \times K$.

1.3. **Orthogonality formulas.**

1.3.1. Let U be a finite-dimensional continuousrepresentation of K. Note that the operator $P_U^{\mathrm{inv}} = T_{\mu_{\mathrm{Haar}}}$ is an idempotent projection onto U^K, the subspace of K-invariant vectors.

For $S \in \mathrm{End}(U)$ we have

$$\int_{k\in K} \mathrm{MC}_{S,U}(k) = \mathrm{Tr}(S \circ P_U^{\mathrm{inv}}, U) = \mathrm{Tr}(P_U^{\mathrm{inv}} \circ S \circ P_U^{\mathrm{inv}}, U^K). \tag{1.5}$$

Proposition 1.3.2. *Let V_1 and V_2 be finite-dimensional continuous irreducible representations. Then we have:*

$$\int_{k\in K} \mathrm{MC}_{S_1,V_1}(k) \cdot \mathrm{MC}_{S_2,V_2}(k) = 0$$

unless $V_2 \simeq V_1^*$ (both reps are irreducible), and for $V_1 = V$ and $V_2 = V^*$, we have

$$\int_{k\in K} \mathrm{MC}_{S_1,V}(k) \cdot \mathrm{MC}_{S_2,V^*}(k) = \frac{1}{\dim(V)} \cdot \mathrm{Tr}(S_1 \circ S_2^*, V).$$

Proof. Consider $U = V_1 \otimes V_2$, and apply (1.5) to $S = S_1 \otimes S_2$. If V_2 is non-isomorphic to V_1^*, then $U^K = 0$, and the assertion follows.

If $V_1 = V$ and $V_2 = V^*$, we identify

$$V \otimes V^* \simeq \mathrm{End}(V),$$

and under this identification, for $S_1 \in \mathrm{End}(V)$ and $S_2 \in \mathrm{End}(V^*)$, the corresponding endomorphism $S_1 \otimes S_2$ of $V \otimes V^*$ corresponds to the endomorphism of $\mathrm{End}(V)$ given by

$$S' \mapsto S_1 \circ S' \circ S_2^*. \tag{1.6}$$

By Schur's lemma, the map

$$\mathbb{C} \to \mathrm{End}(V)^K, \quad 1 \mapsto \mathrm{Id}_V$$

is an isomorphism. The corresponding projector

$$P^{\mathrm{inv}}_{\mathrm{End}(V)} : \mathrm{End}(V) \to \mathrm{End}(V)^K$$

identifies with

$$S' \mapsto \mathrm{Tr}(S', V).$$

The operator

$$S' \mapsto P^{\mathrm{inv}}_{\mathrm{End}(V)} \circ S' \circ P^{\mathrm{inv}}_{\mathrm{End}(V)}, \quad \mathrm{End}(\mathrm{End}(V)) \to \mathbb{C}$$

sends the endomorphism of $\mathrm{End}(V)$ given by (1.6) to

$$\frac{1}{\dim(V)} \cdot \mathrm{Tr}(S_1 \circ S_2^*, V).$$

This establishes the desired identity.

□

1.3.3. Note that
$$\mathrm{MC}_{S^*,V^*}(k) = \mathrm{MC}_{S,V}(k^{-1}).$$

Hence, the orthogonality relation can also be interpreted as a formula for
$$\int\limits_{k\in K} \mathrm{MC}_{S_1,V_1}(k) \cdot \mathrm{MC}_{S_2,V_2}(k^{-1}). \tag{1.7}$$

Namely, (1.7) vanishes unless $V_1 \simeq V_2$, and for $V_1 = V = V_2$, it equals
$$\frac{1}{\dim(V)} \cdot \mathrm{Tr}(S_1 \circ S_2, V).$$

1.3.4. As a corollary of Proposition 1.3.2, we obtain:

Corollary 1.3.5. *For V_1 and V_2 irreducible, the composite*
$$\mathrm{End}(V_1) \overset{\mathrm{Act}_{V_1}}{\longrightarrow} C(K) \hookrightarrow \mathrm{Meas}(K) \overset{\mathrm{MC}_{V_2}}{\longrightarrow} \mathrm{End}(V_2)$$
is zero unless $V_1 \simeq V_2$ and equals $\frac{1}{\dim(V)} \cdot \mathrm{Id}_{\mathrm{End}(V)}$ for $V_1 = V = V_2$.

1.3.6. Using an invariant Hermitian form, we also have
$$\mathrm{MC}_{S,V}(k^{-1}) = \overline{\mathrm{MC}_{S^\dagger,V}(k)}.$$
Thus, we obtain that the images of MC_V for distinct V's are pair-wise orthogonal in $L_2(K)$, and that for an individual V, the map MC_V is an isometric embedding of $\mathrm{End}(V)$ into $L_2(K)$, where we endow $\mathrm{End}(V)$ with a Hermitian structure by the formula
$$(S_1, S_2) = \frac{1}{\dim(V)} \cdot \mathrm{Tr}(S_1 \circ S_2^\dagger). \tag{1.8}$$

1.4. **Characters.**

1.4.1. Let V be a finite-dimensional continuous representation of K. Let $\chi_V \in C(K)$ be equal $\mathrm{MC}_V(\mathrm{Id}_V)$. From Proposition 1.3.2 we obtain:

Corollary 1.4.2. *Let V and W be irreducible. Then*
$$\int\limits_{k\in K} \chi_V(k) \cdot \chi_W(k^{-1}) = \int\limits_{k\in K} \chi_V(k) \cdot \chi_{W^*}(k) = \int\limits_{k\in K} \chi_V(k) \cdot \overline{\chi_W(k)}$$
equals 1 if $V \simeq W$, and vanishes otherwise.

1.4.3. For V irreducible, set $\xi_V := \dim(V) \cdot \chi_V$. From Corollary 1.3.5, we obtain:

Corollary 1.4.4. *For W irreducible, the element $T_{\xi_V} \in \mathrm{End}(W)$ equals Id_W for $W \simeq V$ and zero otherwise.*

From Corollary 1.7, we obtain:

Corollary 1.4.5. *$\xi_V \star \xi_W = 0$ unless $W \simeq V$ and $\xi_V \star \xi_V = \xi_V$.*

2. Week 1, Day 2 (Tue, Jan 26)

2.1. **Continuous representations.**

2.1.1. Let G be a topological group and V a topological vector space. We shall say that a representation of G on V is continuous if the action map
$$G \times V \to V$$
is continuous.

2.1.2. In this course we will restrict our attention to the following class of topological vector spaces: we will assume that they are Hausdorff, 2nd countable, *locally convex* and complete.

Locally convex is equivalent to saying that there is a fundamental system of neighborhoods of $0 \in V$ of the form

$$\{v \in V\,,\, \rho(v) < 1\},$$

where $\rho : V \to \mathbb{R}$ is a continuous map satisfying

$$\rho(c \cdot v) = |c| \cdot \rho(v), \quad \rho(v_1 + v_2) \leq \rho(v_1) + \rho(v_2).$$

Such maps are called semi-norms.

2.1.3. Suppose that G is locally compact. Here is a (tautological) way to reformulate the condition of continuity of action. It amounts to the combination of the following two:

(i) For every $v \in V$, a semi-norm ρ and ϵ, there exists a neighborhood $1 \in U \subset G$ such that

$$\rho(T_g(v) - v) < \epsilon \text{ for } g \in U.$$

(ii) For every compact $\Omega \subset G$ and every semi-norm ρ, there exists a semi-norm ρ' so that

$$\rho(T_g(v)) \leq \rho'(v), \text{ for all } v \in V \text{ and } g \in \Omega.$$

2.1.4. Here are some examples of continuous representations. Let G act continuously on a topological space X.

(a) Let $C(X)$ be the space of continuous functions on G. We topologize by *convergence on compact subsets*. I.e., the fundamental system of neighborhoods of 0 is given by semi-norms

$$\max_{x \in \Omega} |f(x)|,$$

where Ω is a compact subset of X.

We consider the action of G on $V = C(X)$ by

$$(T_g \cdot f)(x) = f(g^{-1} \cdot x).$$

Conditions (i) and (ii) are evident in this case.

(b) Let μ be a (positive) measure on X. I.e., consider $C_c(X)$ equipped with the sup norm, μ is a continuous functional. Assume that G acts on μ continuously: i.e., for every compact $\Omega \subset G$ there exists a scalar $c \in \mathbb{R}^+$ such that

$$g(\mu) \leq c \cdot \mu.$$

Take $V = C_c(X)$, but equip it with the topology induced by the L_p norm with respect to μ. Conditions (i) and (ii) are again evident.

(b') Take $V = L_p(X, \mu)$, i.e., the completion of the space of from the previous example in its norm. Conditions (i) and (ii) follow formally from the fact that they hold on a dense subset.

2.1.5. Let V be a topological vector space as in Sect. 2.1.2. Let X be a topological space, and let

$$f : X \to V$$

be a continuous function.

Proposition 2.1.6. *The assignment*

$$\delta_x \mapsto f(x)$$

uniquely extends to a continuous linear map

$$\mathrm{Meas}_c(X) \to V, \quad \mu \mapsto \int\limits_{\mu} f$$

where $\mathrm{Meas}_c(X) = C(X)^*$*, equipped with the weak topology.*

Proof. Note that the linear span of the elements δ_x, denoted $\mathrm{Meas}^0_c(X)$, is dense in $\mathrm{Meas}_c(X)$, in the weak topology. By the completeness of V, we need to show that the map

$$\mu \mapsto \int\limits_{\mu^0} f, \quad \mathrm{Meas}^0_c(X) \to V$$

is continuous (in the topology on $\mathrm{Meas}^0_c(X)$, induced by the weak topology on $\mathrm{Meas}_c(X)$).

This is equivalent to showing that for every semi-norm ρ on V, there exists a neighborhood $0 \in U^0 \subset \mathrm{Meas}^0_c(X)$, such that

$$\rho(\int\limits_{\mu^0} f) < 1 \text{ for } \mu^0 \in U^0.$$

However, we have

$$\rho(\int\limits_{\mu^0} f) \leq \langle |\mu^0|, \rho(f)\rangle,$$

by convexity.

Hence, we can take U^0 to correspond to the neighborhood $U \subset \mathrm{Meas}_c(X)$ given by

$$\{\mu\, , \, |\langle |\mu|, \rho(f)\rangle| < 1\}.$$

□

2.1.7. As a corollary, we obtain that for a continuous representation V of G and v, we have a well-defined linear map

$$\mu \mapsto \mu \star v : \mathrm{Meas}_c(G) \to V,$$

which is continuous in the weak topology on $\mathrm{Meas}_c(G)$, and such that $\delta_g \star v = T_g(v)$.

Moreover, it is easy to see that the resulting map

$$\mathrm{Meas}_c(G) \times V \to V, \quad (\mu, v) \mapsto \mu \star v$$

is continuous. For a fixed μ, we will denote the corresponding endomorphism of V by T_μ, and the map

$$\mathrm{Meas}_c(G) \to \mathrm{End}(V)$$

by Act_V; it is continuous at least for the weak topology on $\mathrm{End}(V)$.

2.1.8. By a Dirac sequence we will mean a family of continuous compactly supported functions f_n on G such that $f_n \cdot \mu_{\text{Haar}} \to \delta_1$ in the weak topology on $\text{Meas}_c(G)$. The latter condition follows from the following two: $\int f_n \to 1$ and $\text{supp}(f_n) \to \{1\}$.

We obtain that for a continuous representation V and $v \in V$,

$$f_n \star v \to v. \tag{2.1}$$

2.2. The notion of K-finite vector.

2.2.1. Let K be compact, and V a continuous representation of K. We let $V^{K\text{-fin}}$ denote the subspace of V equal to the union of finite-dimensional K-subrepresentations of V.

Clearly, an element $v \in V$ belongs to $V^{K\text{-fin}}$ if and only if the elements $T_k(v)$ span a finite-dimensional subspace of V.

2.2.2. For an irreducible finite-dimensional representation ρ, we let

$$V^\rho \subset V^{K\text{-fin}}$$

be the ρ-isotypic component, i.e., the union of finite-dimensional subrepresentations that are isomorphic to direct sums of copies of ρ.

Clearly, we have

$$V^\rho \simeq \rho \otimes \text{Hom}_K(\rho, V),$$

where $\text{Hom}_K(\rho, V) \subset \text{Hom}(\rho, V)$ inherits a topology from V.

We have

$$V^{K\text{-fin}} = \underset{\rho \in \text{Irrep}(K)}{\oplus} V^\rho.$$

2.2.3. We claim:

Lemma 2.2.4. *Let ρ be an irreducible finite-dimensional representation. Then for any $S \in \text{End}(\rho^*)$, the image*

$$\text{MC}_{S,\rho} \cdot \mu_{\text{Haar}} \in \text{Meas}(K)$$

acting on V belongs to V^ρ.

Proof. Follows from the fact that

$$\delta_k \star (\text{MC}_{S,\rho} \cdot \mu) = \text{MC}_{T_g \cdot S, \rho}.$$

□

Corollary 2.2.5. *For an irreducible representation ρ, the element $\xi_\rho \cdot \mu_{\text{Haar}} \in \text{Meas}(K)$ acts in any continuous representation V as an idempotent with image equal to V^ρ.*

Proof. Follows from Lemma 2.2.4 and Corollary 1.4.4. □

2.3. Peter-Weyl theorem.

2.3.1. Recall that we have an isometric embedding

$$\hat{\bigoplus_{\rho \in \mathrm{Irrep}(K)}} \mathrm{End}(\rho) \to L_2(K), \tag{2.2}$$

where each $\mathrm{End}(\rho)$ is endowed with a Hermitian structure by formula (1.8).

The Peter-Weyl theorem says:

Theorem 2.3.2.

(a) *The map is an isomorphism.*

(b) *The subspace* $\bigoplus_{\rho \in \mathrm{Irrep}(K)} \mathrm{End}(\rho) \subset L_2(K)$ *identifies with the set of K-finite vectors with respect to the action by left translations.*

2.3.3. We will first prove point (b). This consists of the following two statements:

Proposition 2.3.4. *The map* $\mathrm{MC}_\rho : \mathrm{End}(\rho) \to C(K)$ *is an isomorphism onto* $C(K)^{\rho,l}$*, where the superscript l indicates that we are considering $C(K)$ as equipped with the action by left translations.*

Proof. This is just Frobenius reciprocity:

$$\mathrm{Hom}_K(\rho, C(K)) \simeq \rho^*.$$

□

Proposition 2.3.5. *The inclusion* $C(K)^{K\text{-fin},l} \hookrightarrow L_2(K)^{K\text{-fin},l}$ *is an isomorphism.*

Proof. We will need the following lemma:

Lemma 2.3.6. *Let W be a finite-dimensional representation of K. Then* Id_W *lies in the image of* $C(K) \subset \mathrm{Meas}(K)$ *under* Act_W.

Proof. On the one hand, the closure of the image of $C(K)$ contains Id_W by (2.1). On the other hand, the said image is a linear subspace of $\mathrm{End}(W)$, and hence is closed. □

Let f be a (left) K-finite element in $L_2(K)$. By the above lemma, we can find $f_1 \in C(K)$ so that $f_1 \star f = f$ (note that the operator T_{f_1} for $V = L_2(K)$ amounts to the convolution $f_1 \star -$). However, as we shall see below (in Sect. 2.4.3), for any $f_1 \in C(K)$ and $f \in L_1(K)$, the convolution $f_1 \star f$ belongs to $C(K)$.

□

2.4. Proof of density.

2.4.1. To prove point (a) of Theorem 2.3.2, we need to show that for any $v \in L_2(K)$ and ϵ there exists $v' \in L_2(K)^{K\text{-fin},l}$ such that $||v - v'|| < \epsilon$.

We recall that an operator $T : V_1 \to V_2$ between Banach spaces is called *compact* if the closure of the image of the unit ball in V_1 is compact in V_2.

We will use the following key observation:

Theorem 2.4.2. *Let X and Y be compact metric spaces, and let $F(-,-)$ be a continuous function on $X \times Y$. Then for $\mu \in \mathrm{Meas}(X)$, the function*

$$T_{\mu,F}(y) = \int\limits_{x \in X, \mu} F(x,y)$$

is continuous. Moreover, the resulting operator

$$T_F : \mathrm{Meas}(X) \to C(Y),$$

is compact, when $\mathrm{Meas}(X)$ *is considered as equipped with its natural norm.*

Proof. We will show that $T_{\mu,F}$ is bounded and uniformly continuous with the estimates depending only on $||\mu||$. This will prove the theorem in view of the Arzelà-Ascoli theorem.

Indeed, for boundedness we note that for any $y \in Y$

$$|T_{\mu,F}(y)| \leq ||\mu|| \cdot \sup(F).$$

For uniform continuity fix an ϵ. Since F is uniformly continuous, we can find a δ such that

$$\rho(y_1, y_2) < \delta \Rightarrow |F(x, y_1) - F(x, y_2)| < \frac{\epsilon}{||\mu||} \ \forall x \in X.$$

However, this implies that

$$|T_{\mu,F}(y_1) - T_{\mu,F}(y_2)| < \epsilon.$$

□

2.4.3. We apply the above observation to $X = Y = K$, and $F(k_1, k_2) = f(k_1^{-1} \cdot k_2)$, where $f \in C(K)$ We note that the corresponding operator

$$T_F : \mathrm{Meas}(K) \to C(K)$$

equals

$$\mu \mapsto \mu \star f.$$

In particular, we obtain that the latter operator is compact. Hence, the operator

$$T^r_f : L_2(K) \to L_2(K),$$

being equal to the composition

$$L_2(K) \to \mathrm{Meas}(K) \overset{\mu \mapsto \mu \star f}{\longrightarrow} C(K) \to L_2(K),$$

is also compact.

2.4.4. For a given $v \in L_2(K)$, let us choose f to be a continuous function on K such that $||v - v \star f|| < \frac{\epsilon}{2}$; it exists by (2.1), where we regard $L_2(K)$ as equipped with the action on K by *right* translations.

We can furthermore assume that f is such that f is real-valued and satisfies $f(g) = f(g^{-1})$. In this case the operator $T^r_f : L_2(K) \to L_2(K)$ is self-adjoint.

We now quote the following theorem (which is proved in the same way as its finite-dimensional counterpart):

Theorem 2.4.5. *Let T be a compact self-adjoint operator on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ admits a direct sum decomposition*

$$\mathcal{H} \simeq \mathcal{H}^0 \oplus \widehat{\bigoplus_{\lambda}} \mathcal{H}^\lambda,$$

where $\mathcal{H}^\lambda$ is the λ-eigenspace of T. Moreover: (i) for $\lambda \neq 0$ the space $\mathcal{H}^\lambda$ is finite-dimensional; (ii) for every ϵ all but finitely many λ's satisfy $|\lambda| < \epsilon$.

We apply this theorem for $\mathcal{H} = L_2(K)$ and $T = T^r_f$. Note that since left and right multiplications commute, we obtain that each $\mathcal{H}^\lambda$ is left K-invariant. In particular, we obtain that for $\lambda \neq 0$, we have $\mathcal{H}^\lambda \subset L_2(K)^{K\text{-fin},l}$.

Decompose v as $v_0 + \underset{\lambda\neq 0}{\Sigma} v^\lambda$ with each $v^\lambda \in \mathcal{H}^\lambda$. Note that

$$v - v \star f = v_0 + \underset{\lambda\neq 0}{\Sigma} (1-\lambda)\cdot v^\lambda.$$

Hence,

$$||v - v\star f|| < \frac{\epsilon}{2} \Rightarrow ||v - \underset{|\lambda|>\frac{\epsilon}{2}}{\Sigma} v^\lambda|| < \epsilon.$$

Thus, the (finite) sum $\underset{|\lambda|>\frac{\epsilon}{2}}{\Sigma} v^\lambda$ gives the desired approximation to v by left K-finite vectors.

This finishes the proof of Theorem 2.3.2(a).

2.5. **Density in other representations.** We will now prove:

Theorem 2.5.1. *For any continuous representation V of K, the subset $V^{K\text{-fin}} \subset V$ is dense.*

Proof. Let v be a vector in V. Choose a Dirac sequence pf continuous functions f_n so that

$$f_n \star v \to v.$$

By Peter-Weyl, we can choose K-finite functions f'_n such that $||f_n - f'_n||_{L_2} < \frac{1}{n}$. Then the sequence f'_n also converges to δ_1 in the weak topology on $\mathrm{Meas}(K)$. Hence,

$$f'_n \star v \to v.$$

However, each $f'_n \star v$ is K-finite by Lemma 2.2.4.

□

Corollary 2.5.2. *Matrix coefficients of finite-dimensional representations are dense in the max norm in $C(K)$.*

2.5.3. As another corollary we obtain:

Corollary 2.5.4. *Any irreducible continuous representation of K is finite-dimensional.*

Proof. Let V be an irreducible representation. Recall that in the infinite-dimensional setting, "irreducible" means that V does not contain *closed* proper non-zero K-invariant subspaces.

Since $V^{K\text{-fin}}$ is dense in V, it is non-zero. Let V' be an irreducible suprepresentation of $V^{K\text{-fin}}$; it is tautologically finite-dimensional. Hence, V' is closed in V. Hence $V' = V$.

□

2.6. **Plancherel's formula.**

2.6.1. Recall that the Peter-Weyl theorem says that the map

$$\underset{V\in \mathrm{Irrep}(K)}{\hat{\oplus}} \mathrm{End}(V) \to L_2(K)$$

(where $\hat{\oplus}$ means the Hilbert space direct sum) is an isomorphism. Since we already know that it is an isometric embedding, the assertion amounts to the fact that the above map has a dense image.

It follows from Corollary 1.3.5 that the inverse map

$$L_2(K) \to \underset{V\in \mathrm{Irrep}(K)}{\hat{\oplus}} \mathrm{End}(V),$$

sends

$$f \mapsto \hat{\oplus} \dim(V) \cdot T_f. \tag{2.3}$$

In particular, we have:

Theorem 2.6.2. *For $f_1, f_2 \in L_2(K)$, we have*

$$\int\limits_{k \in K} f_1(k) \cdot \overline{f_2(k)} = \sum_{V \in \mathrm{Irrep}(K)} \dim(V) \cdot \mathrm{Tr}(T_{f_1} \circ T_{f_2}^{\dagger}, V), \tag{2.4}$$

where the series on the right is absolutely convergent.

The latter identity is one form Plancherel's formula for compact groups.

2.6.3. Here is another identity (also sometimes referred to as Plancherel's formula):

Theorem 2.6.4. *For a smooth function f on K, we have*

$$f(1) = \sum_{\rho \in \mathrm{Irrep}(K)} \dim(\rho) \cdot \mathrm{Tr}(T_f, \rho), \tag{2.5}$$

where the series on the right is an absolutely convergent.

In order to prove this assertion, one need to recall the notion of *trace class endomorphism* of a Hilbert space. By definition, a *compact* endomorphism $T : \mathcal{H} \to \mathcal{H}$ is said to be *of trace class* if for some choice of orthonormal bases e_i, the series

$$(|T|(e_i), e_i)$$

is absolutely convergent, where $|T| = (T \cdot T^{\dagger})^{\frac{1}{2}}$.

If this happens, one (easily) shows that the above condition and the resulting sum

$$\mathrm{Tr}(T, \mathcal{H}) := (T(e_i), e_i)$$

does not depend on the choice of a basis. Moreover, if $\mathcal{H}$ is represented as $\hat{\bigoplus\limits_i} \mathcal{H}_i$, then

$$\mathrm{Tr}(T, \mathcal{H}) = \sum_i \mathrm{Tr}(T_i, \mathcal{H}_i), \tag{2.6}$$

where T_i is the orthogonal projection of T onto $\mathcal{H}_i$ (part of the statement is that each T_i is trace class and the above series is absolutely convergent).

Let us be in the situation of Theorem 2.4.2 with $X = Y$. Let X be equipped with a Borel measure μ; consider the corresponding embedding

$$C(K) \overset{f \mapsto f \cdot \mu}{\longrightarrow} L_2(K, \mu)$$

and its dual

$$L_2(K, \mu) \hookrightarrow \mathrm{Meas}(K, \mu).$$

For a continuous function F on $X \times X$ consider the corresponding composite (also abusively denoted T_F):

$$L_2(K, \mu) \hookrightarrow \mathrm{Meas}(K) \overset{T_F}{\longrightarrow} C(K) \hookrightarrow L_2(K, \mu).$$

Proposition 2.6.5. *Assume that X is a smooth manifold, and that μ and F are smooth. Under the above circumstances, T_F is trace class and*

$$\mathrm{Tr}(T_F, L_2(X, \mu)) = \int\limits_{x \in X, \mu} F(x, x).$$

Proof of Theorem 2.6.4. We apply Proposition 2.6.5 to $X = K$ and $F(k_1, k_2) = f(k_1 \cdot k_2^{-1})$, so that $T_F = T^l_f$. By Proposition 2.6.5, the operator T^l_f is trace class and its trace equals $f(1)$. However, by (2.3) and (2.6), the RHS in (2.5) also equals $\mathrm{Tr}(T^l_f, L_2(K))$.

□

3. Week 2, Day 1 (Tue, Jan 31)

3.1. **Some differential calculus.**

3.1.1. Let X be a differentiable manifold, and let V be a topological vector space as in Sect. 2.1.2. We shall say that a function

$$F : X \to V$$

is differentiable at x, of there exists a linear map $dF_x : T_xX \to V$ such that for every vector $\xi_x \in T_xX$ and some/any differentiable path

$$\gamma : (-1, 1) \to X, \quad \gamma(0) = x, \quad d\gamma_0 = \xi_x$$

we have

$$\frac{F(\gamma(t)) - F(x)}{t} \to dF_x(\xi_x).$$

Let $\xi^1, ..., \xi^n$ be a vector fields on X such that $\xi^1_x, ..., \xi^n_x$ form a basis of T_xX for every $x \in X$ (such a frame of vector field exists locally on X). We shall say that F is continuously differentiable if the functions

$$\mathrm{Lie}_{\xi^i}(F); \quad x \mapsto dF_x(\xi^i_x), \quad i = 1, ..., n$$

are continuous. This definition is easily seen to be independent of the choice of the frame.

We let $C^1(X, V) \subset C(X, V)$ be the subspace of continuously differentiable functions. We define the spaces $C^m(X, V)$ inductively, by setting

$$C^m(X, V) = \{F \in C^{m-1}(X, V)\,,\, \mathrm{Lie}_{\xi^i}(F) \in C^{m-1}(X, V)\, \forall i = 1, ..., n\}.$$

Set

$$C^\infty(X, V) = \bigcap_m C^m(X, V).$$

3.1.2. We recall the notion of the ring of differential operators on X. Namely, we define

$$\mathrm{Diff}^{\leq n}(X) \subset \mathrm{End}_{\mathbb{C}}(C^\infty(X))$$

inductively as follows:

(i) $\mathrm{Diff}^{\leq n}(X) = 0$ for $n < 0$;
(ii) $T \in \mathrm{Diff}^{\leq n}(X)$ if and only if $[T, f] \in \mathrm{Diff}^{\leq n-1}(X) = 0$ for every $f \in C^\infty(X)$, viewed as an endomorphism $g \mapsto f \cdot g$.

Clearly,

$$\mathrm{Diff}(X) := \bigcup_n \mathrm{Diff}^{\leq n}(X)$$

is a subring of $\mathrm{End}_{\mathbb{C}}(C^\infty(X))$. Each $\mathrm{Diff}^n(X)$ is a bimodule over $C^\infty(X)$.

Thus, $\mathrm{Diff}^{\leq 0}(X) = C^\infty(X)$. One can show that $\mathrm{Diff}^{\leq 1}(X) = C^\infty(X) \oplus \mathrm{Vect}(X)$. Moreover, for a choice of a frame $\xi_1, ..., \xi_n$ as above, every differential operator can be *uniquely* written as

$$\sum_{i_1,...,i_n \geq 0} f_{i_1,...,i_n} \cdot \xi_1^{i_1} \cdot ... \cdot \xi_n^{i_n}, \quad f_{i_1,...,i_n} \in C^\infty(X).$$

Canonically

$$\mathrm{Diff}^{\leq n}(X)/\,\mathrm{Diff}^{\leq n-1}(X) \simeq \mathrm{Sym}^n_{C^\infty(X)}(\mathrm{Vect}(X)).$$

3.1.3. In what follows,for $x \in X$, it will be convenient to consider the vector space

$$\mathrm{Distr}_x(X) := \mathbb{C} \underset{C^\infty(X)}{\otimes} \mathrm{Diff}(X),$$

which is the union of its subspaces $\mathrm{Distr}_x^{\leq n}(X) := \mathbb{C} \underset{C^\infty(X)}{\otimes} \mathrm{Diff}^n(X)$.

Note that for $f \in C^\infty(X)$ and $\mathfrak{d} \in \mathrm{Diff}(X)$, the value of $\mathfrak{d}(f)$ at x only depends on the image of $\mathfrak{d}$ in $\mathbb{C} \underset{C^\infty(X)}{\otimes} \mathrm{Diff}(X)$.

Let Germ_x^n be the quotient ring of $C^\infty(X)$ by the ideal of functions that vanish to the order n at x (so that evaluation at x defines an isomorphism $\mathrm{Germ}_x^0 \simeq \mathbb{C}$).

The map

$$\mathfrak{d}, f \mapsto (\mathfrak{d}(f))(x)$$

defines a pairing

$$\mathrm{Distr}_x^{\leq n}(X) \otimes \mathrm{Germ}_x^n \to \mathbb{C}. \tag{3.1}$$

It is easy to see that (3.1) is a perfect pairing, i.e., identifies $\mathrm{Distr}_x^{\leq n}(X)$ as the dual of Germ_x^n.

Note that for a smooth map

$$F : X \to Y, \quad F(x) = y,$$

we have a canonically defined maps

$$\mathrm{Diff}_x^n(X) \to \mathrm{Diff}_y^n(Y),\ \mathrm{Distr}_x(X) \to \mathrm{Diff}_y(Y), \tag{3.2}$$

where the former map is the dual of the pullback map $F^* : \mathrm{Germ}_y^n \to \mathrm{Germ}_x^n$.

Consider the vector space $C^n(X)$ equipped with its natural topology. Consider its topological dual $C^n(X)^*$. The Taylor expansion at x defines a map

$$C^n(X) \to \mathrm{Germ}_x^n.$$

Hence, we obtain a naturally defined map

$$\mathrm{Distr}_x^{\leq n}(X) \hookrightarrow C^n(X)^*. \tag{3.3}$$

By definition, $C^\infty(X) := \underset{n}{\cap} C^n(X)$, and it is equipped with the inverse image topology. Set

$$\mathrm{Distr}_c(X) := C^\infty(X)^*.$$

From (3.3), we obtain an embedding

$$\mathrm{Distr}_x(X) \hookrightarrow \mathrm{Distr}_c(X). \tag{3.4}$$

3.2. **Differentiable vectors.**

3.2.1. Let G be a Lie group, and let V be a continuous representation. We shall say that a vector $v \in V$ is differentiable if the function

$$F^v : G \to V, , \quad F^v(g) := T_g(v) \tag{3.5}$$

is differentiable at 1.

Lemma 3.2.2. *If $v \in V$ is differentiable, then the function* (3.5) *is continuously differentiable.*

Proof. For $\xi \in \mathfrak{g}$, let ξ^r denote the corresponding *left-invariant* vector field. Note that such fields span the tangent space at every point of G. Hence, it is enough to show that the functions $\mathrm{Lie}_{\xi^r}(F)$ are continuous on G. However, it is easy to see that

$$\mathrm{Lie}_{\xi^r}(F^v) = F^{v'}, \quad v' := dF^v_1(\xi).$$

□

Let $V^1 \subset V$ be the subspace consisting of differentiable vectors. By construction, we have a well defined map

$$\mathfrak{g} \otimes V^1 \to V, \quad \xi, v \mapsto dF^v_1(\xi).$$

For $\xi \in \mathfrak{g}$ denote the corresponding map $V^1 \to V$ by T_ξ.

Define V^n inductively by

$$V^n = \{v \in V^{n-1}\,,\, T_\xi(v) \in V^{n-1}\, \forall \xi \in \mathfrak{g}\}$$

Set

$$V^\infty := \bigcap_n V^n.$$

We topologize V^n inductively as follows as follows. If $\{\rho\}$ is a system of semi-norms for V^{n-1} and ξ^i is a basis for $\mathfrak{g}$, we introduce semi-norms ρ^i on V^n by

$$\rho^i(v) = \rho(T_{\xi^i}(v)).$$

We give V^∞ the inverse limit topology.

In the same way as in Proposition 2.1.6, one shows:

Proposition 3.2.3. *For any m, n there exists a canonically defined continuous map*

$$C^n(G)^* \times V^{m+n} \to V^m, \quad \mu \mapsto T_\mu.$$

It is uniquely determined by the requirement that for $v \in V^{m+n}$ and $v^* \in (V^m)^*$ we have

$$\langle v^*, T_\mu(v)\rangle = \langle \mu, \langle v^*, F^v\rangle\rangle,$$

where the function $\langle v^*, F^v\rangle$ on G is easily seen to be in $C^n(G)$.

As a corollary, we have:

Corollary 3.2.4. *There is a canonically defined continuous map*

$$\mathrm{Distr}_c(G) \times V^\infty \to V^\infty, \quad \mu \mapsto T_\mu,$$

compatible with the convolution algebra structure on $\mathrm{Distr}_c(G)$.

3.2.5. Recall the vector space $\mathrm{Distr}_1(G) = \underset{n}{\cup} \mathrm{Distr}_1^{\leq n}(G)$. The group law on G endows $\mathrm{Distr}_1(G)$ with a structure of associative algebra via (3.2).

We can alternatively view this structure as follows: identify $\mathrm{Distr}_1(G)$ with left (or right) invariant differential operators on G. Then the associative algebra structure on $\mathrm{Diff}(G)$ induces one on $\mathrm{Distr}_1(G)$.

The assignment

$$(\xi \in \mathfrak{g}) \mapsto (f \mapsto df_1(\xi))$$

defines a map

$$\mathfrak{g} \to \mathrm{Distr}_1^{\leq 1}(G) \subset \mathrm{Distr}_1(G),$$

and it is easy to see that it respects the commutator relation. Hence, we obtain a homomorphisms of associative algebras

$$U(\mathfrak{g}) \to \mathrm{Distr}_1(G). \tag{3.6}$$

It is easy to see that (3.6) is an isomorphism. Alternatively, we can view (3.6) as an identification between $U(\mathfrak{g})$ with left (or right) invariant differential operators on G.

Combining with (3.2.3) we obtain canonical maps

$$U(\mathfrak{g})^{\leq n} \otimes V^{m+n} \to V^m, \quad \mathfrak{u} \mapsto T_{\mathfrak{u}}$$

as well as an action of $U(\mathfrak{g})$ on V^∞.

By construction, for $\xi \in \mathfrak{g} \subset U(\mathfrak{g})$, this notation is consistent with the notation T_ξ introduced earlier.

3.2.6. Here comes the arch-important smoothing construction:

Lemma 3.2.7. *Let f be an element of $C_c^k(G)$, and consider $f \circ \mu_{\mathrm{Haar}}$ as an element of $C(G)^*$, where μ_{Haar} is a left-invariant Haar measure. Then for any $v \in V$, the vector $T_{f\cdot\mu_{\mathrm{Haar}}}(v)$ belongs to V^k.*

Proof. It is easy to see that for $\xi \in \mathfrak{g}$,

$$T_\xi(T_{f\cdot\mu_{\mathrm{Haar}}}(v)) = T_{\mathrm{Lie}_\xi(f)\cdot\mu_{\mathrm{Haar}}}(v).$$

□

Corollary 3.2.8. *For $f \in C_c^\infty(G)$ and any $v \in V$, the vector $T_{f\cdot\mu_{\mathrm{Haar}}}(v) \in V$ belongs to V^∞.*

Corollary 3.2.9. *For any V, the subspace $V^\infty \subset V$ is dense.*

Proof. We can choose a Dirac sequence of *smooth* functions f_n. By (2.1), we have

$$T_{f_n}(v) \to v.$$

However, by Corollary 3.2.8, each $T_{f_n}(v)$ is smooth. □

Corollary 3.2.10. *Let V be a continuous finite-dimensional representation. Then $V^\infty = V$. Equivalently, the image of*

$$\mathrm{MC}_V : \mathrm{End}(V) \to C(G)$$

lies in $C^\infty(G)$.

Proof. A dense subspace of a finite-dimensional vector space is everything. □

3.3. Compact real algebraic groups.

3.3.1. Let G be an algebraic group over $\mathbb{R}$. We will say that G is *compact*, if $G(\mathbb{R})$, endowed with its natural structure of Lie group, is compact. Thus, we obtain a functor

$$\text{Compact algebraic groups over } \mathbb{R} \to \text{ Compact Lie groups.} \tag{3.7}$$

The functor (3.7) is *not* an equivalence of categories. Indeed, non-isomorphic compact algebraic groups over $\mathbb{R}$ can give rise to the same Lie group.

Example: Take the group of 3rd roots of unity $x^3 = 1$. The group of its real points is the same as that of the trivial algebraic group.

3.3.2. We shall say that a compact algebraic groups over $\mathbb{R}$ is *relevant* if the map

$$\pi_0(G(\mathbb{R})) \to \pi_0(G)$$

is surjective, where π_0 is the algebro-geometric group of connected components.

We will prove:

Theorem 3.3.3. *The restriction of the functor* (3.7) *to the subcategory of relevant compact algebraic groups is an equivalence from the category of relevant compact algebraic groups over $\mathbb{R}$ to that of compact Lie groups.*

3.3.4. As a first step, we will prove:

Theorem 3.3.5. *Every compact Lie group admits an injective homomorphism into $GL_n(\mathbb{R})$ for some n.*

Proof. Let K be a compact Lie group. The statement of the theorem is equivalent to fact that K admits a faithful finite-dimensional representation. Note that if K is a compact Lie group and

$$K \supset K_1 \supset K_2 ...$$

is sequence of closed subgroups with trivial intersection, then then exists an n such that $K_n = \{1\}$.

The set of isomorphism classes of irreducible representations of K is countable. Enumerate them by $\rho_1, \rho_2, ...$, etc. Set

$$V_i := \rho_1 \oplus ... \oplus \rho_i$$

and $K_i = \ker(K \to GL(V_i))$.

We claim that $\bigcap\limits_i K_i = \{1\}$. Indeed, if $k \in K$ belongs to the above intersection, then its acts trivially in every irreducible representation, and hence, by Corollary 2.5.2, multiplication by k acts as identity on $C(K)$. The latter means that $k = 1$.

Hence, by the above $K_n = \{1\}$ for some n.

□

3.3.6. To prove Theorem 3.3.3 we will need the following construction. Let Z be an affine algebraic variety over $\mathbb{R}$, and let X be a subset of $Z(\mathbb{R})$. Let I_X be the ideal of regular functions on Z that vanish on X, Let $Z' := V(I_X)$ be the corresponding algebraic subvariety in Z. Clearly, $X \subset Z'(\mathbb{R})$.

Moreover, by construction for every connected component of $Z'_0 \subset Z'$, we have

$$X \cap Z'_0(\mathbb{R}) \neq \emptyset. \tag{3.8}$$

3.3.7. Assume now that Z is acted on by a compact Lie group K (i.e., the algebra of regular functions on Z is an algebraic[1] representation of K). Assume that X is a single K-orbit.

[1] algebraic=K-finite

Proposition 3.3.8. *Under the above circumstances, the inclusion $X \subset Z'(\mathbb{R})$ is an isomorphism.*

Proof. Let x' be a point in $Z(\mathbb{R})$ that is not in X. We claim that we can find a regular function that vanishes on X and doesn't vanish at x'. Let X' be the K-orbit of x'. We claim that we can find a K-invariant regular function that takes value 0 on X and value 1 on K'. Indeed, by Stone-Weierstrass, the map

$$\mathbb{C}[Z] \to C(X \sqcup X')$$

is K-equivariant and has a dense image (indeed, embed Z in the affine space $\mathbb{R}^n$ so that $X \sqcup X'$ becomes a compact subset of $\mathbb{R}^n$). Averaging with respect to K, we obtain that the map

$$\mathbb{C}[Z]^K \to C(X \sqcup X')^K$$

also has a dense image. However, $C(X \sqcup X')^K \simeq \mathbb{C} \oplus \mathbb{C}$, so the latter map is surjective, and in particular, its image contains the element $(0, 1)$. □

3.3.9. We are now ready to prove Theorem 3.3.3:

Let now G be a real algebraic group and K a compact subgroup of $G(\mathbb{R})$. Consider the subscheme $G' \subset G$, obtained by the construction in Sect. 3.3.6. By the functoriality of this construction, we obtain that G' is an algebraic subgroup of G. By Proposition 3.3.8, the map $K \to G'(\mathbb{R})$ is an isomorphism, so that G' is a compact real algebraic group, and K is the group of its $\mathbb{R}$-points. Moreover, by (3.8), the compact real group G' is relevant. Combined with Theorem 3.3.5, this implies that that functor in Theorem 3.3.3 is essentially surjective.

Let us now show that the functor in Theorem 3.3.3 is fully faithful. Let G_1 and G_2 be relevant compact real groups, and let $\phi : G_1(\mathbb{R}) \to G_2(\mathbb{R})$ be a homomorphism. We wish to show that ϕ comes from a (unique) homomorphism of algebraic groups. Let

$$K \subset G_1(\mathbb{R}) \times G_2(\mathbb{R})$$

be the graph of ϕ. Let G'_1 be the subgroup of $G_1 \times G_2$ corresponding to K via the construction of Sect. 3.3.6. It suffices to show that the projection $\psi : G'_1 \to G_1$ is an isomorphism.

By Proposition 3.3.8, the map $G'_1(\mathbb{R}) \to G_1(\mathbb{R})$ is an isomorphism. Hence, ψ induces an isomorphism at the level of Lie algebras, and hence also of the (algebraic) connected components of the identity. Since both groups are relevant, this implies that ψ is itself an isomorphism.

□

Note that the above proof actually showed:

Corollary 3.3.10. *Let G_1 and G_2 be real algebraic groups with G_1 compact and relevant. Then the map*

$$\mathrm{Hom}_{\mathrm{AlgGrp}}(G_1, G_2) \to \mathrm{Hom}_{\mathrm{LieGrp}}(G_1(\mathbb{R}), G_2(\mathbb{R}))$$

is an isomorphism.

In particular, taking G_2 to be $GL_{n,\mathbb{C}}|_{\mathbb{R}}$ (here the operation $Z \mapsto Z|_{\mathbb{R}}$ is the operation of restriction of scalars à la Weil, i.e., the standard way of viewing a complex algebraic variety as a real one), we obtain:

Corollary 3.3.11. *For a compact relevant real algebraic group G and its complexification $G_{\mathbb{C}}$, the map*

$$\mathrm{Hom}_{\mathrm{AlgGrp}\,/\mathbb{C}}(G_{\mathbb{C}}, GL_{n,\mathbb{C}}) \to \mathrm{Hom}_{\mathrm{LieGrp}}(G(\mathbb{R}), GL_n(\mathbb{C}))$$

is a bijection.

The last corollary says that for a compact relevant real algebraic group G, the category of algebraic representations of its complexification is canonically equivalent to the category of complex representations of the underlying compact Lie group.

4. Week 2, Day 2 (Thurs, Feb 2)

4.1. **Compact Lie groups vs complex reductive algebraic groups.**

4.1.1. Recall that an algebraic group is said to be *reductive* if its unipotent radical is trivial.

Proposition 4.1.2. *Let G be a compact real algebraic group. Then its compexification $G_{\mathbb{C}}$ is reductive.*

Proof. Let $U(G)$ be the unipotent radical of G. Consider the center of $U(G)$. This is a group over $\mathbb{R}$, whose complexification is isomorphic to $\mathbb{G}_a^n$ for some $n \geq 1$; by Hilbert 90, we obtain that this isomorphism is valid also over $\mathbb{R}$. However, the group of its $\mathbb{R}$-points is then isomorphic to $\mathbb{R}^n$, which cannot be contained as a closed subgroup in a compact group.

□

4.1.3. From the above proposition we obtain that complexification defines a functor

$$\{\text{Compact real algebraic groups}\} \to \{\text{Complex reductive groups}\}.$$

We will study how close this functor is to be an equivalence.

4.2. **The polar decomposition.**

4.2.1. We will prove the following theorem:

Theorem 4.2.2. *Let G a complex algebraic group. Let K be a compact Lie subgroup of $G(\mathbb{C})$. Assume that:*
(i) *The map*

$$\mathfrak{k}_{\mathbb{C}} := \mathbb{C} \underset{\mathbb{R}}{\otimes} \mathfrak{k} \to \mathfrak{g}$$

is an isomorphism (i.e., $\mathfrak{g}$ is a complexification of $\mathfrak{k}$).
(ii) *K intersects non-trivially every connected component of $G(\mathbb{C})$;*

Then the group G carries a unique real structure σ for which $K = G(\mathbb{C})^\sigma$. Moreover: let $\mathfrak{p} \subset \mathfrak{g}$ be the subspace

$$\{\xi \in \mathfrak{g}, \, , \sigma(\xi) = -\xi\}.$$

Then the map

$$K \times \mathfrak{p} \to G(\mathbb{C}), \quad k, p \mapsto k \cdot \exp(p) \tag{4.1}$$

is a diffeomorphism.

In the situation of the theorem we denote by $P \subset G(\mathbb{C})$ the image of $\mathfrak{p} \subset \mathfrak{g}$ under the exponential map. By (4.1), this is a closed submanifold of $G(\mathbb{C})$. By construction

$$P \subset \widetilde{P} := \{g \in G(\mathbb{C}), \, , \sigma(g) = g^{-1}\}. \tag{4.2}$$

Corollary 4.2.3.
(a) *The product map defines a diffeomorphism*

$$\bigsqcup_{k \in K, k^2=1} (\{k\} \times \mathfrak{p}^k) \to \widetilde{P}.$$

(b) *For every $g \in \widetilde{P}$, we have $g^2 \in P$.*

Corollary 4.2.4. *Let G be as in Theorem 4.2.2. Then G is reductive.*

Proof. Follows from Proposition 4.1.2. □

Corollary 4.2.5. *The map $K \to G(\mathbb{C})$ is a homotopy equivalence. In particular $\pi_0(K) \simeq \pi_0(G(\mathbb{C}))$, and $\pi_1(K) \simeq \pi_1(G(\mathbb{C}))$.*

Note that for an algebraic variety Z over $\mathbb{C}$, the topological $\pi_0(Z(\mathbb{C}))$ identifies with the algebro-geometric $\pi_0(Z)$. Also, for a reductive algebraic group G one can define its algebraic $\pi_1(G)$, and for G over the field of complex numbers we have $\pi_1(G(\mathbb{C})) \simeq \pi_1(G)$.

Corollary 4.2.6. *In the situation of Theorem 4.2.2, we have*

$$Z_G(\mathbb{C}) = \mathrm{Centr}_{G(\mathbb{C})}(K) = Z_K \times (Z_G(\mathbb{C}) \cap P). \tag{4.3}$$

If G is semi-simple, $Z_G(\mathbb{C}) = Z_K$.

Proof. By the isomorphism (4.1), in order to prove (4.3), we only need to show that the inclusion $Z_G(\mathbb{C}) \subset \mathrm{Centr}_{G(\mathbb{C})}(K)$ is an equality. If $g \in \mathrm{Centr}(K)$, then its adjoint action on $\mathfrak{k}$ is trivial, and hence its adjoint action on $\mathfrak{g}$ is also trivial (by condition (i)). Therefore $g \in \mathrm{Centr}_{G(\mathbb{C})}(G_0(\mathbb{C}))$. To prove that the adjoint action of g on all of G is trivial, it is therefore enough to show that every connected component of $G(\mathbb{C})$ contains a point that commutes with g. However, every connected component contains a point of K by (ii).

Let us show that if G is semi-simple, then $Z_G(\mathbb{C}) \cap P$ is trivial. Indeed, by the above, $Z_G(\mathbb{C}) \cap P$ equals the set of Ad_K-invariants in P, and the exponential map identifies the latter with $\mathfrak{p}^K$. However, $\mathfrak{p}^K \subset \mathfrak{g}^K = \mathfrak{g}^G$, and the latter is trivial, since G is semi-simple.

□

Corollary 4.2.7. *In the situation of Theorem 4.2.2, we have*

$$\mathrm{Norm}_{G(\mathbb{C})}(K) = K \times (Z_G(\mathbb{C}) \cap P).$$

Proof. It is clear that $\mathrm{Norm}_{G(\mathbb{C})}(K) = K \cdot (\mathrm{Norm}(K) \cap P)$. However, from (4.1) it follows that $\mathrm{Norm}(K) \cap P \subset \mathrm{Centr}_{G(\mathbb{C})}(K)$, and the latter equals $Z_G(\mathbb{C})$. □

4.2.8. *Example.* Let $G = \mathbb{G}_m^n$. Consider the subgroup $K = U_1^n \subset (\mathbb{C}^\times)^n$. It obviously satisfies the assumption of Theorem 4.2.2. The corresponding real structure is

$$\sigma(g) = \overline{g}^{-1}.$$

We claim that this is the *unique* compact real form of $\mathbb{G}_m^n$.

Indeed, the quotient $(\mathbb{C}^\times)^n/U_1^n$ is isomorphic to $\mathbb{R}^n$, and hence contains no non-trivial compact subgroups. Hence, if σ' is another compact real structure, then the corresponding subgroup K' is contained in K. However, since $\dim_{\mathbb{R}}(\mathfrak{k}') = n$, we obtain that $K' = K$. Hence, $\sigma' = \sigma$.

4.3. **Proof of Theorem 4.2.2.**

4.3.1. We first consider the particular case when $G = GL(V)$, where V is a complex vector space. Assume that V is equipped with a Hermitian form. Define a real structure on $G = GL(V)$ by

$$\sigma(S) = (S^\dagger)^{-1}.$$

Then $K = U(V)$ and $\mathfrak{p} = E(V)$ is the space of Hermitian operators. The exponential map defined an isomorphism

$$E(V) \to E^+(V),$$

where the latter is the space of *positive* Hermitian operators. The statement that

$$U(V) \times E^+(V) \to GL(V, \mathbb{C})$$

is an isomorphism is well-known. The inverse map is constructed as follows:

For $S \in GL(V, \mathbb{C})$ consider $S \circ S^\dagger$. This is a positive Hermitian operator; hence it admits a well-defined square root $(S \circ S^\dagger)^{\frac{1}{2}}$. In sought-for inverse map sends S to the pair

$$(S \circ S^\dagger)^{-\frac{1}{2}} \circ S \in U(V), (S \circ S^\dagger)^{\frac{1}{2}} \in E^+(V).$$

4.3.2. Let us now return to the general case of Theorem 4.2.2.

First off, it is clear that the real structure σ is unique: indeed, condition (i) fixes what it is on $\mathfrak{g}$, and hence on G_0, and condition (ii) fixes it on the rest of G.

Let G be as in the theorem, and let $G \to GL(V)$ be a faithful representation. Choose a K-invariant Hermitian structure on V. We claim that the real structure on $GL(V)$ from Sect. 4.3.1 induces one on G.

We have to show that the operation $S \mapsto (S^\dagger)^{-1}$ preserves the image of $G(\mathbb{C})$ in $GL(V, \mathbb{C})$. By construction, it acts as identity on K, and hence preserves $\mathfrak{k}$, and hence by (i) all of $\mathfrak{g}$, and hence $G_0(\mathbb{C})$. By (ii) it preserves all of $G(\mathbb{C})$.

Denote $\widetilde{K} = G(\mathbb{C})^\sigma$. We obviously have $K \subset \widetilde{K}$, and this inclusion is an isomorphism at the level of Lie algebras, and hence on the neutral connected components (we will soon prove that it is an isomorphism also at the group level).

4.3.3. We claim that the mutually inverse maps

$$U(V) \times E^+(V) \rightleftarrows GL(V)$$

send the subsets

$$G(\mathbb{C}) \subset GL(V)$$

and

$$\widetilde{K} \times \mathfrak{p} \subset U(V) \times E^+(V)$$

to one another.

To show this, we only need to see that for $g \in G(\mathbb{C}) \cap E^+(V)$, the elements $g^t \in GL(V, \mathbb{C})$, $t \in \mathbb{R}$ and $\log(g) \in \mathrm{End}(V)$ belong to $G(\mathbb{C})$ and $\mathfrak{g}$, respectively.

If $n \in \mathbb{Z}$, then g^n clearly belongs to $G(\mathbb{C})$. However, for any $S \in GL(V, \mathbb{C})$, the Zariski closure of the elements S^n contains the elements S^t, $t \in \mathbb{R}$ and $\log(S)$. This implies that g^t and $\log(g)$ belong to $G(\mathbb{C})$, since $G(\mathbb{C})$ is Zariski-closed in $GL(V, \mathbb{C})$.

4.3.4. Thus, we have established that the map

$$\widetilde{K} \times \mathfrak{p} \to G(\mathbb{C}).$$

is a diffeomorphism.

In particular $\pi_0(\widetilde{K}) \to \pi_0(G(\mathbb{C}))$ is an isomorphism. However, by assumption, the composite map

$$\pi_0(K) \to \pi_0(\widetilde{K}) \to \pi_0(G(\mathbb{C}))$$

is surjective. Hence, the map

$$K \to \widetilde{K}$$

is a surjective at the level of π_0 components, and hence is an isomorphism.

4.3.5. *Example.* Let V be a real vector space equipped with a positive-definite scalar product. Consider the corresponding algebraic group $O(V)$. Then this is a relevant compact real form of its complexification $O(V)_{\mathbb{C}}$. Indeed, we only need to see that $O(V,\mathbb{R})$ hits every connected component of $O(V,\mathbb{C})$, but the latter is obvious.

5. Week 3, Day 1 (Tue, Feb. 7)

5.1. **Compact Lie groups vs compact Lie algebras.**

5.1.1. Recall that a Lie algebra $\mathfrak{g}$ is semi-simple if and only if the Killing form

$$\mathrm{Kil}(\xi,\eta) := \mathrm{Tr}(\mathrm{ad}_\xi \circ \mathrm{ad}_\eta, \mathfrak{g})$$

is non-degenerate. (This is insensitive to what field we are working over, as long as it is of characteristic 0.)

5.1.2. Let $\mathfrak{k}$ be a real Lie algebra. We shall say that it is *compact* if its Killing form is *negative definite*. By the above, if $\mathfrak{k}$ is compact, then it is semi-simple.

Let $\mathrm{Aut}(\mathfrak{k})$ be the real algebraic group of automorphisms of $\mathfrak{k}$. We claim:

Lemma 5.1.3.

(a) *If $\mathfrak{k}$ is compact, then the Lie group $\mathrm{Aut}(\mathfrak{k})(\mathbb{R})$ is compact.*

(b) *If K is a compact Lie group with trivial center, then its Lie algebra is compact.*

Proof. By definition, $\mathrm{Aut}(\mathfrak{k})(\mathbb{R})$ is a closed subgroup of the group $GL_n(\mathbb{R})$ of linear automorphisms of $\mathfrak{k}$ as a vector space, where $n = \dim(\mathfrak{k})$. However, it is actually contained in the orthogonal group $O_n(\mathbb{R})$ corresponding to Kil. Since the latter is negative-definite, the orthogonal group in question is compact. Hence, $\mathrm{Aut}(\mathfrak{k})(\mathbb{R})$ is compact as well, proving (a).

For (b), since K is compact, we can choose a K-invariant positive-definite scalar product on $\mathfrak{k}$. We have an embedding

$$K \to O_n,$$

Consider the corresponding maps of Lie algebras $\mathfrak{k} \hookrightarrow \mathfrak{o}_n$.

The Killing form on $\mathfrak{k}$ is obtained by restriction form the standard form on $\mathfrak{o}_n$:

$$(S_1, S_2) \mapsto \mathrm{Tr}(S_1 \circ S_2),$$

while the above form is easily see to be negative-definite (skew-symmetric matrices have imaginary eigenvalues).

□

Corollary 5.1.4. *For a compact Lie algebra $\mathfrak{k}$, the neutral (algebraic) connected component $\mathrm{Aut}(\mathfrak{k})_0$ of $\mathrm{Aut}(\mathfrak{k})$ is a relevant compact real algebraic group.*

5.1.5. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, with a chosen Cartan and Boreal subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. For vertex root α let $h_\alpha \in \mathfrak{h}$ be corresponding simple coroot element. Pick a non-zero element e_α in the corresponding root space. Let f_α be the unique element in the corresponding negative root space such that $[e_\alpha, f_\alpha] = h_\alpha$.

Then it is easy to see that there exists a unique real structure σ on $\mathfrak{g}$ such that

$$\sigma(h_\alpha) = -h_\alpha;\ \sigma(e_\alpha) = -f_\alpha;\ \sigma(f_\alpha) = -e_\alpha.$$

We claim that this real structure is compact. Indeed, it is well-known that the Killing form is positive definite on $\mathrm{Span}_{\mathbb{R}}(h_\alpha)$ and

$$\mathrm{Kil}(e_\alpha, f_\alpha) = \frac{\mathrm{Kil}(h_\alpha, h_\alpha)}{2} > 0.$$

Hence, it is negative definite on $i \cdot \mathfrak{h}$ as well as on $e_\alpha - f_\alpha$ and $i \cdot (e_\alpha + f_\alpha)$, while the rest of the scalar products are zero.

5.2. **Existence and uniqueness of the compact real form.**

5.2.1. Let σ and τ be two real structures on a complex Lie algebra. We shall that they are *compatible* if the (sesqui)-linear automorphisms σ and τ commute. This is equivalent to requiring:

(i) $\mathfrak{g}^\sigma = (\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau) \oplus (\mathfrak{g}^\sigma \cap i \cdot \mathfrak{g}^\tau)$;
(i') $\mathfrak{g}^\tau = (\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau) \oplus (i \cdot \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau)$;
(ii) $\sigma(\mathfrak{g}^\tau) = \mathfrak{g}^\tau$;
(ii') $\tau(\mathfrak{g}^\sigma) = \mathfrak{g}^\sigma$.
(iii) The *linear* automorphism $\theta = \tau \circ \sigma$ satisfies $\theta^2 = 1$.

Note that if both σ and τ are compact, then $\sigma = \tau$ (indeed, if the Killing form is negative-definite on $\mathfrak{g}^\sigma$, then it is positive-definite on $i \cdot \mathfrak{g}^\sigma$, so the intersection $i \cdot \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau$ is zero).

5.2.2. We have:

Theorem 5.2.3. *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, and let σ be a compact real structure on $\mathfrak{g}$. Let τ be some other real structure on $\mathfrak{g}$.*
(a) *There exists an element $g \in \mathrm{Aut}(\mathfrak{g})_0(\mathbb{C})$ such that $\mathrm{Ad}_g(\sigma)$ is compatible with τ.*
(b) *If σ_1 and σ_2 be two compact real structures on $\mathfrak{g}$ compatible with a real form ψ, then there exists $g \in \mathrm{Aut}(\mathfrak{g}^\psi)(\mathbb{R})_0$ such that $\mathrm{Ad}_g(\sigma_1) = \sigma_2$.*

Corollary 5.2.4. *If σ_1 and σ_2 are two compact real structures on a semi-simple Lie algebra $\mathfrak{g}$ are conjugate by an element of $\mathrm{Aut}(\mathfrak{g})_0(\mathbb{C})$.*

Proof. Set $G = \mathrm{Aut}(\mathfrak{g})$. Let $K \subset G(\mathbb{C})$ be the group of fixed points of the involution induced by σ. By Lemma 5.1.3, K is compact. We'd like to apply Theorem 4.2.2. For this we need to know that K meets every connected component of $\mathrm{Aut}(\mathfrak{g})$. This is obvious for the compact real form from Sect. 5.1.5. So we argue as follows: we first prove the theorem in this case (below); then deduce Corollary 5.2.4, thereby verifying the condition on π_0 for any given compact form. Then the proof given below applies to this compact form.

Consider the decomposition

$$G(\mathbb{C}) \simeq K \times P.$$

Recall also the subset

$$\widetilde{P} \subset G(\mathbb{C}),$$

see (4.2)

The element $\theta = \tau \circ \sigma \in G(\mathbb{C})$. We have $g \in \widetilde{P}$. Hence, by Corollary 4.2.3, $\theta^2 \in P$. Set $g = \theta^{\frac{1}{4}} \in P$. Then it is easy to see that $\sigma' = \mathrm{Ad}_g(\sigma)$ satisfies

$$\tau \circ \sigma' = \sigma' \circ \tau.$$

This proves point (a).

For point (b), we apply the above construction for $\sigma = \sigma_1$ and $\tau = \sigma_2$. Then all the elements g^t (for $t \in \mathbb{R}$) commutes with ψ, and hence belong to (the identity component of) $\mathrm{Aut}(\mathfrak{g}^\tau)(\mathbb{R})$.

□

5.2.5. We will now prove:

Theorem 5.2.6. *Let G be a complex reductive group.*

(a) *G admits a relevant compact real form, and such forms are in bijection with those of G_0/Z_{G_0}.*

(b) *Any two such real forms are conjugate by an element of $G_0(\mathbb{C})$.*

Proof. Note that if G is adjoint (i.e., maps isomorphically to $\mathrm{Aut}(\mathfrak{g})_0$), then the assertion of Theorem 5.2.6 follows from Sect. 5.1.5 and Corollary 5.2.4.

Assume that G is such that G_0 is adjoint. Choose a compact real form of G_0; let K_0 denote the corresponding compact Lie group. Set $K = \mathrm{Norm}_{G(\mathbb{C})}(K_0)$. Then $K \cdot G_0(\mathbb{C}) = G(\mathbb{C})$ and $K \cap G_0(\mathbb{C}) = K_0$, by Corollaries 5.2.4 and 4.2.7. Applying Theorem 4.2.2, we obtain that the result follows from the adjoint case.

Let G be a torus. Then G admits a unique compact real form by Sect. 4.2.8.

Let G be such that G_0 is a torus. In this case it is still true that G has a unique compact real form. Indeed, the quotient of $G(\mathbb{C})$ by the maximal compact subgroup of $G_0(\mathbb{C})$ is canonically of the form

$$\pi_0(G) \ltimes \mathbb{R}^n.$$

Let G be an arbitrary reductive group. Consider the (surjective) map

$$G \to \widetilde{G} := G/Z_{G_0} \underset{\pi_0(G)}{\times} G/(G_0)'.$$

Choose a relevant compact real structure on G/Z_{G_0} and consider the unique compact real structure on $G/(G_0)'$. Let $\widetilde{K}$ be the corresponding compact Lie subgroup of $\widetilde{G}(\mathbb{C})$. Let K be the preimage of $\widetilde{K}$ in $G(\mathbb{C})$. Applying Theorem 4.2.2, we obtain the result. (Indeed, any real structure on G induces one on G/Z_{G_0} and $G/(G_0)'$ by the canonicity of these quotients.)

□

5.3. **Polar decomposition of a real reductive group.**

5.3.1. Let G be a complex reductive group, equipped with a real form τ. We will write $G_{\mathbb{R}}$ for the corresponding algebraic group over $\mathbb{R}$ and $G(\mathbb{R})$ for its set of real points, i.e.,

$$G(\mathbb{R}) = G(\mathbb{C})^{\tau}.$$

We shall say that a relevant compact real form σ is compatible with τ if $\theta = \tau \circ \sigma = \sigma \circ \tau$ as automorphisms of G. Note that in this case θ defines an involution of $G_{\mathbb{R}}$ as a real algebraic group.

From Theorem 5.2.6 and Theorem 5.2.3 we obtain:

Corollary 5.3.2. *Let G be as above. Then G admits a relevant compact real form compatible with τ; such forms are in bijection with those on G_0/Z_{G_0}. Any two such real forms are conjugate by an element of $G(\mathbb{R})_0$.*

5.3.3. Let $\mathfrak{g}_\mathbb{R}$ be the Lie algebra of $G_\mathbb{R}$, i.e., $\mathfrak{g}_\mathbb{R} := \mathfrak{g}^\tau$. For a compatible compact real form σ, consider the subspaces

$$\mathfrak{k}^\tau = \mathfrak{g}_\mathbb{R}^\theta = (\mathfrak{g}_\mathbb{R} \cap \mathfrak{k}_\mathbb{C}) = \mathfrak{g}_\mathbb{R}^\sigma = \mathfrak{k}^\theta$$

and

$$\mathfrak{p}^\tau = \{\xi \in \mathfrak{g}_\mathbb{R}\,,\, \theta(\xi) = -\xi\} = (\mathfrak{g}_\mathbb{R} \cap \mathfrak{p}_\mathbb{C}) = \{\xi \in \mathfrak{g}_\mathbb{R}\,,\, \sigma(\xi) = -\xi\} = \{\xi \in \mathfrak{p}_\mathbb{C}\,,\, \theta(\xi) = -\xi\},$$

where $\mathfrak{g} \simeq \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_\mathbb{C}$ is the decomposition corresponding to σ.

We have:

$$\mathfrak{g}_\mathbb{R} \simeq \mathfrak{k}^\tau \oplus \mathfrak{p}^\tau.$$

If $\mathfrak{g}$ is semi-simple, then the form

$$-\operatorname{Kil}(\xi, \theta(\eta))$$

is positive-definite on $\mathfrak{g}_\mathbb{R}$.

5.3.4. Let $K = G(\mathbb{C})^\sigma$. Consider the subgroup

$$K^\tau = K^\theta = G(\mathbb{R}) \cap K = G(\mathbb{R})^\theta = G(\mathbb{R})^\sigma.$$

From Theorem 4.2.2 we obtain:

Theorem 5.3.5. *The map*

$$(k, p) \mapsto k \cdot \exp(p)$$

defines a diffeomorphism

$$K^\tau \times \mathfrak{p}^\tau \to G(\mathbb{R}).$$

5.3.6. *Example.* Let V be a real vector space, and $G_\mathbb{R} = GL(V)$. Choose a positive-definite scalar product on V, and endow $V_\mathbb{C}$ with the corresponding Hermitian structure.

The real form τ on $G := GL(V_\mathbb{C})$ is given by $S \mapsto \overline{S}$. The compatible compact form σ is given by $S \mapsto (S^\dagger)^{-1}$. We have:

$$K = U(V) \text{ and } K^\tau = O(V).$$

5.3.7. As in the complex case, we obtain:

Corollary 5.3.8. *The inclusion $K^\tau \to G(\mathbb{R})$ is a homotopy equivalence. In particular $\pi_0(K^\tau) \simeq \pi_0(G(\mathbb{R}))$, and $\pi_1(K^\tau) \simeq \pi_1(G(\mathbb{R}))$.*

Corollary 5.3.9. $\operatorname{Norm}_{G(\mathbb{R})}(K^\tau) = K^\tau \times (Z_{G(\mathbb{R})} \cap P_\mathbb{R})$.

Proof. We only need to show that $(\mathfrak{p}^\tau)^{K^\tau}$ is contained in the center of $\mathfrak{g}_\mathbb{R}$. Thus, we can assume that $\mathfrak{g}_\mathbb{R}$ is semi-simple. Let ξ be an element of $(\mathfrak{p}^\tau)^{K^\tau}$. We need to show that $[\xi, \xi'] = 0$ for any $\xi' \in \mathfrak{p}^\tau$. However, $[\xi, \xi'] \in \mathfrak{k}^\tau$, and it is sufficient to show that $\operatorname{Kil}_\mathfrak{g}([\xi, \xi'], \eta) = 0$ for any $\eta \in \mathfrak{k}^\tau$. However,

$$\operatorname{Kil}_\mathfrak{g}([\xi, \xi'], \eta) = \operatorname{Kil}_\mathfrak{g}(\xi', [\xi, \eta]),$$

while $[\xi, \eta] = 0$ by assumption.

□

6. Week 3, Day 2 (Thurs, Feb. 9)

6.1. **The maximal compact subgroup.**

6.1.1. Let $G_{\mathbb{R}}$ be a real reductive group, and let σ be a compatible relevant compact real form of $G_{\mathbb{C}}$. Consider the corresponding subgroup

$$K^\tau = K \cap G(\mathbb{R}) = G(\mathbb{R})^\sigma$$

and the subset

$$P^\tau = P \cap G(\mathbb{R}).$$

We will prove:

Theorem 6.1.2. *For any compact subgroup $K' \subset G(\mathbb{R})$, there exists an element $g \in P^\tau$ such that $\mathrm{Ad}_g(K') \subset K^\tau$.*

We will deduce this theorem from the next one.

6.1.3. Note that $G(\mathbb{R})$ acts on P^τ by the formula

$$g(p) = g \cdot p \cdot \sigma(g^{-1}).$$

The stabilizer of $1 \in P^\tau$ is by definition $G(\mathbb{R})^\sigma = K^\tau$.

We will prove:

Theorem 6.1.4. *Any compact subgroup $K' \subset G(\mathbb{R})$ has a fixed point on P^τ.*

Let us see how Theorem 6.1.4 implies Theorem 6.1.2:

Proof of Theorem 6.1.2. Let $p \in P^\tau$ be the fixed point of K'. Conjugating K' by p we can assume that K' stabilizes $1 \in P^\tau$. But this means that $K' \subset K^\tau$.

□

6.2. **Proof of Theorem 6.1.4.** To prove the theorem, we can quotient G out by Z_{G_0}, so we can assume that G is semi-simple.

We will define a certain continuous function

$$r : P^\tau \times P^\tau \to \mathbb{R}^{>0}$$

with the following properties:

(i) r is $G(\mathbb{R})$-invariant with respect to the diagonal action of $G(\mathbb{R})$ on $P^\tau \times P^\tau$;

(ii) For every fixed $p' \in P^\tau$ and sufficiently large $R \in \mathbb{R}^{>0}$ (depending on p'), the function

$$p \mapsto r(p, p') : P^\tau \to \mathbb{R}^{>0}$$

takes values $> R$ away from a compact subset.

(iii) For every fixed $p', p \in P^\tau$, the function

$$r(p^t, p') : \mathbb{R} \to \mathbb{R}^{>0}$$

is strictly convex.

6.2.1. Let us prove Theorem 6.1.4 assuming the existence of such a function.

Proof of Theorem 6.1.4. For every compact subset $\Omega \subset P^\tau$, define the function

$$\rho_\Omega : P^\tau \to \mathbb{R}^{>0}, \quad \rho_\Omega(p) = \max_{p' \in \Omega} r(p, p').$$

Lemma 6.2.2. *The function ρ_Ω has a unique point of minimum.*

Let us prove the theorem assuming this lemma. Indeed, take Ω to be the orbit of $1 \in P^\tau$ under K'. Let $p \in P^\tau$ its (unique) point of minimum. Since Ω is K'-invariant, by condition (i), the function ρ_Ω is also K'-invariant. Hence, p is K'-invariant.

□

6.2.3. *Proof of Lemma 6.2.2.* The function ρ_Ω is easily seen to be continuous. By condition (ii), for all sufficiently large $R \in \mathbb{R}^{>0}$ such that the function ρ_Ω takes values $> R$ outside a compact set. Hence, it attains a minimum.

Assume that it has two minima, p_1 and p_2. Translating by means of p_1, we can assume that $p_1 = 1$ and $p_2 = p$. Consider the function

$$\rho_\Omega(p^t), \quad \mathbb{R} \to \mathbb{R}.$$

By assumption, it has a minimum at $t = 0$ and $t = 1$. However, it follows from condition (iii) that the above function is strictly convex, which is a contradiction.

□

6.2.4. We will now construct the desired function r; we will obtain it by restriction on $G(\mathbb{C})$. Choose a $G \to GL(n)$, and choose a K-invariant Hermitian structure on $\mathbb{C}^n$, so that we are in the situation of Sect. 4.3.2. Thus, we can assume that $G = GL(n)$ and $P \simeq E^+(n)$.

Since G was semi-simple, its image in $GL(n)$ in fact belongs to $SL(n)$.

We set

$$r(S_1, S_2) = \mathrm{Tr}(S_1 \circ S_2^{-1}).$$

Property (i) is evident. For property (ii), we claim that for a fixed S_2, we have

$$r(S_1, S_2) > b \cdot ||S_1||$$

for some constant $b > 0$. Indeed, choose an orthonormal basis in which S_1 is diagonal. Then

$$r(S_1, S_2) = \sum_i a_{i,i} \cdot b_{i,i},$$

where $a_{i,i}$ are the diagonal entries of S_1 and $b_{i,i}$ are the diagonal entries of S_2^{-1}. Note that all $b_{i,i}$ are strictly positive, and

$$||S_1|| = \max\, a_{i,i}.$$

This implies the claim: take $b = \min b_{i,i}$ and take the minimum over all possible orthonormal basis (the set of such is compact).

Now, we note that on the set $SL(n, \mathbb{C})$, the sets

$$\{S\,,\, ||S|| \leq c\}$$

are compact (indeed, $SL(n, \mathbb{C})$ is a closed subset of $\mathrm{Mat}(n \times n, \mathbb{C})$.

Property (iii) follows similarly:

$$r(S_1^t, S_2) = \sum_i a_{i,i}^t \cdot b_{i,i}.$$

□

7. Week 4, Day 1 (Tue, Feb. 14)

7.1. **Elements in a reductive group/algebra.**

7.1.1. Let G be a connected reductive group. We will prove:

Theorem 7.1.2. *Every element in a reductive group G (over an algebraically closed field) can be conjugated into a given Borel.*

Proof. Let X be the flag variety (i.e., $X = G/B$), thought of as the variety of Borel subgroups. Over it we consider the bundle, denoted $\widetilde{G}$ (called the Grothendieck alteration) that attaches to a given $x \in X$ the corresponding subgroup B_x. We have the natural forgetful map

$$\pi : \widetilde{G} \to G.$$

Since G acts on X transitively, our assertion is equivalent to the fact that π is surjective. The latter will follow from the combination of the following three observations:

(i) G is reduced and irreducible (obvious);

(ii) The image of π is closed. This follows from the fact that π is proper; in fact, it's the composition of the closed embedding $\widetilde{G} \hookrightarrow X \times G$ and the projection $X \times G \to G$.

(iii) π is generically smooth. To see the latter, take the point of $\widetilde{X}$ of the form (x, ξ), where ξ is regular semi-simple. Consider the map

$$\mathfrak{b}_x \oplus \mathfrak{g} \to T_{(x,\xi)}(X),$$

where the first component is the tangent space to the fiber of $\widetilde{G} \to X$ and the second component is given by the action of G on $\widetilde{G}$. We claim that the composition

$$\mathfrak{b}_x \oplus \mathfrak{g} \to T_{(x,\xi)}(X) \overset{d\pi}{\to} T_\xi(G) \simeq \mathfrak{g}$$

is surjective. Indeed, this composition is given by the inclusion of $\mathfrak{b}_x$ into $\mathfrak{g}$ (the first component), and the map

$$\xi \mapsto \mathrm{Ad}_x(\xi) - \xi$$

(the second component). The assertion follows now from the fact that the action of $\mathrm{Ad}_x - \mathrm{Id}$ on $\mathfrak{g}/\mathfrak{b}_x$ is surjective: indeed the regularity assumption on x means that it has no eigenvalue 1 on Ad_x.

□

7.1.3. The same proof shows that every element in $\mathfrak{g}$ can be conjugated into a given Borel subalgebra.

Recall that an element of G is said to be semi-simple (resp., nilpotent) if its action on $\mathcal{O}_G$ (by left translations) is semi-simple (resp., nilpotent). Equivalently, if its action in every finite-dimensional representation is semi-simple (resp., nilpotent). The latter point of view implies that the condition of being semi-simple (resp., nilpotent) survives under homomorphisms of groups.

Corollary 7.1.4. *Every semi-simple (resp., nilpotent) element of G can be conjugated into T (resp., N).*

Proof. By Theorem 7.1.2, we can conjugate our element into B. Now the claim is that every semi-simple (resp., nilpotent) element of a solvable group can be conjugated into a given maximal torus (resp., is contained in the unipotent radical). □

Corollary 7.1.5. *Every torus on G can be conjugated into T.*

Proof. Every torus is generated (i.e., is the Zariski closure of the abstract group generated) by a semi-simple element. □

7.1.6. Let K be a maximal compact in $G(\mathbb{C})$. Recall that

$$T_K := K \cap B(\mathbb{C})$$

is a maximal compact subgroup of $T(\mathbb{C})$ for *some* Cartan $T \subset B$.

Theorem 7.1.7. *Every element of K can be conjugated into T_K.*

Proof 1. Suppose by contradiction that $k \in K$ is an element that cannot be conjugated into T_K. Consider the action of k on $K/T_K \simeq G(\mathbb{C})/B(\mathbb{C}) = X(\mathbb{C})$. We obtain that this action has no fixed points. Now, the Lefschetz fixed point theorem implies that the action of k on $H^\bullet(X(\mathbb{C}), \mathbb{R})$ has a zero trace. Now, it is easy to see that since K is connected, the traces of all elements of K on $H^\bullet(X(\mathbb{C}), \mathbb{Z})$ are equal (if two endomorphisms of a space are homotopic, they induce the same maps on cohomology). Hence, they are all equal to zero.

Let us take $k' = 1$. Then the trace in question equals $\chi(X(\mathbb{C})) = |W| \neq 0$, which is a contradiction.

Alternatively, we can take k' to be a regular element in T_K. The automorphism of $X(\mathbb{C})$ defined by such a k' preserves the orientation and has isolated fixed points (the W Borels that contain T). Hence, the Lefschetz number equals $|W|$.

□

Proof 2, due to Sherry Gong. We will emulate the proof of Theorem 7.1.2. Set $\widetilde{K}$ be the fibration over $X(\mathbb{C})$ that attaches to every x the intersection $B_x \cap K$. We have the evident map

$$\pi : \widetilde{K} \to K.$$

Since the action of K on $X(\mathbb{C})$ is transitive, it is enough to show that π is surjective. Note that this is a map between orientable compact manifolds. In the proof of Theorem 7.1.2 we saw that π is an orientation-preserving local isomorphism on a neighborhood of a regular element, and the preimage of such an element has has $|W|$ many preimages.

We now have the following general assertion (well-definedness of degree):

Lemma 7.1.8. *Let us be given a map $f : Y \to Z$ between orientable compact manifolds. Suppose that Z contains a a point z such that f is an orientation-preserving local isomorphism on a neighborhood of z. Then f is surjective.*

□

7.2. **Real tori.** Let T be a real torus; let $G_{\mathbb{C}}$ denote its complexification. We denote by $\mathfrak{t}$ the Lie algebra of T and by $\mathfrak{t}_{\mathbb{C}}$ its complexification, i.e., the Lie algebra of $T_{\mathbb{C}}$.

7.2.1. We have:

$$T_{\mathbb{C}} \simeq \mathbb{G}_m \underset{\mathbb{Z}}{\otimes} \Lambda$$

for a canonically defined lattice Λ. The real structure τ on $T_{\mathbb{C}}$ corresponds to an involution θ on Λ:

$$\tau(c \otimes \lambda) = \bar{c} \otimes \theta(\lambda),$$

so that $\tau \circ \theta$ is the unique compact real structure on $T_{\mathbb{C}}$.

Write $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$ as $U(1) \times \mathbb{R}^{>0,\times}$. Hence,

$$T(\mathbb{R}) = (U(1) \underset{\mathbb{Z}}{\otimes} \Lambda)^\tau \times (\mathbb{R}^{>0,\times} \underset{\mathbb{Z}}{\otimes} \Lambda)^\tau.$$

Note that τ acts on $U(1) \underset{\mathbb{Z}}{\otimes} \Lambda$ by

$$\tau(c \otimes \lambda) = c^{-1} \otimes \theta(\lambda)$$

and on $(\mathbb{R}^{>0,\times} \underset{\mathbb{Z}}{\otimes} \Lambda)^\tau$ by

$$\tau(c \otimes \lambda) = c \otimes \theta(\lambda).$$

The exponential map identifies $\mathbb{R}^{>0,\times}$ with the additive group $\mathbb{R}$, so that

$$\mathbb{R}^{>0,\times} \underset{\mathbb{Z}}{\otimes} \Lambda \simeq \mathbb{R} \underset{\mathbb{Z}}{\otimes} \Lambda,$$

where the latter is a vector space, equipped with a linear involution θ. Thus

$$(\mathbb{R}^{>0,\times} \underset{\mathbb{Z}}{\otimes} \Lambda)^\tau \simeq (\mathbb{R} \underset{\mathbb{Z}}{\otimes} \Lambda)^\theta,$$

where the latter is the subspace of θ-invariants.

7.2.2. We denote

$$A := (\mathbb{R}^{>0,\times} \underset{\mathbb{Z}}{\otimes} \Lambda)^\tau \text{ and } \mathfrak{a} := (\mathbb{R} \underset{\mathbb{Z}}{\otimes} \Lambda)^\tau,$$

so that the exponential map identifies $\mathfrak{a}$ and A.

$$K_T := (U(1) \underset{\mathbb{Z}}{\otimes} \Lambda)^\tau,$$

i.e., $K_T \subset T(\mathbb{R})$ is the maximal compact. Thus:

$$T(\mathbb{R}) = K_T \times A. \tag{7.1}$$

This is the polar decomposition of Theorem 5.3.5 for $T(\mathbb{R})$.

7.2.3. We shall say that an element in $T(\mathbb{R})$ is *split* if it belongs to the A factor in (7.1). We shall say that a subtorus $T' \subset T$ is split if $\theta|_{T'}$ is trivial (equivalently, $K_T \cap T'(\mathbb{R})$ is finite).

Consider the corresponding decomposition

$$\mathfrak{t} \simeq \mathfrak{k}_T \oplus \mathfrak{a}. \tag{7.2}$$

We shall say that an element of $\mathfrak{k}$ is *split* if it belongs to the $\mathfrak{a}$ factor in (7.2). We shall say that a subspace $\mathfrak{v} \subset \mathfrak{t}$ is split if all of its elements are split. Note that a subtorus T' is split if and only its Lie algebra $\mathfrak{t}'$ is split.

From the above analysis it is easy to deduce the following:

Proposition 7.2.4.

(a) *An element in $T(\mathbb{R})$ is split if and only if the algebraic group that it generates is connected (and hence is a torus), whose Lie algebra is split.*

(a') *An element in $T(\mathbb{R})$ is split if and only if its action on every (algebraic) representation has positive real eigenvalues.*

(b) *An element in $\xi \in \mathfrak{t}$ is split if and only if the smallest subtorus $T' \subset T$ such that $\xi \in \mathfrak{t}'$, is split.*

(b') *An element in $\mathfrak{t}$ is split if and only if its action on every (algebraic) representation has real eigenvalues.*

7.3. **Split elements in a real reductive group.** In this subsection we let G be a real reductive group, let $G_\mathbb{C}$ denote its complexification. We denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{g}_\mathbb{C}$ its complexification, i.e., the Lie algebra of $G_\mathbb{C}$.

7.3.1. Let $g \in G$ be an element. We shall say that g is split if: (i) it is semi-simple, (ii) the algebraic group that it generates is a split torus.

Let $\xi \in \mathfrak{g}$ be an element. We shall say that ξ is split if it is semi-simple and smallest torus $T' \subset G$ such that $\xi \in \mathfrak{t}'$, is split.

We shall say that a subalgebra $\mathfrak{v} \subset \mathfrak{g}$ is $\mathbb{R}$-diagonalizable if it is abelian and consists of split elements.

From Proposition 7.2.4 we obtain:

Corollary 7.3.2.

(a) *An element $g \in G$ if and only if its action on every (algebraic) representation of G is diagonalizable (over $\mathbb{R}$) with positive eigenvalues.*

(b) *A subalgebra $\mathfrak{v} \subset \mathfrak{g}$ is $\mathbb{R}$-diagonalizable if and only if its action on every (algebraic) representation of G is simultaneously is simultaneously diagonalizable (over $\mathbb{R}$).*

(c) *For an $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{v} \subset \mathfrak{g}$, the smallest subtorus $T_{\mathfrak{v}} \subset G$ such that $\mathfrak{v} \subset T_{\mathfrak{v}}$, is split.*

7.3.3. We now claim:

Proposition 7.3.4.

(a) *Let $\mathfrak{v} \subset \mathfrak{g}$ be an $\mathbb{R}$-diagonalizable subalgebra. Then there exists a compatible compact form on $G_{\mathbb{C}}$ such that $\mathfrak{v} \subset \mathfrak{p}$.*

(b) *If $\mathfrak{v}$ is contained in $\mathfrak{p}$, then $\mathfrak{v}$ is $\mathbb{R}$-diagonalizable.*

Proof. We will use the following statement, which can be proved by an argument similar to that of Theorem 5.2.3:

Theorem 7.3.5. *Let $\phi : G^1 \to G^2$ be a homomorphism of real reductive groups. Then given a compact form σ_1 on $G^1_{\mathbb{C}}$ compatible with G_1, one can find a compact form σ_2 on $G^2_{\mathbb{C}}$ that is compatible with G^2 and $\sigma_2 \circ \phi = \phi \circ \sigma_1$.*

For point (a), let G^1 be its centralizer in G, and let $\mathfrak{g}'$ be its Lie algebra. Then G^1 is a reductive group. By Theorem 7.3.5 it suffices to find a compatible compact form for G^1.

We have $\mathfrak{g}^1 \simeq \mathfrak{z}_{\mathfrak{g}^1} \oplus (\mathfrak{g}^1)'$. Choose any compact form on G^1. We have

$$\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{z}_{\mathfrak{g}^1}) \oplus (\mathfrak{p} \cap (\mathfrak{g}^1)').$$

Then by Proposition 7.2.4 and (7.2), we have $\mathfrak{v} \subset (\mathfrak{p} \cap \mathfrak{z}_{\mathfrak{g}^1}) \subset \mathfrak{p}$.

For point (b), consider a representation $G \to GL(V)$. It suffices to show that every element of $\mathfrak{p}$ acts by an $\mathbb{R}$-diagonalizable operator on V. By Theorem 7.3.5, we can choose a positive definite scalar product on V such that $\mathfrak{p}$ maps to $E(V)$. This implies the claim.

□

8. Week 4, Day 2 (Thurs, Feb. 16)

8.1. **The maximal $\mathbb{R}$-diagonalizable subalgebra.**

8.1.1. We shall say that a subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is a *maximal* $\mathbb{R}$-diagonalizable subalgebra if it is an $\mathbb{R}$-diagonalizable subalgebra and is not properly contained in other such.

Let $\mathfrak{a}$ be a *maximal* $\mathbb{R}$-diagonalizable subalgebra. Let $T_{\mathfrak{a}}$ be the smallest split subtorus of G such that $\mathfrak{a} \subset \mathrm{Lie}(T_{\mathfrak{a}})$. By Corollary 7.3.2 and the maximality assumption the above inclusion is in fact an equality

$$\mathfrak{a} = \mathrm{Lie}(T_{\mathfrak{a}}).$$

This establishes a bijection between *maximal* $\mathbb{R}$-diagonalizable subalgebras of $\mathfrak{g}$ and maximal split subtori in G.

Set

$$A := T_{\mathfrak{a}}(\mathbb{R})_0;$$

this is the subgroup of $T_{\mathfrak{a}}(\mathbb{R})$ as in (7.1).

8.1.2. We claim:

Proposition 8.1.3. *Given a maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a} \subset \mathfrak{g}$ and an arbitrary $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{v} \subset \mathfrak{g}$, there exists an element of $G(\mathbb{R})_0$ that conjugates $\mathfrak{v}$ into $\mathfrak{a}$.*

Proof. By Proposition 7.3.4(a), we can assume that both $\mathfrak{v}$ and $\mathfrak{a}$ are contained in $\mathfrak{p}$. Choosing a generic element $\xi \in \mathfrak{v}$, its suffices to show that there exists $k \in K$ such that $\mathrm{Ad}_k(\xi) \in \mathfrak{a}$; by the maximality of $\mathfrak{a}$, the latter is equivalent to the condition that $[\mathrm{Ad}_k(\xi), \eta] = 0$ for a given generic element $\eta \in \mathfrak{a}$. Consider the function

$$f(k) = (\mathrm{Ad}_k(\xi), \eta).$$

By compactness, choose a point $k \in K$, on which it attains a maximum. We claim that this point will do. Indeed, conjugating ξ by k, we can assume that $k = 1$. We obtain that the function f has a zero differential at $k = 1$. However, the differential is given by

$$\mathsf{k} \mapsto ([\mathsf{k}, \xi], \eta) = (\mathsf{k}, [\xi, \eta]).$$

We have $[\xi, \eta] \in \mathfrak{k}$, and $(-,-)$ is non-degenerate on $\mathfrak{k}$. Hence, $[\xi, \eta] = 0$.

□

8.1.4. We shall say that the real group G is split if $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. One can show that every connected complex reductive group has a unique split form (up to conjugacy).

8.2. The centralizer of a maximal $\mathbb{R}$-diagonalizable subalgebra.

8.2.1. As is the case of any abelian semi-semi subalgebra in $\mathfrak{g}$, the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ is the Lie algebra $\mathfrak{m}$ of a Levi subgroup M.

Proposition 8.2.2. $\mathfrak{m} \simeq \mathfrak{a} \oplus (\mathfrak{m} \cap \mathfrak{k})$.

Proof. Let $\xi + \eta$ be an element of $\mathfrak{g}$ that centralizes $\mathfrak{a}$, with $\xi \in \mathfrak{k}$ and $\eta \in \mathfrak{p}$. It suffices to show that ξ and η each centralize $\mathfrak{a}$. For $a \in \mathfrak{a}$ we have

$$0 = [a, \xi + \eta] = [a, \xi] + [a, \eta],$$

where $[a, \xi] \in \mathfrak{p}$ and $[a, \eta] \in \mathfrak{k}$. Hence, both are zero. □

Corollary 8.2.3.
(a) *The map $A \times (K \cap M) \to M$ is the polar decomposition of Theorem 5.3.5 for M.*
(b) *The map $A \times (K \cap Z_M) \to Z_M$ is the polar decomposition of Theorem 5.3.5 for Z_M.*

Corollary 8.2.4. *The group M is compact modulo its center.*

8.3. **The minimal parabolic.**

8.3.1. By assumption, the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ is diagonalizable. Write

$$\mathfrak{g} = \mathfrak{m} \bigoplus \underset{\alpha}{\oplus}\, \mathfrak{g}_\alpha,$$

where $\alpha \in \mathfrak{a}^*$. Note that since σ acts as -1 on $\mathfrak{a}$, we have

$$\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}.$$

One can show that the collection of those α that appear forms a root system.

8.3.2. In terms of this decomposition, $\mathfrak{k}$ identifies with the direct sum of $\mathfrak{m}$ and the span of elements

$$(\xi_\alpha + \xi_{-\alpha}), \quad \xi_\alpha \in \mathfrak{g}_\alpha, \quad \xi_{-\alpha} \in \mathfrak{g}_{-\alpha}, \quad \tau(\xi_\alpha) = \xi_{-\alpha}.$$

8.3.3. Choose an element $a \in \mathfrak{a}$ such that $\alpha(a) \neq 0$ for any α. Then the subspace

$$\mathfrak{q} := \mathfrak{m} \bigoplus \underset{\alpha, \alpha(a)>0}{\oplus} \mathfrak{g}_\alpha,$$

is the Lie algebra of a parabolic subgroup P of G. (This manipulation is valid for any abelian semi-simple subalgebra in $\mathfrak{g}$.)

Let $\mathfrak{n}$ denote the Lie algebra of $N(Q)$. We have:

$$\mathfrak{n} = \underset{\alpha, \alpha(a)>0}{\oplus} \mathfrak{g}_\alpha.$$

Proposition 8.3.4. *The subgroup Q is minimal among parabolics defined over $\mathbb{R}$.*

Proof. The statement is equivalent to the fact that M has no proper parabolics defined over $\mathbb{R}$. This follows from the fact that $M/Z(M)$ is compact (compact real groups do not have proper parabolics because they have no unipotent elements). □

8.3.5. Consider the subgroup

$$Q^0 \subset P(\mathbb{R})$$

equal to the preimage of $A \subset T_{\mathfrak{a}}(\mathbb{R}) \subset Z_M(\mathbb{R}) \subset M(\mathbb{R})$ under the projection

$$Q(\mathbb{R}) \to M(\mathbb{R}).$$

By construction, P^0 is an extension

$$1 \to N_{\min}(\mathbb{R}) \to Q^0 \to A \to 1.$$

8.3.6. We now claim:

Theorem 8.3.7. *Any unipotent subgroup of G can be conjugated into $N_{\min}$ by an element of $G(\mathbb{R})$.*

Proof. Let $N' \subset G$ be a unipotent subgroup. By Hilbert 90, it is enough to see that $N'(\mathbb{R})$ can be conjugated into $N(\mathbb{R})$. Consider the quotient $G(\mathbb{R})/Q(\mathbb{R})$. We claim that is it enough to show that $N'(\mathbb{R})$ has a fixed point on $G(\mathbb{R})/Q(\mathbb{R})$. Indeed, such a point, will conjugate $N'(\mathbb{R})$ into $Q(\mathbb{R})$. Now, any unipotent element of $Q(\mathbb{R})$ is contained in $N_{\min}(\mathbb{R})$. Indeed, the quotient $Q(\mathbb{R})/N_{\min}(\mathbb{R}) \simeq M(\mathbb{R})$ contains no unipotent elements.

To prove the existence of the fixed point, we will show that for a proper variety X and a unipotent group N' acting on (everything is defined over $\mathbb{R}$), every connected component of $X(\mathbb{R})$ contains an $N'(\mathbb{R})$-fixed point. The proof is obtained as in the case of algebraically closed fields:

Reduce by induction to the case when $N' = \mathbb{G}_a$. Then the claim is that for any $x \in X(\mathbb{R})$, the map

$$\mathbb{A}^1 \to X, \quad a \mapsto a \cdot x$$

extends to a map $\mathbb{P}^1 \to X$ (by the valuative criterion). Its value on $\infty \in \mathbb{P}^1(\mathbb{R})$ is the desired fixed point.

□

8.4. **The Iwasawa decomposition.**

8.4.1. We will prove:

Theorem 8.4.2. *The product map*

$$Q^0 \times K \to G(\mathbb{R})$$

is a diffeomorphism.

8.4.3. For the proof we first notice that the corresponding assertion does take place at the Lie algebra level:

Proposition 8.4.4. *The sum map*

$$\mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{k} \to \mathfrak{g}$$

is an isomorphism.

Proof. Follows from Sect. 8.3.2.

□

We can now prove Theorem 8.4.2:

Proof. We need to show that the map

$$K \to G(\mathbb{R})/Q^0, \tag{8.1}$$

given by the action on the element $1 \in G(\mathbb{R})/Q^0$ is a diffeomorphism.

Since K is compact, the image is closed. We now claim that the map in question is submersive. This is an assertion at the level of tangent spaces, and follows from the surjectivity of the map in Proposition 8.4.4. Hence, the image is open. Being both open and closed, it is the union of certain connected components. However, the map (8.1) is a bijection at the level of connected components (by Theorem 5.3.5).

Thus, we obtain that the action of K on $G(\mathbb{R})/Q^0$ is transitive. It remains to see that $\mathrm{Stab}_K(1) = 1$. However, this is just the fact that $K \cap P^0 = 1$.

□

Remark 8.4.5. The action of K on $G(\mathbb{R})/Q(\mathbb{R})$ comes from an action of the complex algebraic group $K_\mathbb{C} = G_\mathbb{C}^\theta$ on the complex algebraic variety $(G/Q)_\mathbb{C}$. We note, however, that the latter action is *not necessarily* transitive. However, the above proof shows that the orbit of the element $1 \in (G/Q)_\mathbb{C}$ under this action is *open*.

Here is how it looks in an example $G = SL_{2,\mathbb{R}}$. We have $Q = B$; $K = U(1)$. At the complexified level $G_\mathbb{C} = SL_2$ and $K_\mathbb{C} = \mathbb{G}_m$; $(G/P)_\mathbb{C} \simeq \mathbb{P}^1$. The action of $\mathbb{G}_m$ on $\mathbb{P}^1$ is the usual one, and the open orbit in question is $\mathbb{P}^1 - \{0, \infty\}$.

8.5. **The Cartan decomposition.**

8.5.1. Consider the product map

$$K \times A \times K \to G(\mathbb{R}). \tag{8.2}$$

The Cartan decomposition is the following statement:

Theorem 8.5.2. *The map* (8.2) *is surjective and proper (the preimage of every compact is compact).*

The meaning of this theorem is that $G(\mathbb{R})$ "looks like A modulo something compact".

Remark 8.5.3. The map (8.2) is (obviously) not bijective. For example, the subset

$$K \times \{1\} \times K \subset K \times A \times K \to G(\mathbb{R})$$

gets collapsed onto just one copy of K.

8.5.4. *Proof of properness.* It suffices to show that the map $K \times A \to G(\mathbb{R})/K$ is proper. This follows from the fact that the map $A \to G(\mathbb{R})/K$ is proper (by Theorem 8.4.2) and the fact that K is compact.

□

8.5.5. *Proof of surjectivity.* By the polar decomposition, the assertion is equivalent to the fact that the adjoint action of K on P defines a surjection

$$K \times A \to P,$$

or, equivalently,

$$K \times \mathfrak{a} \to \mathfrak{p}.$$

I.e., we need to show that every element of $\mathfrak{p}$ can be conjugated by means of K into $\mathfrak{a}$. But we have shown that in Proposition 8.1.3.

□

9. Week 5, Day 1 (Tue, Feb. 21)

9.1. **Admissible representations.** Let G be a real reductive group. We will assume that G is connected as an algebraic group (which does *not* imply that the corresponding Lie group is connected. To save the notation we will denote by the same symbol G (rather than $G(\mathbb{R})$) the corresponding Lie group. Let $K \subset G$ be its maximal compact. Note that K may be disconnected.

9.1.1. Let V be a continuous representation of G. Recall that V is acted on by the algebra $\text{Meas}_c(G)$ by convolutions.

Recall that we denote by $V^\infty \subset V$ the vector subspace consisting of smooth vectors. We have shown that it is dense. We recall also that V^∞ is acted on by a larger algebra, namely $\text{Distr}_c(G)$. The action of elements of $\text{Meas}_c(G)$ on V^∞ factors through the tautological map $\text{Meas}_c(G) \to \text{Distr}_c(G)$, dual to the inclusion $C^\infty(G) \to C(G)$.

9.1.2. Recall also that for an irreducible representation ρ of K, we denote by

$$V^\rho := \rho \otimes \mathrm{Hom}_K(V^\rho, V)$$

the ρ-isotypic component in V. The operator $T_{\xi_\rho \cdot \mu_{\mathrm{Haar}_K}}$ acts as a projector on V^ρ: i.e., its image is in V^ρ, and its restriction to V^ρ is the identity.

Recall the notation

$$V^{K\text{-fin}} := \bigoplus_\rho V^\rho;$$

this is the space of K-finite vectors V. Its tautological map into V is injective and has a dense image, called the space of K-finite vectors.

We claim:

Lemma 9.1.3. *For every ρ, the subspace $V^\infty \cap V^\rho$ is dense in V^ρ.*

Proof. Choose a Dirac sequence of smooth functions $f_n \to \delta_1$. Then for every $v \in V^\rho$, the elements

$$T_{\xi_\rho \cdot \mu_{\mathrm{Haar}_K}} \star T_{f_n \cdot \mu_{\mathrm{Haar}_G}}(v)$$

belong to both V^ρ and V^∞ and converge to v. □

9.1.4. A representation V of G is said to be *admissible* if for every ρ, the vector space V^ρ is finite-dimensional.

A key observation is:

Proposition 9.1.5. *Let V be admissible. Then $V^{K\text{-fin}} \subset V^\infty$.*

Proof. It is enough to show that $V^\rho \subset V^\infty$ for every ρ. I.e., we need to show that $V^\infty \cap V^\rho$ equals all of V^ρ. However, this subspace is dense by Lemma 9.1.3, and since V^ρ is finite-dimesnional, it must be the whole thing. □

9.1.6. As a result we obtain that if V is admissible, the Lie algebra $\mathfrak{g}$ acts on $V^{K\text{-fin}}$. Note that the actions of K and $\mathfrak{g}$ (on all of V^∞) are compatible in the following sense:

(i) For $k \in K$ and $\eta \in \mathfrak{g}$,

$$T_k \cdot T_\eta \cdot T_{k^{-1}} = T_{\mathrm{Ad}_k(\eta)};$$

(ii) The action of $\mathfrak{k}$ on V^∞ arising from the action of K on V (any vector smooth with respect to G is obviously smooth with respect to K) equals the restriction of the action of $\mathfrak{g}$ along the tautological map $\mathfrak{k} \to \mathfrak{g}$.

9.1.7. We define the notion of $(\mathfrak{g}, K)$-module to be a $\mathbb{C}$-vector space, equipped with an *algebraic* (i.e., locally finite) action of K and an action of $\mathfrak{g}$ that are compatible in the sense that (i) and (ii) above hold.

Let $K_\mathbb{C}$ be the complex algebraic group corresponding to K. Note that $(\mathfrak{g}, K)$-modules can be equivalently defined as complex vector spaces equipped with an algebraic action of $K_\mathbb{C}$ and an action of $\mathfrak{g}_\mathbb{C}$ that are compatible in the similar sense.

Thus, $(\mathfrak{g}, K)$-modules form a purely algebraic category; we denote it by $(\mathfrak{g}, K)$-mod.

Let

$$(\mathfrak{g}, K)\text{-mod}_{\mathrm{adm}} \subset (\mathfrak{g}, K)\text{-mod}$$

denote the full subcategory of admissible $(\mathfrak{g}, K)$-modules. The assignment $V \mapsto V^{K\text{-fin}}$ is a functor

$$\mathrm{Rep}(G)_{\mathrm{adm}} \to (\mathfrak{g}, K)\text{-mod}_{\mathrm{adm}}. \tag{9.1}$$

9.1.8. Let M be an admissible $(\mathfrak{g}, K)$-module. We can form its *algebraic* dual

$$(M^*)^{\text{alg}} := \bigoplus_{\rho} (M^\rho)^*,$$

which is equipped with a natural action on $\mathfrak{g}$.

Equivalently, $(M^*)^{\text{alg}}$ is the subspace of K-finite vectors in the full linear dual M^*, the latter being $\prod\limits_{\rho} (M^\rho)^*$.

9.2. How well is a representation approximated by its $(\mathfrak{g}, K)$-module?

9.2.1. Following Harish-Chandra, we shall sat that two admissible representations V_1 and V_2 of G are *infinitesimally equivalent* if $V_1^{K\text{-fin}}$ and $V_2^{K\text{-fin}}$ are isomorphic as $(\mathfrak{g}, K)$-modules.

It is (obviously) not true that if two admissible representations V_1 and V_2 of G are *infinitesimally equivalent* then they are isomorphic. For example, take $G = K$ and $V_1 = L_2(K)$ and $V_2 = C(K)$.

The above example may not be too interesting since the representations V_1 and V_2 are highly reducible. We will bring more interesting examples when we introduce principal series representations. But in case, the phenomenon has to with the fact that a given $(\mathfrak{g}, K)$-module can be topologized in many different ways so that its completion carries an action of G.

9.2.2. Nonetheless, we have:

Theorem 9.2.3.
(a) *Let V_1 and V_2 be two admissible representations of G. Let $S : V_1 \to V_2$ be a continuous map of the underlying vector spaces. Assume that S sends $V_1^{K\text{-fin}}$ to $V_2^{K\text{-fin}}$ and that the resulting map $V_1^{K\text{-fin}} \to V_2^{K\text{-fin}}$ is compatible with $(\mathfrak{g}, K)$-module structures. Then the initial S was a map of G-representations.*
(b) *For $V \in \text{Rep}(G)_{\text{adm}}$ and $M := V^{K\text{-fin}}$, for every $(\mathfrak{g}, K)$-submodule $M_1 \subset M$, its closure $\overline{M_1}_\subset V$ is a G-subrepresentation.*
(c) *The assignments*

$$(V_1 \subset V) \mapsto (V_1)^{K\text{-fin}} \subset M \text{ and } M_1 \mapsto \overline{M_1} \subset V$$

define mutually inverse bijections between the set of closed G-subrepresentations of V and the set of $(\mathfrak{g}, K)$-submodules of M.

Corollary 9.2.4. *An admissible representation V is irreducible (i.e., contains no closed G-invariant subspaces) if and only if $V^{K\text{-fin}}$ is irreducible as a $(\mathfrak{g}, K)$-module.*

It is of course not true that every G-representation is admissible (take an infinite direct sum of copies of the same representation). But one can ask: is it true that irreducible G-representations are admissible. This was conjectured by Harish-Chandra, but many years later, W. Soergel produced a counterexample. However, we have the following theorem (to be proved next week):

Theorem 9.2.5. *Every* unitary *irreducible representation of G is admissible.*

In Corollary 10.1.6 we will see that a unitary irreducible representation of G is completely determined by its space of K-finite vectors.

Theorem 9.2.5 has a converse (also due to Harish-Chandra):

Theorem 9.2.6. *Let M be an irreducible $(\mathfrak{g}, K)$-module equipped with an* invariant *Hermitian positive-definite form. Then the Hilbert space completion V of M carries a unique unitary G-representation such that $V^{K\text{-fin}} = M$ as $(\mathfrak{g}, K)$-modules.*

In the above theorem a form $(-,-)$ is called *invariant* if

$$(k \cdot m_1, k \cdot m_2) = (m_1, m_2),\ \forall k \in K \text{ and } (\xi \cdot m_1, m_2) = -(m_1, \xi \cdot m_2),\ \forall \xi \in \mathfrak{g}.$$

As a formal corollary, one obtains:

Corollary 9.2.7. *Let M be an admissible $(\mathfrak{g}, K)$-module equipped with an* invariant *Hermitian positive-definite form. Then the Hilbert space completion V of M carries a unique unitary G-representation such that $V^{K\text{-fin}} = M$ as $(\mathfrak{g}, K)$-modules.*

Proof. By admissibility, M is an orthogonal direct sum of irreducible $(\mathfrak{g}, K)$-modules. □

9.2.8. The proof of Theorem 9.2.3 is based on the following:

Proposition 9.2.9. *Let V be an admissible representation of V, and let v be an element of $V^{K\text{-fin}}$. Then for every $\eta \in V^*$, the function on G*

$$g \mapsto \eta(g(v))$$

is real analytic.

Let us prove Theorem 9.2.3 using this proposition:

Proof. For point (a), it is enough to show that for $v_1 \in (V_1)^{K\text{-fin}}$ we have

$$T_g \circ S(v_1) = S \circ T_g(v_1).$$

For that it suffices to show that for any $\eta \in V_2^*$, we have

$$\eta(T_g \circ S(v_1)) = \eta(S \circ T_g(v_1)).$$

Both sides are analytic functions in g. Hence, it is enough to show that all of their derivatives at $1 \in G$ are equal, and that they agree on at least one point on each connected component of G.

The first condition follows from the compatibility with the action of $U(\mathfrak{g})$. The second condition follows from the compatibility with the action of K.

For point (b), we recall that by the Hahn-Banach theorem, the closure of a subspace $M_1 \subset V$ equals $((M_1)^\perp)^\perp$, where $(M_1)^\perp \subset V^*$ is the annihilator of M_1. Hence, it suffices to show that for any $\eta \in (M_1)^\perp$ we have $g \cdot \eta \in (M_1)^\perp$, i.e., for every $v_1 \in M_1$ we have $\eta(g(v_1)) = 0$. This follows by analyticity in the same way as above.

To prove point (c), we note that for a subrepresentation $V_1 \subset V$, the subspace $V_1^{K\text{-fin}}$ is dense in V_1, so its closure is all of V_1.

Vice versa, for a sudmodule $M_1 \subset M$, it suffices to show that for every ρ, the image of $T_{\xi_\rho \cdot \mu_{\mathrm{Haar}_K}}$ on $\overline{M_1}$ lies in M_1^ρ. However, $T_{\xi_\rho \cdot \mu_{\mathrm{Haar}_K}}(M_1) \subset M_1^\rho$, and the assertion follows by continuity.

□

9.3. **Proof of analyticity.** The goal of this subsection is to prove Proposition 9.2.9. With no restriction of generality, we can assume that $v \in V^\rho$ for some $\rho \in \mathrm{Irrep}(K)$.

9.3.1. Let D be a differential operator of order n on a differentialble (resp., real analytic) manifold X. Let $\sigma_n(D)$ be its symbol, i.e., the image of D under the projection

$$\mathrm{Diff}^{\leq n}(X) \to \mathrm{Diff}^{\leq n}(X)/\,\mathrm{Diff}^{\leq n-1}(X) \simeq \mathrm{Sym}^n_{C^\infty(X)}(\mathrm{Vect}(X)).$$

We can regard $\sigma_n(D)$ as a function on $T^*(X)$, which is homogenous of degree n along the fibers.

Recall that D is said to be *elliptic* if $\sigma_n(D)(\eta) > 0$ for all $0 \neq \eta \in T^*_x(X)$.

We have the following basic result (elliptic regularity):

Theorem 9.3.2. *Let ϕ be an element of* $\mathrm{Distr}(X)$ *(where* $\mathrm{Distr}(X)$ *is the topological dual of* $C^\infty_c(X)$*). If $D(\phi) = 0$ (i.e., $\phi(D(f)) = 0$ for all $f \in C^\infty_c(X)$). Then:*

(a) *The distribution ϕ is smooth. I.e., it given by a smooth function (times a smooth measure on X).*

(b) *If X is real analytic, and the coefficients of D are real analytic, then ϕ is real analytic. I.e., it is given by an analytic function (times an analytic measure on X).*

9.3.3. We will apply Theorem 9.3.2(b) to $X = G$ and ϕ being the (continuous) function (times μ_{Haar_G}) given by $g \mapsto \eta(g(v))$. Thus, our goal is to find an elliptic operator D with analytic coefficients that annihilates this function.

We will define D as a left-invariant differential operator corresponding to a certain element $u \in U(\mathfrak{g})^{\leq n}$. The analyticity is then guaranteed by the construction.

The ellipticity will amount to checking that the image $\sigma_n(u)$ of u in

$$U(\mathfrak{g})^{\leq n}/U(\mathfrak{g})^{\leq n-1} \simeq \mathrm{Sym}^n(\mathfrak{g})$$

satisfies $\sigma_n(u)(\eta) > 0$ for any $0 \neq \eta \in \mathfrak{g}^*$.

9.3.4. Write $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}_\mathfrak{g}$. Let $(-,-)$ be a bilinear form on $\mathfrak{g}$, which is the Killing form on $\mathfrak{g}'$ and a form on $\mathfrak{z}_\mathfrak{g}$ that is positive-definite on the split part of $\mathfrak{z}_\mathfrak{g}$ (i.e., $\mathfrak{z}_\mathfrak{g} \cap \mathfrak{p} = \mathfrak{z}_\mathfrak{g} \cap \mathfrak{a}$) and negative-definite on the compact part of $\mathfrak{z}_\mathfrak{g}$ (i.e., $\mathfrak{z}_\mathfrak{g} \cap \mathfrak{k}$).

Then $(-,-)$ is the direct sum of a positive-definite form on $\mathfrak{p}$ and a negative-definite form on $\mathfrak{k}$.

9.3.5. Let $C_\mathfrak{g} \in Z(\mathfrak{g}) \subset U(\mathfrak{g})$ be the Casimir element corresponding to $(-,-)$. I.e., if e_i is an orthonormal basis in $\mathfrak{g}$ with respect to $(-,-)$, then

$$C_\mathfrak{g} = \sum_i (e_i)^2.$$

It is an elementary fact that $C_\mathfrak{g}$ indeed belongs to $Z(\mathfrak{g})$ and is independent of the choice of a basis.

Consider also the element $C_\mathfrak{k} \in U(\mathfrak{k}) \subset U(\mathfrak{g})$. Set

$$\widetilde{u} = C_\mathfrak{g} - 2 \cdot C_\mathfrak{k}.$$

Since $Z(\mathfrak{g})$ commutes with the action of K, the action of $\widetilde{u}$ on V^∞ preserves V^ρ. Since V^ρ is finite-dimensional, we can find a monic polynomial p (of some degree n) such that $p(T_{\widetilde{u}})$ annihilates V^ρ. Set

$$u = p(\widetilde{u}).$$

By construction u annihilates v, and hence annihilates our function ϕ. It remains to check the ellipticity. Take $n = 2d$.We have $\sigma_n(u) = (\sigma_2(\widetilde{u}))^d$. Hence, it suffices to show that

$$\sigma_2(\widetilde{u}) \in \mathrm{Sym}^2(\mathfrak{g})$$

is elliptic.

Choose an orthonormal bases for $\mathfrak{g}$ to be of the form $\{e'_i\} \cup \{e''_j\}$, where $\{e'_i\}$ is an orthonormal basis for $\mathfrak{k}$, and $\{e''_j\}$ is an orthonormal basis for $\mathfrak{p}$. Then

$$\sigma_2(\widetilde{u}) = \sum_j (e''_j)^2 - \sum_i (e''_i)^2.$$

I.e., $\sigma_2(\widetilde{u})$ is the quadratic form corresponding to the bilinear form

$$(-,-)|_{\mathfrak{p}} - (-,-)|_{\mathfrak{k}},$$

and the result follows from the fact that the latter is positive-definite.

9.4. $(\mathfrak{g}, K)$-modules vs $\mathfrak{g}$-modules.

9.4.1. Consider the forgetful functor

$$(\mathfrak{g}, K)\text{-mod} \to \mathfrak{g}\text{-mod}.$$

We can factor it as

$$(\mathfrak{g}, K)\text{-mod} \to (\mathfrak{g}, K_0)\text{-mod} \to \mathfrak{g}\text{-mod},$$

where K_0 is the neutral connected component of K.

Lemma 9.4.2. *The functor* $(\mathfrak{g}, K_0)$-mod $\to \mathfrak{g}$-mod *is fully faithful. Its essential image is stable with respect to taking submodules.*

Proof. The action of $\mathfrak{g}$ on a vector space determines that of $\mathfrak{k}$, and if the latter extends to an action of K_0, it does so in a unique way (in this case we say that the action of $\mathfrak{k}$ *integrates* to that of K_0). This happens of and only if the following happens:

(i) The action of the derived Lie algebra $\mathfrak{k}'$ is locally finite;

(ii) the resulting action of the simply-connected cover of K'_0 factors through that of K'_0.

(iii) The abelian algebra $\mathfrak{z}_{\mathfrak{k}}$ acts semi-simply with characters given by characters of the torus $(Z_{K_0})_0$.

(iv) The two resulting actions of $Z_{K'_0} \cap (Z_{K_0})_0$ agree.

It is clear that these conditions are stable with respect to taking submodules.

□

Corollary 9.4.3. *If* $M \in (\mathfrak{g}, K_0)$-mod *is such that the underlying* $\mathfrak{g}$*-module is irreducible, then* M *itself is irreducible.*

9.4.4. Recall that an object M of an abelian category $\mathcal{A}$ is said to be finitely generated if for every ascending chain

$$M_1 \subset M_2 \subset ... \subset M$$

with $\underset{i}{\cup} M_i = M$, we have $M_i = M$ for some i.

We say that $\mathcal{A}$ is Noetherian if a subobject of a finitely generated object is finitely generated. If $\mathcal{A} = A$-mod for an associative algebra A, this property is equivalent to A being (left)-Noetherian. In particular, this is the case for $\mathcal{A} = \mathfrak{g}$-mod, since $U(\mathfrak{g})$ is Noetherian (this follows from the fact that $\mathrm{gr}(U(\mathfrak{g})) \simeq \mathrm{Sym}(\mathfrak{g})$ is Noetherian).

We say that $\mathcal{A}$ is Artinian, if every finitely generated object has finite length.

Lemma 9.4.5. *The forgetful functor* $(\mathfrak{g}, K)$-mod $\to \mathfrak{g}$-mod *sends finitely generated objects to finitely generated objects.*

Proof. By Lemma 9.4.2, it suffices to prove the corresponding fact for the restriction functor $(\mathfrak{g}, K)$-mod $\to (\mathfrak{g}, K_0)$-mod.

Let M be a finitely generated $(\mathfrak{g}, K)$-module, and let

$$M_1 \subset M_2 \subset ... \subset M, \quad \underset{i}{\cup} M_i = M$$

be a chain of $(\mathfrak{g}, K)$-submodules. Pick representatives $k \in K$ for each element of $\pi_0(K)$. Then each

$$M_i' = \sum_{k \in K} k \cdot (M_i)$$

is a $(\mathfrak{g}, K)$-submodule of M. Hence, $M_i' = M$ for some i. Pick j large enough so that $k \cdot (M_i) \subset M_j$. Then $M_j = M$.

□

Corollary 9.4.6. *The category* $(\mathfrak{g}, K)$-mod *is Noetherian.*

9.4.7. We now claim:

Proposition 9.4.8. *For an irreducible* $(\mathfrak{g}, K)$*-module, the underlying* $\mathfrak{g}$*-module is a direct sum of finitely many irreducibles.*

Proof. Let M be an irreducible $(\mathfrak{g}, K)$-module. By Lemma 9.4.2, it suffices to show that M is a direct sum of finitely many irreducibles $(\mathfrak{g}, K_0)$-modules.

Let $M' \subset M$ be a maximal $(\mathfrak{g}, K_0)$-submodule with $M/M' \neq 0$ (such always exists by Zorn's lemma). Thus, $N = M/M'$ is a non-zero irreducible $(\mathfrak{g}, K_0)$-module. Pick a representative $k \in K$ for each element of $\pi_0(K)$. Consider

$$M'' := \underset{k}{\cap} k(M') \subset M.$$

Then M'' is a proper $(\mathfrak{g}, K)$-submodule of M, and hence equals zero. Hence, the map

$$M \to \underset{k}{\oplus} (M')^k$$

is injective, where $(M')^k$ denotes M' with the action twisted by the automorphism k. Thus, M, when viewed as a $(\mathfrak{g}, K_0)$-module is a submodule of a semi-simple module, and hence is semi-simple.

□

10. Week 5, Day 2 (Thurs, Feb. 23)

10.1. Schur's lemma for $(\mathfrak{g}, K)$-modules.

10.1.1. Here is version of Schur's lemma for Lie algebras:

Theorem 10.1.2. *Let M be an irreducible* $\mathfrak{g}$*-module. Then the map* $\mathbb{C} \to \mathrm{End}_{\mathfrak{g}}(M)$ *is an isomorphism.*

Proof. Let S be an endomorphism of M. Suppose that it is not a scalar. Consider the map

$$\mathbb{C}[s] \to \mathrm{End}_{\mathfrak{g}}(M), \quad s \mapsto S.$$

We claim that it is injective. Indeed, if it was not, it would contain an element of the form $\prod\limits_i (s - a_i)^{n_i}$ in its kernel, which would mean that one of operators $S - a_i \cdot \mathrm{Id}$) is non-invertible. Since M is irreducible, this would mean that $S = a_i \cdot \mathrm{Id}$, contradicting the assumption.

Thus, the above map extends to an (automatically injective) map from the fraction field $\mathbb{C}(s)$ of $\mathbb{C}[s]$ to $\text{End}_{\mathfrak{g}}(M)$. Note, however, that $\mathbb{C}(s)$, when viewed as a vector space over $\mathbb{C}$ has dimensional at least continuum: indeed, the elements $\frac{1}{s-a}$ are all linearly independent. However, we claim that $\text{End}_{\mathfrak{g}}(M)$ is countable-dimensional. Indeed, for any non-zero vector $m \in M$, evaluation on m defines an injective map

$$\text{End}_{\mathfrak{g}}(M) \to M,$$

while M itself is countable-dimensional, being the quotient of $U(\mathfrak{g})$ by $\text{Ann}(m)$.

□

10.1.3. Note that the same proof applies for M being an irreducible object in the category $(\mathfrak{g}, K)$-mod. We will soon see that any irreducible $(\mathfrak{g}, K)$-module is admissible (Theorem 10.3.3). Now, for admissible $(\mathfrak{g}, K)$-modules, one can give a simpler proof of Schur:

Proposition 10.1.4. *Any endomorphism of an irreducible admissible* $(\mathfrak{g}, K)$*-module is a scalar.*

Proof. It is enough to show that any endomorphism S of an admissible $(\mathfrak{g}, K)$-module M has a non-zero eigenspace. Pick ρ such that M^ρ is non-zero. Then S preserves M^ρ, and since the latter is finite-dimensional, the assertion follows. □

10.1.5. Here is an application of Schur's lemma. This is a statement complementary to Theorem 9.2.5. It says that a unitary irreducible representation is fully determined by the space of its K-finite vectors, viewed as a $(\mathfrak{g}, K)$-module:

Corollary 10.1.6. *Let V_1 and V_2 be two irreducible* unitary *representations that are infinitesimally equivalent. Then they are isomorphic.*

Proof. First off, by Theorem 9.2.5, V_1 and V_2 are admissible, so we can talk about the corresponding $(\mathfrak{g}, K)$-modules.

Set $M_i := (V_i)^{K\text{-fin}}$. We can recover V_i as the Hilbert space completion of M_i equipped with the induced Hermitian form.

Note that if M is equipped with a $(\mathfrak{g}, K)$-invariant Hermitian form then we have a canonical identification

$$M \simeq (M^\dagger)^{\text{alg}},$$

where † means "take dual+complex conjugate".

Fix an isomorphism $S : M_1 \simeq M_2$. It does not necessarily respect the given inner form, but it does so up to multiplication by a (positive) scalar. Indeed, consider the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{S} & M_2 \\ \sim\downarrow & & \downarrow\sim \\ ((M_1)^\dagger)^{\text{alg}} & \xleftarrow{S^\dagger} & ((M_2)^\dagger)^{\text{alg}}. \end{array}$$

The composite automorphism of M_1 respects the $(\mathfrak{g}, K)$-action, and hence by Schur's lemma and Corollary 9.2.4, it is given by multiplication by a scalar.

Dividing the initial ϕ by the square root of this scalar, we can thus assume that ϕ respects the Hermitian structures. Thus, it defines an isomorphism of Hilbert spaces $V_1 \to V_2$. Now the result follows from Theorem 9.2.3(a).

□

10.2. **Action of the center of $U(\mathfrak{g})$.**

10.2.1. Denote by $Z(\mathfrak{g})$ the center of the associative algebra $U(\mathfrak{g})$. We can identify $Z(\mathfrak{g})$ with invariants in $U(\mathfrak{g})$ with respect to the adjoint action of $\mathfrak{g}$ (or G itself). It is known to be "large":

Consider the filtration on $Z(\mathfrak{g})$, induced by the PBW filtration on $U(\mathfrak{g})$.

Lemma 10.2.2. *The map from* $\mathrm{gr}_n(Z(\mathfrak{g})) := Z(\mathfrak{g})^{\leq n}/Z(\mathfrak{g})^{\leq n-1}$ *to the space of* $\mathrm{ad}_{\mathfrak{g}}$*-invariants in*

$$\mathrm{gr}^n(U(\mathfrak{g})) := U(\mathfrak{g})^{\leq n}/U(\mathfrak{g})^{\leq n-1} \simeq \mathrm{Sym}^n(\mathfrak{g})$$

is an isomorphism.

Proof. Follows from complete reducibility of algebraic $\mathfrak{g}$-representations: the functor of $\mathfrak{g}$-invariants is exact, so sends cokernels to cokernels. □

Hence, $\mathrm{gr}(Z(\mathfrak{g}))$ identifies as a commutative algebra with $\mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}}$, and the latter identifies with $\mathrm{Sym}(\mathfrak{h})^W$, and is known to be isomorphic to be a polynomial algebra of rank equal to the rank of $\mathfrak{g}$ (i.e., $\dim(\mathfrak{h})$). Lifting the generators, we can therefore find a (non-canonical) isomorphism between $Z(\mathfrak{g})$ with the same polynomial algebra.

However, such isomorphism can be made canonical: this is the Harish-Chandra isomorphism

$$Z(\mathfrak{g}) \simeq \mathrm{Sym}(\mathfrak{h})^W, \tag{10.1}$$

to be discussed later.

10.2.3. We claim:

Proposition 10.2.4. *Let M be an admissible $(\mathfrak{g}, K)$-module. Then the action of $Z(\mathfrak{g})$ is locally finite. I.e., M splits as a direct sum* $M \simeq \underset{\chi \in \mathrm{Spec}(Z(\mathfrak{g}))}{\oplus} M_\chi$*, such that $Z(\mathfrak{g})$ acts on each M_χ be a generalized character χ.*

Proof. Since the action of G (and hence K) commutes with that of $Z(\mathfrak{g})$, we obtain that $Z(\mathfrak{g})$ preserves each K-isotypic componnent M^ρ. Since the latter are finite-dimensional, the assertion follows.

□

10.2.5. For a given element $\chi \in \mathrm{Spec}(Z(\mathfrak{g}))$, let $(\mathfrak{g}, K)\text{-mod}_\chi$ be the full subcategory of $(\mathfrak{g}, K)$-mod consisting of modules, on which $Z(\mathfrak{g})$ acts with a generalized character χ, i.e., for every $m \in M$ there exists a power n such that the ideal $(\ker(\chi))^n \subset Z(\mathfrak{g})$ annihilates m.

Note that by Schur's lemma, every irreducible object in $(\mathfrak{g}, K)$-mod belongs to $(\mathfrak{g}, K)\text{-mod}_\chi$, for a uniquely defined χ.

We have:

Theorem 10.2.6. *The category $(\mathfrak{g}, K)\text{-mod}_\chi$ has only finitely many isomorphism classes of irreducible objects.*

Remark 10.2.7. This is a rather deep theorem. Although it is completely algebraic, the initial proof by Harish-Chandra was very indirect and used analysis. Later in the semester we will supply a proof using the *localization theory* of Beilinson-Bernstein.

Assuming the above theorem, we obtain:

Theorem 10.2.8. *For an object $M \in (\mathfrak{g}, K)\text{-mod}_\chi$ the following conditions are equivalent:*

(i) *M is finitely generated;*

(ii) *M is of finite length;*

(iii) *M is admissible.*

Proof. The implication (ii) $\Rightarrow$ (i) is evident. Let us prove that (iii) implies (ii). Let M be an admissible object of $(\mathfrak{g}, K)$-mod$_\chi$, and let

$$0 = M_0 \subset M_1 \subset M_2 ... \subset M_n = M$$

be a chain of submodules. We will effectively bound the integer n.

Let L_α be the irreducible objects of $(\mathfrak{g}, K)$-mod$_\chi$; by the second statement in Theorem 10.2.6, there are finitely many of them. For each α, pick $\rho_\alpha \in \text{Irrep}(K)$ so that $L_\alpha^{\rho_\alpha} \neq 0$. Let $\rho = \underset{\alpha}{\oplus} \rho_\alpha$.

For each $i = 1, ..., n$ there exists an index α so that L_α is a subquotient of M_i/M_{i_1}. Hence

$$\dim(\text{Hom}_K(\rho, M_i/M_{i_1})) \geq 1.$$

Hence,

$$\dim(\text{Hom}_K(\rho, M)) \geq n.$$

Let us show that (i) implies (iii). This follows from the next assertion (of independent interest):

Proposition 10.2.9. *Let M be a finitely generated $(\mathfrak{g}, K)$-module. Then for any ρ, the isotypic component M^ρ is finitely generated over $Z(\mathfrak{g})$.*

□

Proof of Proposition 10.2.9. Let M be a finitely generated $(\mathfrak{g}, K)$-module. Then it receives a surjection from a $(\mathfrak{g}, K)$-module of the form $U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} \rho'$ for ρ' a finite-dimensional representation of K. Hence, it is enough to show that

$$\text{Hom}_K(\rho, U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} \rho')$$

is finitely generated over the center. The latter assertion is enough to check at the associated graded level. I.e., it is enough to check that

$$(\text{Sym}(\mathfrak{g}/\mathfrak{k}) \otimes \text{Hom}(\rho, \rho'))^K$$

is finitely generated as a module over $\text{Sym}(\mathfrak{g})^G$.

Let us identify $\mathfrak{g}$ with its dual via the Killing form. Denote $W := \text{Hom}(\rho, \rho')$; this is a finite-dimensional K-representation. Thus, we need to show that

$$(\mathcal{O}_\mathfrak{p} \otimes W)^K$$

is finitely-generated over $\mathcal{O}_\mathfrak{g}^G$, where $\mathcal{O}_\mathfrak{g}^G$ maps to $\mathcal{O}_\mathfrak{p}$ via the restriction map.

By Sect. 8.5.5, the K-orbit of $\mathfrak{a}$ is dense in $\mathfrak{p}$; henve the restriction map under $\mathfrak{a} \subset \mathfrak{p}$ defines an injection

$$(\mathcal{O}_\mathfrak{p} \otimes W)^K \hookrightarrow \mathcal{O}_\mathfrak{a} \otimes W.$$

Now the assertion follows from the fact that $\mathcal{O}_\mathfrak{a}$ is finite as a module over $\mathfrak{g}//G$. Indeed, $\mathfrak{a}$ is a closed subvariety in $\mathfrak{h}$, and $\mathfrak{g}//G$ identifies with $\mathfrak{h}//W$, and the assertion follows from the fact that $\mathfrak{h}$ is finite over $\mathfrak{h}//W$.

□

10.3. **Some consequences about properties of the category $(\mathfrak{g}, K)$-mod.**

10.3.1. The implication (i) $\Rightarrow$ (ii) in Theorem 10.2.8, we obtain:

Corollary 10.3.2. *The category* $M \in (\mathfrak{g}, K)\text{-mod}_\chi$ *is Artinian.*

Combining the implication (ii) $\Rightarrow$ (iii) in Theorem 10.2.8 with Theorem 10.1.2 we also obtain:

Theorem 10.3.3. *Every irreducible* $(\mathfrak{g}, K)$*-module is admissible.*

We note that the proof of Theorem 10.3.3 *does not* use the (more difficult) theorem Theorem 10.2.6.

10.3.4. Combining the above results, we obtain:

Corollary 10.3.5. *For a* $(\mathfrak{g}, K)$*-module the following conditions are equivalent:*
(i) M *is finitely generated and admissible;*
(ii) M *is finitely generated and its support over* $\operatorname{Spec}(Z(\mathfrak{g}))$ *is finite;*
(iii) M *is admissible and its support over* $\operatorname{Spec}(Z(\mathfrak{g}))$ *is finite;*
(iv) M *is of finite length.*

We will refer to $(\mathfrak{g}, K)$-modules satisfying the equivalent conditions of Corollary 10.3.5 as Harish-Chandra modules.

10.4. **Admissibility of unitary representations.** In this subsection we will begin the proof of Theorem 9.2.5, which says that an irreducible *unitary* representation of G is admissible. We will prove a sharper result:

Theorem 10.4.1. *Let* V *be an irreducible unitary representation of* G*. Then for any* $\rho \in \operatorname{Irrep}(K)$*, we have:*

$$\dim(V^\rho) \leq \dim(\rho)^2.$$

10.4.2. Let V be a topological vector soace. Recall that among the various topologies that exist on the vector space $\operatorname{End}(V)$ of continuous endomorphisms of V, there exists what is called the *strong* topology:

The fundamental system of neighborhoods of 0 in V is given by finite collections

$$((v_1, U_1), ..., (v_n, U_n)),$$

where $v_i \in V$ and U_i are neighborhoods of zero in V. The corresponding neighborhood in V consists of those S such that

$$S(v_i) \in U_i, \ \forall i = 1, ..., n.$$

Let A be an associative algebra acting on V, i.e., we have a homomorphism $A \to \operatorname{End}(V)$. We shall say that V is *strongly irreducible* if the image of A in $\operatorname{End}(V)$ is dense in the strong topology.

10.4.3. Theorem 10.4.1 follows from the following two statements:

Theorem 10.4.4. *Let* V *be an irreducible unitary representation of* G*. Then* V*, viewed as acted on by* $\operatorname{Meas}_c(G)$*, is strongly irreducible.*

Theorem 10.4.5. *Let* V *be a representation of* G*, which is strongly irreducible when viewed as acted on by* $\operatorname{Meas}_c(G)$*. Then for any* $\rho \in \operatorname{Irrep}(K)$*, we have:*

$$\dim(V^\rho) \leq \dim(\rho)^2.$$

10.5. **Proof of Theorem 10.4.4.**

10.5.1. Let H be a Hilbert space. Let A be a subalgebra in $\mathrm{End}(H)$ closed under the operation $S \mapsto S^\dagger$. Let $\overline{A}$ be the strong closure of A in $\mathrm{End}(H)$. Let $A^c \subset \mathrm{End}(H)$ be the commutant of A, i.e.,

$$A^c = \{S' \in \mathrm{End}(V),\ S' \circ S = S \circ S' \text{ for all } S \in A\}.$$

We will use the following result of von Neumann:

Theorem 10.5.2. *The inclusion* $\overline{A} \subset (A^c)^c$ *is an equality.*

Corollary 10.5.3. *If the inclusion* $\mathbb{C} \hookrightarrow A^c$ *is an equality, then the action of* A *on* V *is strongly irreducible.*

10.5.4. Let A be the image of $\mathrm{Meas}_c(G)$ in $\mathrm{End}(V)$. The operation

$$f(g) \mapsto \overline{f(g^{-1})}$$

defines an involution on $C(G)$, and hence on $\mathrm{Meas}_c(G)$, denoted $\phi \mapsto \phi^\dagger$. We have

$$T_{\phi^\dagger} = (T_\phi)^\dagger.$$

Thus, we obtain that Theorem 10.4.4 follows from the next result, which can be viewed as a Hilbert space version of Schur's lemma:

Theorem 10.5.5. *Let* V *be an irreducible unitary representation of* G*. Then any endomorphism of* V *is a scalar.*

10.5.6. For the proof of Theorem 10.5.5, we will use the following weak form of the spectral theorem. Let H be a Hilbert space, and S be a continuous self-adjoint endomorphism of H. Then for any $\lambda \in \mathbb{R}$ there exists an idempotent π_λ of H with the following properties:

(i) Any endomorphism of H that commutes with S commutes also with π_λ;

(ii) For any $v \in \mathrm{Im}(\pi_\lambda)$ we have $(S(v), v) \leq \lambda(v, v)$;

(iii) For any $v \in \ker(\pi_\lambda)$ we have $(S(v), v) \geq \lambda(v, v)$.

10.5.7. *Proof of Theorem 10.5.5.* Let S be an endomorphism of V. First off, we can assume that S is self-adjoint. Indeed, if S is an endomorphism, then so is $S^\dagger$. Then

$$S + S^\dagger \text{ and } i(S - S^\dagger)$$

are self-adjoint, and if we can prove that each of them is a scalar, then so is the initial S.

Foe every λ, let π_λ be the corresponding idempotent of V. By (i), it commutes with the action of G. Hence, by the irreducibility we either have $\mathrm{Im}(\pi_\lambda) = V$ or $\pi_\lambda = 0$.

Let λ_0 be the infimum of all λ such that

$$(S(v), v) \leq \lambda(v, v);$$

it exists because S is bounded (and non-zero, which we can assume). Then from (ii) we obtain that for all $\lambda < \lambda_0$, $\pi_\lambda = 0$. Hence, by (iii) we obtain that

$$(S(v), v) \geq \lambda(v, v),\ \forall v \in V.$$

Thus, we obtain that $(S(v), v) = \lambda_0(v, v)$, i.e., $S = \lambda_0 \cdot \mathrm{Id}$.

□

10.6. **Proof of Theorem 10.4.5.**

10.6.1. For $\rho \in \text{Irrep}$ consider the corresponding projector ξ_ρ on V^ρ. Set

$$A_\rho = \xi_\rho \cdot \text{Meas}_c(G) \cdot \xi_\rho.$$

This is a subalgebra in $\text{Meas}_c(G)$ and it acts on V^ρ. It is easy to see that if the action of $\text{Meas}_c(G)$ on V is strongly irreducible, then the action of A_ρ on V^ρ is strongly irreducible.

Theorem 10.4.5 follows from the combination of the following two statements:

Proposition 10.6.2. *There exists a family of finite-dimensional representations π_ρ of A_ρ, such that:*
(i) *Each π is of dimension $\leq n$ for $n = \dim(\rho)^2$;*
(ii) *For every element $a \in A_\rho$ there exists a π such that the action of a in π is non-zero.*

Proposition 10.6.3. *Let A be an associative algebra equipped with a family of finite-dimensional modules satisfying conditions (i) and (ii) from Proposition 10.6.2. Then if V is a topological vector spaced equipped with a strongly irreducible A-action, then* $\dim(V) \leq n$.

In the rest of this subsection we will prove Theorem 10.6.3.

10.6.4. For an associative algebra A and a positive integer r consider the map

$$P_r : A^{\otimes r} \to A, \quad (a_1, ..., a_r) \mapsto \sum_{\sigma \in \Sigma_r} (-1)\sigma \cdot a_{\sigma(1)} \cdot ... \cdot a_{\sigma(n)}.$$

Suppose that $A = \text{End}(\mathbb{C}^n)$. It is clear that $P_r = 0$ for $r \geq n^2$. Let $r(n)$ be the minimal integer such that $P_{r(n)}$ vanishes. It is easy to see that the function

$$n \mapsto r(n)$$

is strictly increasing.

Remark 10.6.5. The theorem of Amitzur-Levitzky says that $r = 2n$.

10.6.6. Let A be as in Proposition 10.6.3 it follows that $P_{r(n)}$ vanishes on A. Let now V be a topological vector space equipped with a strongly irreducible action of A. By density, we obtain that $P_{r(n)}$ vanishes also on $\text{End}(V)$.

Assume for the sake of contradiction that $\dim(V) > n$. Then we can split V as a direct sum

$$V = \mathbb{C}^{n+1} \oplus V',$$

where V' is some topological vector space. Then $\text{End}(V)$ contains a subalgebra isomorphic to $\text{End}(\mathbb{C}^{n+1})$. However, $P_{r(n)}$ is non-zero on $\text{End}(\mathbb{C}^{n+1})$, since $r(n+1) > r(n)$. This is a contradiction.

□

10.7. **Proof of Proposition 10.6.2.**

10.7.1. Let π be the set of all irreducible finite-dimensional (hence, algebraic) representations of G. Since G is linear (and so admits a faithful finite-dimensional representation), it is clear that for any $\phi \in \text{Meas}_c(G)$, there exists a π on which it acts non-trivially.

Take $\pi_\rho := \pi^\rho$, the ρ-isotypic component in π. We obtain that this family satisfies condition (ii). Hence, it remains to prove the following:

Theorem 10.7.2. *For every irreducible finite-dimensional representation π of G, the dimension of the space π^ρ is $\leq \dim(\rho)^2$.*

The rest of this subsection is devoted to the proof of this theorem.

10.7.3. Consider the corresponding complex groups $G_{\mathbb{C}}$ and $K_{\mathbb{C}}$. Let X be the flag variety of $G_{\mathbb{C}}$. We will use the following fundamental result (the Bott-Borel-Weil theorem):

Theorem 10.7.4. *Every irreducible representation of $G_{\mathbb{C}}$ can be realized as the space of regular sections of a G-equivariant line $\mathcal{L}$ bundle on X.*

10.7.5. Recall that the Iwasawe decomposition said that K acted transitively on G/P. This implies that the orbit of the unit point under the action of $K_{\mathbb{C}}$ on $(G/P)_{\mathbb{C}}$ is *open*. Since M is compact modulo its center, we obtain that the action of $K_{\mathbb{C}}$ on X also has an open orbit; denote it X^0.

The map

$$\Gamma(X, \mathcal{L}) \hookrightarrow \Gamma(X^0, \mathcal{L})$$

is an injective map of $K_{\mathbb{C}}$-representations.

It remains to show that for any ρ,

$$\dim(\mathrm{Hom}_{K_{\mathbb{C}}}(\rho, \Gamma(X^0, \mathcal{L}))) \leq \dim(\rho).$$

However, this is true for any $K_{\mathbb{C}}$-homogeneous space equipped with an equivariant line bundle. Indeed, $\Gamma(X^0, \mathcal{L})$ is a subrepresentation inside the regular representation $\mathrm{Reg}(K_{\mathbb{C}})$ of $K_{\mathbb{C}}$ (i.e., the space of regular functions on $K_{\mathbb{C}}$), and

$$\mathrm{Hom}_{K_{\mathbb{C}}}(\rho, \mathrm{Reg}(K_{\mathbb{C}})) \simeq \rho^*,$$

by Frobenius reciprocity.

□

11. Week 6, Day 1 (Thurs, Feb. 28)

11.1. More on the notion of infinitesimal equivalence.

11.1.1. Let V_1 and V_2 be two admissible G-representations, and let $S : M_1 \to M_2$ be a map between the underlying $(\mathfrak{g}, K)$-modules. It is (obviously) *not* true that one can continuously extend S to a map from V_1 to V_2. But one can do so "up to a hut":

Proposition 11.1.2. *There exists a (canonically defined) admissible G-representations $V_{1,2}$ equipped with maps*

$$V_1 \overset{S_1}{\longleftarrow} V_{1,2} \overset{S_2}{\to} V_2,$$

such that S_1 induces an isomorphism $M_{1,2} \to M_1$, and such that the resulting map

$$M_1 \overset{S_1}{\simeq} M_{1,2} \overset{S_2}{\longrightarrow} M_2$$

is the original S.

Proof. Consider the graph of S, which is a map

$$M_1 \to M_1 \oplus M_2$$

, and compose it with $M_1 \oplus M_2 \to V_1 \oplus V_2$.

Let $V_{1,2}$ be the closure of the image of M_1 in $V_1 \oplus V_2$. By Theorem 9.2.3, $V_{1,2}$ is a G-subrepresentation of $V_1 \oplus V_2$, with the required properties. □

11.1.3. Let $\mathcal{H}_G := \mathrm{Distr}_c(G)^{K\times K\text{-fin}}$ be the subspace of $\mathrm{Distr}_c(G)$ that consists of distributions that are K-finite with respect to both left and right translations. We have

$$\mathrm{Distr}_c(G)^{K\times K\text{-fin}} = \underset{\rho_1,\rho_2}{\oplus} \xi_{\rho_1} \star \mathrm{Distr}_c(G) \star \xi_{\rho_2},$$

where the notation ξ_ρ is as in Sect. 1.4.3 (it acts as a projector on the ρ-isotypic component).

The subspace $\mathcal{H}(G)$ is closed under convolutions, so it is a subalgebra in $\mathrm{Distr}_c(G)$. Note, however, that it is *non-unital*.

If V is an admissible G-representation, we have a canonically defined action of $\mathcal{H}(G)$ on $V^{K\text{-fin}}$. From Proposition 11.1.2 we obtain:

Corollary 11.1.4. *Let V_1 and V_2 be two admissible G-representations, and let $S : M_1 \to M_2$ be a map between the underlying $(\mathfrak{g}, K)$-modules. Then S intertwines the actions of $\mathcal{H}(G)$ on M_1 and M_2.*

Proof. The assertion is obvious when S comes from a map $V_1 \to V_2$. Now, Proposition 11.1.2 reduces us to this situation. □

In particular:

Corollary 11.1.5. *The action of $\mathcal{H}(G)$ on $V^{K\text{-fin}}$ depends only on the class of infinitesimal equivalence of V.*

11.1.6. Let V be an admissible G-representation, and let V^* be its dual. Note that

$$(V^*)^{K\text{-fin}} \simeq (V^{K\text{-fin}})^{*,\mathrm{alg}}.$$

Denote $M := V^{K\text{-fin}}$. Thus, for $m \in M$ and $m^* \in M^{*,\mathrm{alg}}$, we obtain the matrix coefficient function $\mathrm{MC}_{V,m\otimes m^*}$

$$g \mapsto \langle m^*, g \cdot m,$$

which is a C^∞-function on G

From Corollary 11.1.5 we obtain:

Corollary 11.1.7. *The function $\mathrm{MC}_{V,m\otimes m^*}$ only depends on the infinitesimal equivalence class of V.*

11.2. **Proof of Theorem 9.2.6.** :

11.2.1. Let M be an irreducible admissible $(\mathfrak{g}, K)$-module, equipped with a $(\mathfrak{g}, K)$-invariant Hermitian form. We want to show that there exists unitary representation of G such that M is its space of K-finite vectors.

The proof is based on the following proposition:

Proposition 11.2.2. *There exists a Banach realization V on M, such that*

$$(m, m) \leq ||m||^2.$$

Proof. We will use the fact that any irreducible $(\mathfrak{g}, K)$-module admits a Banach realization (to br proved later). Let V_1 be such a realization, and let $V_1^\dagger$ be its complex-conjugate dual. We have

$$M \simeq V_1^{K\text{-fin}} \text{ and } M^{\dagger,\mathrm{alg}} \simeq (V_1^\dagger)^{K\text{-fin}}.$$

However, the Hermitian form on M gives rise to an identification $M \simeq M^{\dagger,\mathrm{alg}}$. Thus, we obtain two embeddings

$$\imath : M \to V_1 \text{ and } \imath^\dagger : M \to V_1^\dagger.$$

Let V be the closure of the image of M under the diagonal embedding

$$M \to M \oplus M \stackrel{\imath \oplus \imath^\dagger}{\longrightarrow} V_1 \oplus V_1^\dagger.$$

Then V is a G-subrepresentation of $V_1 \oplus V_1^\dagger$, by Theorem 9.2.3. It is easy to see that it satisfies the requirements since

$$(m,m) \leq ||\imath(m)|| \cdot ||\imath^\dagger(m)|| \leq (||\imath(m)|| + ||\imath^\dagger(m)||)^2.$$

□

11.2.3. Let V be a Banach representation, supplied by Proposition 11.2.2. By definition, the Hermitian form $(-,-)$ extends continuously to V. We claim that it is G-invariant.

This would imply the assertion of Theorem 9.2.6 as the desired unitary representation can be defined as the completion of V with respect to $(-,-)$.

11.2.4. To prove the invariance, it suffices to show that for $m_1, m_2 \in M$, the function

$$f(g) = (g \cdot m_1, m_2) - (m_1, g^{-1} \cdot m_2) = 0.$$

Note that $(m_1, -)$ and $(-, m_2)$ are continuous functionals on V. Now, by Proposition 9.2.9, it suffices to show that all the derivatives of f at $1 \in G$ vanish. However, this follows from the $\mathfrak{g}$-invariance of $(-,-)$ on M.

□

11.3. **An algebra that controls (some) $(\mathfrak{g}, K)$-modules.**

11.3.1. Let ρ be an irreducible K-representation. Set

$$\mathbb{M}_\rho := U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} \rho.$$

This is a $(\mathfrak{g}, K)$-module, where we make K act by

$$k \cdot (u \otimes m) = \mathrm{Ad}_k(u) \otimes m.$$

It is is easy to see that

$$\mathrm{Hom}_{(\mathfrak{g},K)\text{-mod}}(\mathbb{M}_\rho, M) \simeq M^\rho.$$

Set

$$A_\rho := (\mathrm{End}_{(\mathfrak{g},K)\text{-mod}}(\mathbb{M}_\rho))^{\mathrm{op}}.$$

11.3.2. We have a pair of mutually adjoint functors

$$\Phi : A_\rho\text{-mod} :\rightleftarrows (\mathfrak{g}, K)\text{-mod} : \Psi,$$

where Ψ sends $M \in (\mathfrak{g}, K)$-mod to $\mathrm{Hom}_{(\mathfrak{g},K)\text{-mod}}(\mathbb{M}_\rho, M)$, viewed as a module over A_ρ, and Φ sends $Q \in A_\rho$-mod to

$$\mathbb{M}_\rho \underset{A_\rho}{\otimes} Q.$$

The functor Ψ is exact, and the functor Φ is only right exact.

Lemma 11.3.3. *The unit of the adjunction*

$$\mathrm{Id} \to \Psi \circ \Phi$$

is an isomorphism.

Proof. For $Q \in A_\rho$-mod, since the functor $M \mapsto M^\rho$ is exact, we have

$$(\mathbb{M}_\rho \underset{A_\rho}{\otimes} Q)^\rho \simeq (\mathbb{M}_\rho)^\rho \underset{A_\rho}{\otimes} Q \simeq A_\rho \underset{A_\rho}{\otimes} Q \simeq Q.$$

□

Corollary 11.3.4. *The functor Φ is fully faithful.*

11.3.5. We now claim:

Proposition 11.3.6.

(a) *If M is an irreducible $(\mathfrak{g}, K)$-module with $M^\rho \neq 0$, then $\Psi(M)$ is an irreducible A_ρ-module.*

(b) *If Q is an irreducible A_ρ-module, then $\Phi(Q)$ has a unique irreducible quotient, to be denoted M_Q. The composite map*

$$Q \to \Psi \circ \Phi(Q) \to \Psi(M_Q)$$

is an isomorphism.

Proof. For point (a), let Q be a submodule of $\Psi(M)$. By adjunction, we have a non-zero map $\Phi(Q) \to M$. Since M is irreducible, the above map is surjective. Since Ψ is exact, the map

$$Q \simeq \Psi \circ \Phi(Q) \to M$$

is surjective. Hence Q is all of $\Psi(M)$.

For point (b), by Zorn's lemma, $\Phi(Q)$ admits some irreducible quotient, denote it M. By adjunction, we have a non-zero map

$$Q \to \Psi(M).$$

However, by point (a), $\Psi(M)$ is irreducible, so the above map is an isomorphism.

Suppose now that $\Phi(Q)$ admits a surjective map to a direct sum $M_1 \oplus M_2$. We claim that one of these modules must be zero. Indeed, applying Ψ, we obtain a (still surjective) map

$$Q \simeq \Psi \circ \Phi(Q) \to \Psi(M_1) \oplus \Psi(M_2).$$

But Q is irreducible, so one of these maps, say $Q \to \Psi(M_2)$ must be zero. However, by adjunction, this means that the initial map $\Phi(Q) \to M_2$ was zero.

□

11.3.7. Let M be the Levi subgroup of the minimal parabolic Q of G. Denote $K_M := K \cap M$. This is the maximal compact subgroup in M. Recall that

$$M = K_M \times A.$$

In the next lecture we will construct an injective algebra homomorphism

$$(A_\rho)^{\mathrm{op}} \to \mathrm{End}_{K_M}(\rho) \otimes U(\mathfrak{a}). \tag{11.1}$$

11.4. **Admissibility of irreducible $(\mathfrak{g}, K)$-modules.** Our goal on this subsection is to give another proof Theorem 10.3.3, by a method that we will employ again.

11.4.1. Let us decompose ρ into irreducible K_M-modules:

$$\rho \simeq \underset{i}{\oplus} \pi_i^{\oplus n_i}.$$

Let $n = \max(n_i)$. We will show that any irreducible representation of A_ρ is finite-dimensional of dimension $\leq n$. This would prove Theorem 10.3.3 in view of Proposition 11.3.6.

11.4.2. Recall the notations of Sect. 10.6.4. Since $\mathfrak{a}$ is commutative, the polynomial $P_{r(n)}$ vanishes on $\mathrm{End}_{K_M}(\rho) \otimes U(\mathfrak{a})$. The existence of the homomorphism (11.1) implies that $P_{r(n)}$ vanishes on A_ρ.

We now claim:

Proposition 11.4.3. *Let A be an associative algebra such that $P_{r(n)}$ vanishes on A. Then any irreducible representation of A is finite-dimensional of dimension $\leq n$.*

Proof. We claim that if M is an A module that contains $n+1$ linearly independent vectors, then A has a subquotient isomorphic to $\mathrm{Mat}_{n+1,n+1}$. This would be a contradiction since $P_{r(n)}$ is non-zero on $\mathrm{Mat}_{n+1,n+1}$.

The existence of a subquotient follows from Burnside's theorem: if M is an irreducible A-module, and $m_1, ..., m_k \in M$ linearly independent vectors, and $m'_1, ..., m'_k$ some k-tuple of vectors, then there exists an element $a \in A$ such that

$$a \cdot m_i = m'_i, \quad i = 1, ..., k.$$

□

11.5. **Construction of the homomorphism** (11.1)**.**

11.5.1. Consider the tensor product

$$F_\rho := U(\mathfrak{m}) \underset{U(\mathfrak{q})}{\otimes} \mathbb{M}_\rho,$$

which we can also think of as

$$\mathbb{C} \underset{U(\mathfrak{n})}{\otimes} \mathbb{M}_\rho.$$

It is acted on by $\mathfrak{m}$ via the left action on the first factor. Moreover, it is easy to see as in Sect. 11.3.1 that F_ρ is naturally an $(\mathfrak{m}, K_M)$-module.

In addition, F_ρ carries a commuting action of $(A_\rho)^{\mathrm{op}}$ via the second factor.

11.5.2. Using the $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n} \oplus \mathfrak{k}$ decomposition, we can write

$$F_\rho := U(\mathfrak{m}) \underset{U(\mathfrak{q})}{\otimes} U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} \rho \simeq U(\mathfrak{m}) \underset{U(\mathfrak{q})}{\otimes} U(\mathfrak{q}) \underset{U(\mathfrak{k}_M)}{\otimes} \rho \simeq U(\mathfrak{m}) \underset{U(\mathfrak{k}_M)}{\otimes} \rho$$

as an $(\mathfrak{m}, K_M)$-module. So, this is the object of the same nature as $\mathbb{M}_\rho$ but for the pair $(\mathfrak{m}, K_M)$ and the K_M-representation ρ.

Further, writing $U(\mathfrak{m})$ (as an algebra) as

$$U(\mathfrak{m}) \simeq U(\mathfrak{a}) \otimes U(\mathfrak{k}_M),$$

we obtain that F_ρ is isomorphic to

$$U(\mathfrak{a}) \otimes \rho,$$

as an $(\mathfrak{m}, K_M)$-module, where $\mathfrak{m} \simeq \mathfrak{a} \oplus \mathfrak{k}_M$ acts via the action of $\mathfrak{a}$ on the first factor and $\mathfrak{k}_M$ on the second factor.

11.5.3. Thus, we obtain that the action of $(A_\rho)^{\mathrm{op}}$ on F_ρ gives rise to a map

$$(A_\rho)^{\mathrm{op}} \to \mathrm{End}_{(\mathfrak{m},K)\text{-mod}}(F_\rho) \simeq U(\mathfrak{a}) \otimes \mathrm{End}_{K_M}(\rho).$$

This is the desired map (11.1). Let us now prove that it is injective.

11.5.4. Consider the algebra $U(\mathfrak{g})^K$ of Ad_K-invariants in $U(\mathfrak{g})$. We claim that it naturally acts on $\mathbb{M}_\rho$ on the right by $(\mathfrak{g}, K)$-module endomorphisms. Indeed, this action is given by

$$(u \otimes v) \cdot u' \mapsto u \cdot u' \otimes v.$$

Hence, we obtain an algebra map

$$U(\mathfrak{g})^K \to A_\rho.$$

Let I_ρ be the kernel of the action map $U(\mathfrak{k}) \to \mathrm{End}(\rho)$. This is a two-sided ideal in $U(\mathfrak{k})$. Consider the left ideal $U(\mathfrak{g}) \cdot I_\rho \subset U(\mathfrak{g})$.

Note, however, that the intersection

$$J_\rho := U(\mathfrak{g})^K \cap U(\mathfrak{g}) \cdot I_\rho$$

is a two-sided ideal in $U(\mathfrak{g})^K$.

It is easy to see that the above map $U(\mathfrak{g})^K \to A_\rho$ factors through a map

$$U(\mathfrak{g})^K / J_\rho \to A_\rho. \tag{11.2}$$

Proposition 11.5.5.
(a) *The map* (11.2) *is an isomorphism.*
(b) *The composite map*

$$(U(\mathfrak{g})^K / J_\rho)^{\mathrm{op}} \to (A_\rho)^{\mathrm{op}} \to U(\mathfrak{a}) \otimes \mathrm{End}_{K_M}(\rho)$$

is injective.

Clearly, Proposition 11.5.5 implies that the map (11.1) is injective.

Proof of Proposition 11.5.5. Note that all objects in sight carry a filtration, induced by the canonical filtration on $U(\mathfrak{g})$. To prove the proposition, it suffices to show that both (a) and (b) hold at the associated graded level.

We have:

$$\mathrm{gr}(U(\mathfrak{g})^K / J_\rho) \simeq (\mathrm{gr}(U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} U(\mathfrak{k})/I_\rho))^K \simeq (\mathrm{gr}(U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} \mathrm{End}(\rho)))^K \simeq (\mathrm{Sym}(\mathfrak{g}/\mathfrak{k}) \otimes \mathrm{End}(\rho))^K$$

and

$$\mathrm{gr}\, A_\rho \simeq \mathrm{gr}(\mathrm{Hom}(\rho, U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} \rho)^K) \simeq (\mathrm{gr}(\mathrm{Hom}(\rho, U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} \rho))^K \simeq (\mathrm{Sym}(\mathfrak{g}/\mathfrak{k}) \otimes \mathrm{End}(\rho))^K,$$

and it is easy to see that under these identifications, the associated graded of (11.2) is the identity map on $(\mathrm{Sym}(\mathfrak{g}/\mathfrak{k}) \otimes \mathrm{End}(\rho))^K$. This proves point (a).

Similarly, the associated graded of the map (11.1) is the map

$$(\mathrm{Sym}(\mathfrak{g}/\mathfrak{k}) \otimes \mathrm{End}(\rho))^K \to (\mathrm{Sym}(\mathfrak{a}) \otimes \mathrm{End}(\rho))^{K_M}, \tag{11.3}$$

where the map $\mathfrak{g}/\mathfrak{k} \to \mathfrak{a}$ is

$$\mathfrak{g}/\mathfrak{k} \simeq \mathfrak{q}/\mathfrak{k}_M \to \mathfrak{m}/\mathfrak{k}_M \simeq \mathfrak{a}.$$

Thus, it remains to see that (11.3) is injective. Identifying $\mathfrak{g}$ with its dual via the Killing form and denoting $W := \mathrm{End}(\rho)$, interpret the above map as

$$(\mathcal{O}_\mathfrak{p} \otimes W)^K \to (\mathcal{O}_\mathfrak{a} \otimes W)^{K_M},$$

given by restriction along $\mathfrak{a} \hookrightarrow \mathfrak{p}$.

Now, the assertion follows from the fact that under the adjoint action of K on $\mathfrak{p}$, the orbit of $\mathfrak{a}$ is dense in $\mathfrak{p}$, see Sect. 8.5.5.

□

12. Week 6, Day 2 (Thurs, March 2)

12.1. **Induced representations.**

12.1.1. Let Q' be a parabolic subgroup of G; let N' denote its unipotent radical, and let M' denote its Levi quotient. Let W be a representation of M', which we regard as a representation of Q'.

We define the induced representation $I^G_{Q'}(W)$ as follows: its space consists of continuous functions

$$f : G \to W$$

that satisfy

$$f(g \cdot q) = q^{-1} \cdot f(g), \quad g \in G,\ q \in Q'.$$

Equivalently, we can identify this space with the space of functions $f : G/N' \to W$ that satisfy

$$f(g \cdot m) = m^{-1} \cdot f(g), \quad g \in G,\ m \in M',$$

since Q' acts on W via M'.

Frobenius reciprocity implies:

Lemma 12.1.2. *For a G-representation V, the space of continuous maps of G-representations $V \to I^G_{Q'}(W)$ identifies with the space of maps of Q'-representations $V \to W$.*

12.1.3. Let us identify what $I^G_{Q'}(W)$ looks like as a K-representation.

From the fact that $G = K \cdot Q'$ and $K \cap Q' = K \cap M' =: K_{M'}$ is the maxmal compact in M', we obtain that the restriction of $I^G_{Q'}(W)$ identifies with the space of functions

$$f : K \to W \text{ such that } f(k \cdot m) = m^{-1} \cdot f(k), \quad k \in G,\ m \in K_{M'},$$

i.e., we have a canonical isomorphism

$$\mathrm{Res}^G_K \circ I^G_{Q'} \simeq I^K_{K_{M'}} \circ \mathrm{Res}^{M'}_{K_{M'}} \tag{12.1}$$

This point of view lets us endow $I^G_{Q'}(W)$ with a variety of different topologies, and pass to the corresponding completions. For example if W is finite-dimensional, we can consider the L_p topology on (12.1). It is easy to see that the action of G will still be continuous (but it will not preserve the L_p norm; only the action of K does).

12.1.4. We claim:

Proposition 12.1.5. *Suppose that W is admissible. Then $I^G_{Q'}(W)$ is admissible.*

Proof. By (12.1) For a finite-dimensional representation ρ of K, we have

$$\mathrm{Hom}_K(\rho, I^G_{Q'}(W)) \simeq \mathrm{Hom}_{K_{M'}}(\rho, W).$$

□

12.2. **Induction for $(\mathfrak{g}, K)$-modules.**

12.2.1. We will now describe the algebraic counterpart of the induction construction. It will use D-modules. Consider the category $(\mathfrak{m}', K_{M'})$-mod. We have a functor

$$r^G_{M'} : (\mathfrak{g}, K)\text{-mod} \to (\mathfrak{m}', K_{M'})\text{-mod}, \quad M \mapsto M_{\mathfrak{n}'},$$

where the subscript $\mathfrak{n}'$ means taking coinvariants with respect to the Lie algebra $\mathfrak{n}'$. The result carries an action of $(\mathfrak{m}', K_M)$ because M' normalizes $\mathfrak{n}'$.

We can also rewrite $r^G_{M'}$ as

$$M \mapsto U(\mathfrak{m}') \underset{U(\mathfrak{q}')}{\otimes} M.$$

It is called the Jacquet functor.

12.2.2. The functor $r^G_{M'}$ admits a right adjoint by general nonsense. We will denote it by $i^G_{Q'}$.

We observe:

Proposition 12.2.3. *The natural transformations*

$$\mathrm{Res}^{(\mathfrak{g},K)}_K \circ i^G_{Q'} \to \mathrm{Ind}^K_{K_{M'}} \circ \mathrm{Res}^{(\mathfrak{m}',K_{M'})}_{K_{M'}}$$

and

$$\mathrm{coInd}^{(\mathfrak{m}',K_{M'})}_{K_{M'}} \circ \mathrm{Res}^K_{K_{M'}} \to r^G_{Q'} \circ \mathrm{coInd}^{(\mathfrak{g},K)}_K,$$

induced by the tautological natural transformation

$$\mathrm{Res}^K_{K_{M'}} \circ \mathrm{Res}^{(\mathfrak{g},K)}_K \to \mathrm{Res}^{(\mathfrak{m}',K_{M'})}_{K_{M'}} \circ r^G_{M'}$$

are isomorphisms.

In other words, whatever the functor $i^G_{Q'}$ happens to be, we know what it does at the level of K-representations: this is just induction from $K_{M'}$ to K, i.e., the following diagram commutes:

$$\begin{array}{ccc} (\mathfrak{g},K)\text{-mod} & \xleftarrow{i^G_{Q'}} & (\mathfrak{m}',K_{M'})\text{-mod} \\ {\scriptstyle \mathrm{Res}^{(\mathfrak{g},K)}_K}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}^{(\mathfrak{m}',K_{M'})}_{K_{M'}}} \\ K\text{-mod} & \longleftarrow & K_{M'}\text{-mod}. \end{array}$$

Note this is an algebraic counterpart of (12.1).

Proof. It suffices to prove the second isomorphism, as the first one is obtained by passing to right adjoints.

For the second isomorphism, we note that the functor $\mathrm{coInd}^{(\mathfrak{g},K)}_K$ is given by

$$\rho \mapsto U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} \rho$$

Using the fact that

$$\mathfrak{k} \oplus \mathfrak{q}' \twoheadrightarrow \mathfrak{g} \text{ and } \mathfrak{q}' \cap \mathfrak{k} = \mathfrak{k}_{M'},$$

we obtain:

$$U(\mathfrak{m}') \underset{U(\mathfrak{q}')}{\otimes} U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} \rho \simeq U(\mathfrak{m}') \underset{U(\mathfrak{q}')}{\otimes} U(\mathfrak{q}') \underset{U(\mathfrak{k}_{M'})}{\otimes} \rho \simeq U(\mathfrak{m}') \underset{U(\mathfrak{k})}{\otimes} \rho,$$

as desired.

□

Corollary 12.2.4. *The functor $i^G_{Q'}$ preserves admissibility.*

12.2.5. Let W be an admissible M'-representation. Let L denote the underlying $(\mathfrak{m}', K_{M'})$-module. We claim:

Proposition 12.2.6. *The $(\mathfrak{g}, K)$-module underlying $I^G_{Q'}(W)$ identifies canonically with $i^G_{Q'}(L)$.*

Proof. We first construct a map of $(\mathfrak{g}, K)$-modules

$$(I^G_{Q'}(W))^{K\text{-fin}} \to i^G_{Q'}(L). \tag{12.2}$$

By the definition of $i^G_{Q'}(L)$, the datum of such a map is equivalent to that of a map of $(\mathfrak{q}', K_{M'})$-modules

$$(I^G_{Q'}(W))^{K\text{-fin}} \to L. \tag{12.3}$$

By Frobenius reciprocity (i.e., Lemma 12.1.2) we have a map of Q'-representations

$$I^G_{Q'}(W) \to W.$$

Under this map the subspace $I^G_{Q'}(W))^{K\text{-fin}}$ gets sent to $W^{K_{M'}\text{-fin}} = L$. This defines the desired map (12.3). It is easy to see that it indeed respect the $(\mathfrak{q}', K_{M'})$-action.

Thus, by adjunction we also obtain the map (12.2). In order to see that this map is an isomorphism, it suffices to show that it is such at the level of K-representations.

We have:

$$\text{Res}^{(\mathfrak{g},K)}_K((I^G_{Q'}(W))^{K\text{-fin}}) \simeq (\text{Res}^G_K(I^G_{Q'}(W)))^{K\text{-fin}},$$

which by (12.1) identifies with

$$(I^K_{K_{M'}} \circ \text{Res}^{M'}_{K_{M'}}(W))^{K\text{-fin}},$$

and the latter is easily seen to identify with $\text{Ind}^K_{K'_M}(L)$.

Now,

$$\text{Res}^{(\mathfrak{g},K)}_K(i^G_{Q'}(L)) \simeq \text{Ind}^K_{K'_M}(L)$$

by Proposition 12.2.3.

By unwinding the definitions, it is easy to see that the resulting map from $\text{Ind}^K_{K'_M}(L)$ to itself is the identity.

□

12.3. **Explicit description of the induction functor on $(\mathfrak{g}, K)$-modules.** We will now describe the functor $i^G_{Q'}$ explicitly. In doing so we will used modules over algebraic differential operators, a.k.a., D-modules.

For the duration of this subsection we be in the context of algebraic geometry, so will write G, K, etc, for their complexifications.

12.3.1. Recall that the K-orbit of 1 in $X' := G/Q'$ is open; denote it by X'_0. This is an affine scheme. Denote

$$\widetilde{X}' := G/N';$$

this is a M'-torsor over X'. Let $\widetilde{X}'_0$ be the preimage of X'_0 in $\widetilde{X}'_0$. This is also an affine scheme.

The scheme $\widetilde{X}'$ carries an action of G (in particular K) by left translations, and a commuting action of M' by right translations.

Consider the ring of differential operators $D(\widetilde{X}'_0)$. The above action of G on $\widetilde{X}'$ gives rise to an algebra homomorphism

$$\iota^l : U(\mathfrak{g}) \to D(\widetilde{X}') \subset D(\widetilde{X}'_0)$$

and the right action of M' gives rise to an algebra homomorphism

$$\iota^r : U(\mathfrak{m}') \to D(\widetilde{X}') \subset D(\widetilde{X}'_0).$$

The images of these two algebras commute with each other. In particular, every D-module on $\widetilde{X}'_0$ carries a left action of $\mathfrak{g}$ and a right-action of $\mathfrak{m}$.

12.3.2. We let Y be the closed subscheme of $\widetilde{X}'_0$ equal to the orbit of 1 under the K-action; note Y is isomorphic to K as $K \cap N' = \{1\}$.

We will now consider a particular D-module on $\widetilde{X}'_0$, denoted it by $\delta_{Y,\widetilde{X}'_0}$, to be thought of as the δ-distribution[2] on Y inside $\widetilde{X}'_0$.

Explicitly, $\delta_{Y,\widetilde{X}'_0}$ is obtained by quotienting $D(\widetilde{X}'_0)$ by the left ideal gererated by:

(i) $I_Y \subset \mathcal{O}_{\widetilde{X}'_0} \subset D(\widetilde{X}'_0)$, where I_Y is the ideal of Y in $\widetilde{X}'_0$;

(ii) $\iota^l(\mathfrak{k}) \subset D(\widetilde{X}'_0)$.

Note that since Y is invariant under left translations by K, the elements from $\iota^l(\mathfrak{k})$ normalize the ideal I_Y.

12.3.3. We claim that the left action of $\mathfrak{g}$ on $\delta_{Y,\widetilde{X}'_0}$ extends to a structure of $(\mathfrak{g}, K)$-module, and the above action of $\mathfrak{m}'$ extends to a structure of $(\mathfrak{m}', K_{M'})$-module.

Indeed, the action of K on $\widetilde{X}'_0$ defines an action of K on $D(\widetilde{X}'_0)$. The derivative of this action is given by

$$\xi \in \mathfrak{k},\ d \in D(\widetilde{X}'_0) \mapsto \iota^l(\xi) \cdot d - d \cdot \iota^l(\xi).$$

This action of K preserves the ideal defining $\delta_{Y,\widetilde{X}'_0}$. Now, if $\xi \in \mathfrak{k}$, then the element $\iota^l(\xi)$ belongs to this ideal and so the above action of $\xi \in \mathfrak{k}$ on $\delta_{Y,\widetilde{X}'_0}$ is given by left multiplication by $\iota^l(\xi)$.

Similarly, the right action of M' on $\widetilde{X}'_0$ defines an action of M' on $D(\widetilde{X}'_0)$. The derivative of this action is given by

$$\xi \in \mathfrak{m}',\ d \in D(\widetilde{X}'_0) \mapsto \iota^r(\xi) \cdot d - d \cdot \iota^r(\xi).$$

The induced action of $K_{M'}$ preserves the ideal defining $\delta_{Y,\widetilde{X}'_0}$. Further, we claim that elements of the form $\iota^r(\xi)$ for $\xi \in \mathfrak{k}_{M'}$ belong to this ideal. Indeed, this follows from the fact that the restriction of the vector field $\iota^r(\xi)$ to Y is of the form

$$\sum_i f_i \cdot \iota^l(\xi_i), \quad f_i \in \mathcal{O}_Y,\ \xi_i \in \mathfrak{k}_{M'}.$$

Hence, the above action of $\xi \in \mathfrak{k}_{M'}$ on $\delta_{Y,\widetilde{X}'_0}$ is given by left multiplication by $\iota^r(\xi)$.

[2] In the D-module language, it is the direct image under $Y \hookrightarrow \widetilde{X}'_0$ of the D-module $\mathcal{O}_Y$ tensored with the line $(\Lambda^{\mathrm{top}}(\mathfrak{k}/\mathfrak{k}_{M'}))^{\otimes -1}$.

12.3.4. In what follows we will need the following construction. Let M_1 and M_2 be a right and a left $(\mathfrak{g}, K)$-modules respectively. In this case we can form the vector space[3]

$$M_1 \overset{K}{\underset{\mathfrak{g}/\mathfrak{k}}{\otimes}} M_2 := (M_1 \underset{U(\mathfrak{g})}{\otimes} M_2)^K.$$

Note that when $\mathfrak{g} = \mathfrak{k}$, the projection

$$(M_1 \otimes M_2)^K \to M_1 \overset{K}{\underset{\mathfrak{k}}{\otimes}} M_2$$

is an isomorphism (this can be seen by identifying K-invariants with K-coinvariants).

So, we should think of the functor

$$M_1, M_2 \mapsto M_1 \overset{K}{\underset{\mathfrak{g}/\mathfrak{k}}{\otimes}} M_2$$

as the operation of taking invariants in the K-direction and coinvariants with respect to $\mathfrak{g}/\mathfrak{k}$.

For example, if

$$M_1(\rho) = \mathrm{coInd}_K^{(\mathfrak{g},K)} := \rho \underset{U(\mathfrak{k})}{\otimes} U(\mathfrak{g}),$$

we have

$$M_1 \overset{K}{\underset{\mathfrak{g}/\mathfrak{k}}{\otimes}} M_2 \simeq (\rho \otimes M_2)^K. \tag{12.4}$$

Remark 12.3.5. In the particular case when G is compact modulo its center (a situation that will be of particular interest for us, because we will take $G = M$, the Levi of the minimal parabolic), the operation $\overset{K}{\underset{\mathfrak{g}/\mathfrak{k}}{\otimes}}$ can be described in simpler terms. Namely,

$$M_1 \overset{K}{\underset{\mathfrak{g}/\mathfrak{k}}{\otimes}} M_2 \simeq (M_1 \underset{U(\mathfrak{a})}{\otimes} M_2)^K.$$

12.3.6. We take the module $\delta_{Y,\widetilde{X}'_0}$, and given $L \in (\mathfrak{m}', K_{M'})$-mod we form

$$'i^G_{Q'}(L) := \delta_{Y,\widetilde{X}'_0} \overset{K_{M'}}{\underset{\mathfrak{m}'/\mathfrak{k}_{M'}}{\otimes}} L.$$

The action of $\mathfrak{g}$ via ι^l makes it into an object of $(\mathfrak{g}, K)$-mod. We will prove:

Theorem 12.3.7. *There is a canonical isomorphism of functors*

$$i^G_{Q'} \simeq {}'i^G_{Q'}, \quad (\mathfrak{m}', K_{M'})\text{-mod} \to (\mathfrak{g}, K)\text{-mod}.$$

12.4. **Proof of Theorem 12.3.7.**

[3]The definition given below is fine when K is reductive, and when we are interested in the non-derived functor. In general, more care is needed.

12.4.1. We first claim that there exists a canonical isomorphism of functors

$$\operatorname{Res}^{(\mathfrak{g},K)}_K \circ {'i}^G_{Q'} \to \operatorname{Ind}^K_{K_{M'}} \circ \operatorname{Res}^{(\mathfrak{m}',K_{M'})}_{K_{M'}}, \tag{12.5}$$

i.e., that at the level of K-representations, ${'i}^G_{Q'}$ does the right thing.

Indeed, we claim that $\delta_{Y,\widetilde{X}'_0}$ when regarded as a K-module with respect to the left action and a $(\mathfrak{m}', K_{M'})$-module with respect to the right action, identifies canonically with

$$\mathcal{O}_Y \underset{U(\mathfrak{k}_{M'})}{\otimes} U(\mathfrak{m}).$$

This would imply the isomorphism (12.5) in view of (12.4), since $Y \simeq K$ as a $K \times K_{M'}$-scheme.

To establish the desired description of $\delta_{Y,\widetilde{X}'_0}$ we note that there is a canonical map

$$\mathcal{O}_Y \to \delta_{Y,\widetilde{X}'_0}$$

as $K \times K_{M'}$-modules. By adjunction, this gives rise to a map

$$\mathcal{O}_Y \underset{U(\mathfrak{k}_{M'})}{\otimes} U(\mathfrak{m}) \to \delta_{Y,\widetilde{X}'_0}$$

as K-modules and $(\mathfrak{m}', K_{M'})$-modules. To check that this map is an isomorphism, it is enough to do so at the associate graded level.

The latter, however, is easily seen to be the identity map on

$$\mathcal{O}_Y \otimes \operatorname{Sym}(\mathfrak{m}'/\mathfrak{k}_{M'}).$$

12.4.2. We next consider the case of $Q' = G$. We claim that ${'i}^G_G$ is indeed isomorphic to the identity functor on $(\mathfrak{g}, K)$-mod. I.e., we claim:

Proposition 12.4.3. *The functor*

$$M \mapsto \delta_{K,G} \overset{K}{\underset{\mathfrak{g}/\mathfrak{k}}{\otimes}} M$$

is isomorphic to the identity functor on $(\mathfrak{g}, K)$-mod.

Proof. For $M \in (\mathfrak{g}, K)$-mod, the action of K on M gives rise to a map

$$M \to \mathcal{O}_K \otimes M.$$

Its image lies in $(\mathcal{O}_K \otimes M)^K$, where we consider the diagonal action of K comprised from the given action on M and the action of K on $\mathcal{O}_K$ by right translations.

Composing, we obtain a map

$$M \to (\mathcal{O}_K \otimes M)^K \to (\delta_{K,G} \otimes M)^K \to \delta_{K,G} \overset{K}{\underset{\mathfrak{g}/\mathfrak{k}}{\otimes}} M.$$

It is straightforward to check that the above map is a map of $(\mathfrak{g}, K)$-modules. This defines a natural transformation

$$\operatorname{Id} \to {'i}^G_G.$$

To check that it is an isomorphism, it suffices to show that the induced natural transformation

$$\operatorname{Res}^{(\mathfrak{g},K)}_K \to \operatorname{Res}^{(\mathfrak{g},K)}_K \circ {'i}^G_G$$

is an isomorphism.

However, it is straightforward to check that the above map is the isomorphism of (12.5) in the particular case of $Q' = G$.

□

12.4.4. Next we construct a natural transformation

$$r^G_{Q'} \circ {}'i^G_{Q'} \to \mathrm{Id}\,. \tag{12.6}$$

In view of Proposition 12.4.3, this amounts to constructing a map of $(\mathfrak{q}', K_{M'})$-modules

$$\delta_{Y,\widetilde{X}'_0} \to \delta_{K_{M'},M'}$$

that respects the $(\mathfrak{q}', K_{M'})$-action on the left and the $(\mathfrak{m}', K_{M'})$-action on the right.

Consider M' as a subscheme in $\widetilde{X}'_0$ (i.e., the fiber over 1 of $G/N' \to G/Q'$) and consider the tensor product

$$\mathcal{O}_{M'} \underset{\mathcal{O}_{\widetilde{X}'_0}}{\otimes} \delta_{Y,\widetilde{X}'_0}.$$

This is a D-module on M'. The projection

$$\delta_{Y,\widetilde{X}'_0} \to \mathcal{O}_{M'} \underset{\mathcal{O}_{\widetilde{X}'_0}}{\otimes} \delta_{Y,\widetilde{X}'_0}$$

respects the $(\mathfrak{q}', K_{M'})$-action on the left and the $(\mathfrak{m}', K_{M'})$-action on the right.

Thus, it suffices to construct an isomorphism of D-modules on M'

$$\delta_{K_{M'},M'} \simeq \mathcal{O}_{M'} \underset{\mathcal{O}_{\widetilde{X}'_0}}{\otimes} \delta_{Y,\widetilde{X}'_0}.$$

We construct a map

$$D(M) \to \mathcal{O}_{M'} \underset{\mathcal{O}_{\widetilde{X}'_0}}{\otimes} \delta_{Y,\widetilde{X}'_0}$$

by sending the generator to $1 \in \mathcal{O}_{M'}$. It is easy to see that this map factors via a map

$$\delta_{K_{M'},M'} \to \mathcal{O}_{M'} \underset{\mathcal{O}_{\widetilde{X}'_0}}{\otimes} \delta_{Y,\widetilde{X}'_0}. \tag{12.7}$$

To show that the latter map is an isomorphism, we do it at the associated graded level. However, it is easy to see that gr of both sides identifies naturally with

$$\mathcal{O}_{K_{M'}} \otimes \mathrm{Sym}(\mathfrak{m}/\mathfrak{k}_{M'})$$

so that (12.7) induces the identity map.

12.4.5. By adjunction, the map (12.6) constructed above, gives rise to a natural transformation

$${}'i^G_{Q'} \to i^G_{Q'}.$$

We claim that this natural transformation is an isomorphism. To prove this, it is sufficient to show that the induced natural transformation

$$\mathrm{Res}^{(\mathfrak{g},K)}_K \circ {}'i^G_{Q'} \to \mathrm{Res}^{(\mathfrak{g},K)}_K \circ i^G_{Q'}$$

is an isomorphism.

However, by (12.5), the left-hand side identifies with $\mathrm{Ind}^K_{K_{M'}} \circ \mathrm{Res}^{(\mathfrak{m}',K_{M'})}_{K_{M'}}$, and the right-hand side identifies with the same by Proposition 12.2.3.

Now, by unwinding the constructions, we see that the resulting natural transformation from $\mathrm{Ind}^K_{K_{M'}} \circ \mathrm{Res}^{(\mathfrak{m}',K_{M'})}_{K_{M'}}$ to itself is the identity map.

□

13. Week 7, Day 1 (Tue., March 7)

13.1. **Harish-Chandra homomorphism.**

13.1.1. Let Q' be a parabolic in G. Consider the tensor product

$$F := U(\mathfrak{m}') \underset{U(\mathfrak{p}')}{\otimes} U(\mathfrak{g}),$$

as a right $\mathfrak{g}$-module and a left $\mathfrak{m}'$-module.

We claim:

Proposition 13.1.2. *The action of $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ on F factors through a uniquely defined map*

$$\phi : Z(\mathfrak{g}) \to Z(\mathfrak{m}')$$

and the action of $Z(\mathfrak{m}') \subset U(\mathfrak{m})$. Furthermore, ϕ is injective and $Z(\mathfrak{m}')$ is finite as a $Z(\mathfrak{g})$-module.

Remark 13.1.3. The above map is most commonly used when $Q' = B$, and so $\mathfrak{m}' = \mathfrak{h}$. It is usually referred to as the Harish-Chandra homomorphism.

Proof of Proposition 13.1.2. An element $z \in Z(\mathfrak{g})$ defines an endomorphism of F as a $\mathfrak{g}$-module and as a $\mathfrak{m}'$-module. We claim that for any endomorphism S of F as an $\mathfrak{g}$-module, the image $S(1)$ of the element $1 \in F$ lies in

$$U(\mathfrak{m}') \simeq U(\mathfrak{m}') \underset{U(\mathfrak{p}')}{\otimes} U(\mathfrak{p}') \subset U(\mathfrak{m}') \underset{U(\mathfrak{p}')}{\otimes} (U(\mathfrak{p}') \otimes U(\mathfrak{n}'^{-})) \simeq U(\mathfrak{m}') \underset{U(\mathfrak{p}')}{\otimes} U(\mathfrak{g}) = F.$$

Indeed, $S(1)$ is invariant under the adjoint action of $\mathfrak{z}_{M'} \subset \mathfrak{m}'$, but the subspace of such elements in $U(\mathfrak{n}'^{-})$ equals $\mathbb{C}$.

Hence $1 \cdot z = \phi(z) \cdot 1$ for a well-defined element $\phi(z) \in U(\mathfrak{m})$. For $u \in U(\mathfrak{m}')$ we have

$$u \cdot \phi(z) = u \cdot z = z \cdot u = \phi(z) \cdot u,$$

(the second equality since $z \in Z(\mathfrak{g})$), as elements in $U(\mathfrak{m}') \subset F$. Hence, $\phi(z) \in Z(\mathfrak{m}')$. A similar argument shows that ϕ is an algebra homomorphism.

To show that ϕ injective and $Z(\mathfrak{m}')$ is finite as a $Z(\mathfrak{g})$-module it is enough to do so at the associated graded level. By transitivity, it is enough to consider the case of $Q' = B$. The corresponding map is

$$\mathrm{Sym}(\mathfrak{g})^G \to \mathrm{Sym}(\mathfrak{g}/\mathfrak{n})^T \simeq \mathrm{Sym}(\mathfrak{h}).$$

Identifying $\mathfrak{g}$ with its dual via the Killing form, the latter map is the restriction map

$$\mathrm{Sym}(\mathfrak{g})^G \to \mathrm{Sym}(\mathfrak{h}),$$

which is injective because the orbit if $\mathfrak{h}$ in $\mathfrak{g}$ under the adjoint action is dense.

□

13.1.4. Consider the functor $r^G_{Q'}$. We claim:

Lemma 13.1.5. *For $\mathcal{M} \in (\mathfrak{g}, K)$-mod, the action of $Z(\mathfrak{g})$ on $r^G_{Q'}(\mathcal{M})$ induced by the $Z(\mathfrak{g})$-action on $\mathcal{M}$ equals the action obtained from $\phi : Z(\mathfrak{g}) \to Z(\mathfrak{m}')$ and the action of $Z(\mathfrak{m}')$ on $r^G_{Q'}(\mathcal{M})$ as an $\mathfrak{m}'$-module.*

Proof. Follows from the fact that the functor $\mathfrak{g}$-mod $\to \mathfrak{m}'$-mod underlying $r^G_{Q'}$ is given by

$$\mathcal{M} \mapsto F \underset{U(\mathfrak{g})}{\otimes} \mathcal{M}.$$

□

By adjunction, we obtain:

Corollary 13.1.6. *For $\mathcal{L} \in (\mathfrak{m}', K_{M'})$-mod, the action of $Z(\mathfrak{g})$ on $i^G_{Q'}(\mathcal{L})$ as a $(\mathfrak{g}, K)$-module equals the action induced by the action of $Z(\mathfrak{g})$ on $\mathcal{L}$ via ϕ.*

13.1.7. We claim:

Proposition 13.1.8.
(a) *The functor $r^G_{M'}$ sends finitely generated objects in* $(\mathfrak{g}, K)$-mod *to finitely generated objects in* $(\mathfrak{m}, M')$-mod.
(b) *The functor $r^G_{M'}$ sends objects of finite length in* $(\mathfrak{g}, K)$-mod *to objects of finite length in* $(\mathfrak{m}, M')$-mod.

Proof. Point (a) follows from the second isomorphism in Proposition 12.2.3. Point (b) follows from point (a) and Corollary 10.3.5. □

Remark 13.1.9. The proof of Proposition 13.1.8 used Corollary 10.3.5, and that in turned used the (non-trivial) Theorem 10.2.6, applied to the reductive group M'. However, when Q' is the minimal parabolic Q, so that the Levi $M' = M$ is compact modulo its center, the conclusion of Theorem 10.2.6 is evident.

So, we obtain that the corresponding functor r^G_Q sends $(\mathfrak{g}, K)$-modules of finite length to finite-dimensional $(\mathfrak{m}, K_M)$-modules.

The proof amounts to the fact that for $\mathcal{M} \in (\mathfrak{g}, K)$-mod of finite length, the $(\mathfrak{m}, K_M)$-module $r^G_M(\mathcal{M})$ is finitely generated and has finite support over $Z(\mathfrak{g})$. This implies that its support over $Z(\mathfrak{m})$ is also finite. The latter means that its support in $\operatorname{Spec}(U(\mathfrak{a}))$ is finite and that it has only finitely many K_M-isotypics components.

13.2. **The subquotient theorem.**

13.2.1. In this subsection we will prove the following fundamental result, known as the Subquotient Theorem:

Theorem 13.2.2. *Every irreducible $(\mathfrak{g}, K)$-module can be realized as a subquotient of $i^G_P(\mathcal{L})$, where P is the minimal parabolic and $\mathcal{L}$ is a finite-dimensional $(\mathfrak{m}, K_M)$-module.*

The particular significance of this theorem is explained by the following corollary:

Theorem 13.2.3. *Every irreducible $(\mathfrak{g}, K)$-module can be realized as the $(\mathfrak{g}, K)$-module underlying an admissible representation of G on a Banach (and even Hilbert) space.*

Let us see how Theorem 13.2.2 implies Theorem 13.2.3:

Proof of Theorem 13.2.3. Let us realize a given irreducible $(\mathfrak{g}, K)$-module M as a subquotient of $i^G_Q(\mathcal{L})$. By Proposition 12.2.6 we can realize $i^G_Q(\mathcal{L})$ as the space of K-finite vectors in $I^G_Q(\mathcal{L})$, which is a Banach representation of G (and also in the L_2-version of $I^G_Q(\mathcal{L})$, which is a Hilbert (but not unitary!) representation of G).

Hence, by Theorem 9.2.3, M can be realized as a subquotient of $I^G_Q(L)$ (or its L_2 version). □

Remark 13.2.4. Note that Theorem 13.2.2 does not explicitly mention the fact that the $(\mathfrak{g}, K)$-modules $i^G_Q(L)$ have a finite length. The latter is true, but is a hard theorem, essentially equivalent to Theorem 10.2.6, see below.

13.2.5. Let us make the following observation on the structure of the theory:

Proposition 13.2.6. *The following assertions are logically equivalent:*

(i) *For an irreducible* $(\mathfrak{m}, K_M)$*-module* $\mathcal{L}$*, the* $(\mathfrak{g}, K)$*-module* $i^G_Q(\mathcal{L})$ *is of finite length.*

(i') *For a finite-dimensional* $(\mathfrak{m}, K_M)$*-module* $\mathcal{L}$*, the* $(\mathfrak{g}, K)$*-module* $i^G_Q(\mathcal{L})$ *is finitely generated.*

(ii) *The assertion of Theorem 10.2.6 holds, i.e., for every character* χ *of* $Z(\mathfrak{g})$ *there are only finitely many classes of irreducible objects in* $(\mathfrak{g}, K)\text{-mod}_\chi$.

Proof. Let us show that (i) implies (ii). By Theorem 13.2.2, we know that every irreducible $(\mathfrak{g}, K)$-module can be realized as a subquotient of $i^G_Q(\mathcal{L})$ for some irreducible $(\mathfrak{m}, K_M)$-module $\mathcal{L}$. By Corollary 13.1.6, if $\mathcal{L}$ is irreducible, then $Z(\mathfrak{g})$ acts on $i^G_Q(\mathcal{L})$ by a single character, obtained via the Harish-Chandra homomorphism $\phi : Z(\mathfrak{g}) \to Z(\mathfrak{g})$ from the character by which $Z(\mathfrak{m})$ acts on $\mathcal{L}$.

Hence, it suffices to show that, given χ, there are only finitely many irreducible $(\mathfrak{m}, K_M)$-modules $\mathcal{L}$, on which $Z(\mathfrak{g})$ acts by χ. Since the map $\text{Spec}(Z(\mathfrak{m})) \to \text{Spec}(Z(\mathfrak{g}))$ is finite, it suffices to show that for a given character χ' of $Z(\mathfrak{m})$, there are at most finitely many irreducible $(\mathfrak{m}, K_M)$-modules $\mathcal{L}$, on which $Z(\mathfrak{m})$ acts by χ'. In fact, there is at most one such module:

Te datum of χ' determines the action of $\mathfrak{a}$, and it is known that finite-dimensional representations of a compact group (in our case K_M) are distinguished by the action of the center[4] of the universal enveloping algebra.

Let us show that (ii) implies (i). We have seen that Theorem 10.2.6 implies Corollary 10.3.5. In particular, in order to prove that $i^G_Q(\mathcal{L})$ is of finite length it is sufficient to it is admissible (which follows from Corollary 12.2.4) and that its support over $Z(\mathfrak{g})$ is finite (which follows from Corollary 13.1.6).

Clearly (i) implies (i'). For the inverse implication we will use the fact (to be proved next time) that the algebraic dual of $i^G_Q(\mathcal{L})$ is isomorphic to $i^G_Q(\mathcal{L}^*)$, where $\mathcal{L}^*$ is the dual of $\mathcal{L}$ (up to a ρ-shift). If

$$... \subset \mathcal{M}_i \subset \mathcal{M}_{i+1} \subset ...$$

is an infinite chain of subobjects of $i^G_Q(\mathcal{L})$ (with non-zero subquotients), consider the corresponding chain

$$... \subset (\mathcal{M}_{i+1})^\perp \subset (\mathcal{M}_i)^\perp \subset ...$$

in $i^G_Q(\mathcal{L}^*)$. Assuming (i') and using the fact that the category $(\mathfrak{g}, K)$-mod is Noetherian, we obtain that both these chains stabilize on the right. This is a contradiction.

□

13.3. **Proof of the subquotient theorem.**

13.3.1. Let ρ be an irreducible representation of K. Recall the algebra A_ρ from Sect. 11.3. We claim that it suffices to show that every irreducible A_ρ-module can be realized as a subquotient of $\Psi(i^G_Q(\mathcal{L})) = \text{Hom}_K(\rho, i^G_Q(\mathcal{L}))$ for some finite-dimensional representation L of M.

Indeed, let $\mathcal{M}$ be an irreducible $(\mathfrak{g}, K)$-module. Let ρ be such that $\mathcal{M}^\rho \neq 0$. Denote $\mathcal{Q} := \Psi(\mathcal{M})$. Let L be such that $\mathcal{Q}$ can be realized as a subquotient of $\Psi(i^G_Q(\mathcal{L}))$. I.e., we have submodules

$$\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \Psi(i^G_Q(\mathcal{L}))$$

and an isomorphism $\mathcal{Q} \simeq \mathcal{Q}_2/\mathcal{Q}_1$.

[4]If we just consider the action of the Casimir, for every eigenvalue there will be at most finitely many irreducibles on which it acts with this eigenvalue.

Consider the corresponding maps

$$\Phi(\mathcal{Q}_1) \to \Phi(\mathcal{Q}_2) \to i_Q^G(\mathcal{L}).$$

We claim that the composition

$$\Phi(\mathcal{Q}_2) \to i_Q^G(\mathcal{L}) \to \operatorname{coker}(\Phi(\mathcal{Q}_1) \to i_Q^G(\mathcal{L}))$$

is non-zero. Indeed, if it were, then applying the functor Ψ we would obtain that the composition

$$\Psi \circ \Phi(\mathcal{Q}_2) \to \Psi(i_Q^G(\mathcal{L})) \to \operatorname{coker}(\Psi \circ \Phi(\mathcal{Q}_1) \to \Psi(i_Q^G(\mathcal{L})))$$

is zero. However, since $\Psi \circ \Phi \simeq \mathrm{Id}$, the latter map identifies with

$$\mathcal{Q}_2 \to \Psi(i_Q^G(\mathcal{L})) \to \operatorname{coker}(\mathcal{Q}_1 \to \Psi(i_Q^G(\mathcal{L}))),$$

which was non-zero by assumption.

Therefore, since Φ is right exact, we obtain that $\Phi(\mathcal{Q})$ surjects onto a subquotient, (denote it $\mathcal{M}'$) of $i_Q^G(L)$. We claim that in this case $\mathcal{M}$ is a quotient of $\mathcal{M}'$. Indeed, this follows from the fact that $\Phi(\mathcal{Q})$ has a unique irreducible quotient (Proposition 11.3.6(b)).

13.3.2. Let us now write down explicitly the functor $\Psi \circ i_Q^G$. (This will be a lot easier than writing down i_Q^G itself).

By definition, the functor $\Psi \circ i_Q^G$ is the right adjoint of the functor $r_Q^G \circ \Phi$. Now, the latter functor sends

$$\mathcal{Q} \mapsto U(\mathfrak{m}) \underset{U(\mathfrak{p})}{\otimes} U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} \rho \underset{A_\rho}{\otimes} \mathcal{Q} \simeq F_\rho \underset{A_\rho}{\otimes} \mathcal{Q},$$

where we regard F_ρ as an $(\mathfrak{m}, K_M)$-module equipped with a commuting right action of A_ρ. Note that F_ρ, veiwed as an $(\mathfrak{m}, K_M)$-module identifies with $r_Q^G(U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} \rho)$.

Hence, the functor $\Psi \circ i_Q^G$ is given by

$$\mathcal{L} \mapsto \operatorname{Hom}_{(\mathfrak{m},K_M)\text{-mod}}(F_\rho, \mathcal{L}).$$

Recall that F_ρ, when viewed as an $(\mathfrak{m}, K_M)$-module, isomorphic to

$$U(\mathfrak{m}) \underset{U(\mathfrak{k}_M)}{\otimes} \rho.$$

So, the composition of $\Psi \circ i_Q^G$ with the forgetful functor $A_\rho\text{-mod} \to \mathrm{Vect}$ is just

$$\mathcal{L} \mapsto \operatorname{Hom}_{K_M}(\rho, \mathcal{L}).$$

We also note:

Lemma 13.3.3. *The module F_ρ are finitely generated over $Z(\mathfrak{g})$.*

Proof. Note that by Lemma 13.1.5 the $Z(\mathfrak{g})$-action on F_ρ, equals the action obtained from $\phi : Z(\mathfrak{g}) \to Z(\mathfrak{m})$ and the $Z(\mathfrak{m})$ action on F_ρ as a $(\mathfrak{m}, K_M)$-module. Clearly, F_ρ is finitely generated as a module over $U(\mathfrak{a})$, and hence over all of $Z(\mathfrak{m})$. Now the assertion follows from the fact that $Z(\mathfrak{m})$ is finitely generated as a module over $Z(\mathfrak{g})$.

□

13.3.4. Since the (right) action of A_ρ on F_ρ is faithful by Sect. 11.5, (and F_ρ is finitely generated over $U(\mathfrak{a})$), we can find an embedding

$$A_\rho \hookrightarrow F_\rho^{\oplus k} \tag{13.1}$$

as right A_ρ-modules, for some k.

Let $\mathcal{Q}$ be an irreducible (automatically, finite-dimensional) module over A_ρ; consider its linear dual $\mathcal{Q}^*$ as a right A_ρ-module; it is irreducible. In particular, it can be realized as a quotient of A_ρ. It suffices to show that $\mathcal{Q}$ can be realized as a subquotient of $\mathrm{Hom}_{(\mathfrak{m},K_M)\text{-mod}}(F_\rho^{\oplus k}, \mathcal{L})$ for some $\mathcal{L}$.

Let χ be the character of $Z(\mathfrak{g})$ through which it acts on $\mathcal{Q}$. Then $Z(\mathfrak{g})$ acts by the same character on $\mathcal{Q}^*$. Let $I_\chi \subset Z(\mathfrak{g})$ be the corresponding ideal. Recall that by Lemma 13.3.3, F_ρ is finitely generated as a $Z(\mathfrak{g})$-module. It follows from Artin-Rees applied to the embedding (13.1) that A_ρ/I_χ is a subquotient of $F' := F_\rho^{\oplus k}/I_\chi^n$ for some n.

This F' is an $(\mathfrak{m}, K_M)$-module that carries a commuting right action of A_ρ. It suffices to show that $\mathcal{Q}$ can be realized as a subquotient of $\mathrm{Hom}_{(\mathfrak{m},K_M)\text{-mod}}(F', \mathcal{L})$ for some finite-dimensional $(\mathfrak{m}, K_M)$-module $\mathcal{L}$.

Note that F' is finite-dimensional as a vector space. Take $\mathcal{L} = F' \otimes (F')^*$, considered as a module over $(\mathfrak{m}, K_M)$ via the action on the first factor. We claim that it does the job.

Indeed, the evaluation map $F' \otimes (F')^* \to \mathbb{C}$ defines a surjection

$$\mathrm{Hom}_{(m,M)}(F', \mathcal{L}) \to (F')^*$$

as left A_ρ-modules. However, $(F')^*$ contains $\mathcal{Q}$ as a subquotient, by construction.

14. Week 7, Day 2 (Tue., March 9)

14.1. **Normalized induction and duality on induced representations.**

14.1.1. Consider the adjoint action of $\mathfrak{q}'$ on $\mathfrak{q}'$, and let us consider the top exterior power of this action. We obtain an (algebraic) character of the group Q', i.e., a homomorphism

$$Q' \to \mathbb{G}_m,$$

which automatically factors via a homomorphism

$$M' \to \mathbb{G}_m. \tag{14.1}$$

In its turn, the homomorphism (14.1) factors through a homomorphism

$$M'/[M', M'] \to \mathbb{G}_m. \tag{14.2}$$

Note that $M'/[M', M']$ is a connected torus defined over $\mathbb{R}$. Therefore it (rather, the group of its $\mathbb{R}$-points) can be written as

$$K_{M'/[M',M']} \times A_{M'/[M',M']}.$$

(Note that when Q' is the minimal parabolic Q, then $A_{M'/[M',M']} \simeq A$.)

Since $K_{M'/[M',M']}$ is connected, (14.2) factors through a homomorphism

$$A_{M'/[M',M']} \to \mathbb{R}^{>0,\times}. \tag{14.3}$$

We denote the latter by $2\rho_{Q'}$. By a slight abuse of notation, we will denote by the same symbol the composite homomorphism

$$M'(\mathbb{R}) \to M'/[M', M'](\mathbb{R}) \to A_{M'/[M',M']} \to \mathbb{R}^{>0,\times}$$

and also the corresponding characters of Lie algebras.

Explicitly, let us write $\mathfrak{n}'$ as a sum of $\mathfrak{a}_{M'/[M',M']}$-eigenspaces

$$\mathfrak{n}' \simeq \bigoplus_{\alpha'} (\mathfrak{n}')_{\alpha'}.$$

Then

$$2\rho_{Q'} = \sum_{\alpha'} \alpha' \cdot \dim((\mathfrak{n}')_{\alpha'}).$$

14.1.2. Let us once and for all choose a trivialization of the line $\Lambda^{top}(\mathfrak{g}/\mathfrak{q}^-)$. The key observation is the following:

Proposition 14.1.3. *There exists a canonically defined G-invariant functional*

$$\int_{G/Q'} : I^G_{Q'}(2\rho_{Q'}) \to \mathbb{C}.$$

Proof. By definition $I^G_{Q'}(2\rho_{Q'})$ is the space of scalar-valued functions on G that satisfy

$$f(g \cdot p) = (-2\rho_{Q'})(p^{-1}) \cdot f(g).$$

We claim that this vector space identifies canonically with the space of top continuous degree differential forms on G/Q'. This follows from the fact that the action of Q' on the top exterior power of $T_e(G/Q') \simeq \mathfrak{g}/\mathfrak{q}'$ is given by $2\rho_{Q'}$.

Now the sought-for functional is given by integration.

□

14.1.4. The homomorphism (14.3) has a well-defined square root, which we denote by $\rho_{Q'}$. We denote the *normalized induction* functor

$$I^{n,G}_{Q'} : \mathrm{Rep}(M') \to \mathrm{Rep}(G')$$

by

$$I^{n,G}_{Q'}(W) := I^G_{Q'}(W \otimes (-\rho_{Q'})).$$

Note that $I^{n,G}_{Q'}(W)$ and $I^G_{Q'}(W)$ are the same as K-representations. This is because $-\rho_{Q'}$ is trivial on $K_{M'}$.

14.1.5. Let W be a representation of M and let W^* be its dual. From Proposition 14.1.3 we obtain that there exists a canonically defined G-invariant pairing

$$I^{n,G}_{Q'}(W) \times I^{n,G}_{Q'}(W^*) \to \mathbb{C}, \quad f_1, f_2 \mapsto \int_{G/Q'} \langle f_1, f_2 \rangle. \tag{14.4}$$

We clam:

Proposition 14.1.6. *Let W is admissible. Then the induced pairing*

$$(I^{n,G}_{Q'}(W))^{K\text{-fin}} \otimes (I^{n,G}_{Q'}(W^*))^{K\text{-fin}} \to \mathbb{C}$$

is perfect, i.e., identifies $(I^{n,G}_{Q'}(W^*))^{K\text{-fin}}$ with

$$((I^{n,G}_{Q'}(W))^{K\text{-fin}})^{*,\mathrm{alg}}.$$

Proof. It is enough to verify the statement at the level of K-representations. In the latter case, the pairing in question is given by

$$\mathrm{Ind}_{K_M}^K(\mathcal{L}) \otimes \mathrm{Ind}_{K_M}^K(\mathcal{L}^{*,\mathrm{alg}}) \to \mathrm{Ind}_{K_M}^K(\mathbb{C}) \to \mathbb{C},$$

where the latter map is the K-invariant (!) integration over $K/K_M \simeq G/P$, or which is the same as unique splitting of the map

$$\mathbb{C} \hookrightarrow \mathrm{Ind}_{K_M}^K(\mathbb{C})$$

as K-representations.

Now, for a given $\rho \in \mathrm{Irrep}(K)$, we have

$$(\mathrm{Ind}_{K_M}^K(\mathcal{L}))^\rho \simeq \rho \otimes \mathrm{Hom}_{K_M}(\rho, \mathcal{L})$$

and similarly

$$(\mathrm{Ind}_{K_M}^K(\mathcal{L}^{*,\mathrm{alg}}))^{\rho^*} \simeq \rho^* \otimes \mathrm{Hom}_{K_M}(\rho^*, \mathcal{L}^{*,\mathrm{alg}}),$$

and our pairing is the perfect pairing induced by

$$\rho \otimes \rho^* \to \mathbb{C}$$

and

$$\mathrm{Hom}_{K_M}(\rho, \mathcal{L}) \otimes \mathrm{Hom}_{K_M}(\rho, \mathcal{L}^{*,\mathrm{alg}}) \to \mathbb{C}.$$

□

14.1.7. Let $i_Q^{n,G}$ denote the functor $(\mathfrak{m}', K_{M'})$-mod $\to (\mathfrak{g}, K)$-mod given by

$$i_{Q'}^{n,G}(W) := i_{Q'}^G(W \otimes (-\rho_{Q'})).$$

Note that $i_{Q'}^{n,G}(W)$ and $i_{Q'}^G(W)$ are the same as K-representations.

Combining Proposition 14.1.6 with Theorem 14.3.6 (see below) and Proposition 11.1.2, we obtain:

Corollary 14.1.8. *Let $\mathcal{L}$ be an admissible $(\mathfrak{m}', K_{M'})$-module. Then there exists a canonical isomorphism between $(i_Q^{n,G}(\mathcal{L}))^{*,\mathrm{alg}}$ and $i_Q^{n,G}(\mathcal{L}^{*,\mathrm{alg}})$.*

Remark 14.1.9. The assertion of Corollary 14.1.8 is purely algebraic. It amounts to the fact that the canonical K-invariant functional

$$i_{Q'}^G(-2\rho_{Q'}) \simeq \mathcal{O}_{K/K_M} \to \mathbb{C}$$

is G-invariant. However, I was not able to give an algebraic proof of this fact: I needed to appeal to Proposition 14.1.6 that involves integration over the real manifold G/Q'.

14.1.10. Let us return to the situation of Sect. 14.1.5. Suppose that W is *unitary.*

In this case, we have a G-invariant sesquilinear pairing

$$I_{Q'}^{n,G}(W) \times I_{Q'}^{n,G}(W) \to \mathbb{C}, \quad \int_{G/Q'} (f_1, f_2). \tag{14.5}$$

Interpreting the topologigal space underlying $I_{Q'}^{n,G}(W)$ as the space of continuous functions

$$f : K \to W, \quad f(k \cdot m) = m^{-1} \cdot f(k), \quad m \in K_M,$$

we obtain that (14.5) equips $I_{Q'}^{n,G}(W)$ positive-definite scalar product, which is continuous in the original topology on $I_{Q'}^{n,G}(W)$.

Let ${}^{L_2}I^{n,G}_{Q'}(W)$ denote the completion of $I^{n,G}_{Q'}(W)$ with respect to the resulting L_2-norm. We obtain that ${}^{L_2}I^{n,G}_{Q'}(W)$ is a unitary representation of G. Thus, normalized induction maps unitary representations to unitary representations.

14.2. **Casselman's submodule theorem.**

14.2.1. We will prove the following:

Theorem 14.2.2. *Every irreducible* $(\mathfrak{g}, K)$*-module can be realized as a* submodule *of some* $i^G_Q(\mathcal{L})$ *for an irreducible finite-dimensional* $(\mathfrak{m}, K_M)$*-module* $\mathcal{L}$*.*

By adjunction, this theorem is equivalent to:

Theorem 14.2.3. *Let* $\mathcal{M}$ *be a finitely generated admissible* $(\mathfrak{g}, K)$*-module. Then the space* $\mathcal{M}_{\mathfrak{n}}$ *of* $\mathfrak{n}$*-coinvariants is non-zero.*

There exist two algebraic proofs of this theorem. One by O. Gabber and another by Beilinson-Bernstein via localization theory. We will present an analytic proof. It is based on considering the growth of matrix coefficients of K-finite vectors in a realization of our $(\mathfrak{g}, K)$-module.

14.2.4. Let a be a vector in $\mathfrak{a}$. Let V be a G-representation on a Banach space.

Lemma 14.2.5. *There exists a real number* λ *such that for all* $v \in V$ *and* $v^* \in V^*$ *there exists a constant* C *so that we have*

$$|\langle v^*, \exp(t \cdot a) \cdot v\rangle| \leq C \cdot \exp(t \cdot \lambda), \quad t \in \mathbb{R}^{\geq 0}.$$

Proof. Follows from the fact that the operator $T_{\exp(a)}$ is bounded, and we take λ it be the log of its norm. □

14.2.6. We shall say that a is *regular* if all of its eigenvalues on $\mathfrak{n}$ are non-zero. We shall say that a is *regular dominant* if all of its eigenvalues on $\mathfrak{n}$ are strictly positive.

We have the following key assertion:

Theorem 14.2.7. *Let* $a \in \mathfrak{a}$ *be regular. Let* V *be admissible, and let* v *and* v^* *be* K*-finite such that* $\langle v^*, \exp(t \cdot a) \cdot v\rangle$ *is not identically equal to zero. Then the set of* $\lambda \in \mathbb{R}$ *such that*

$$|\langle v^*, \exp(t \cdot a) \cdot v\rangle| \leq C \cdot \exp(t \cdot \lambda), \quad t \in \mathbb{R}^{\geq 0} \text{ for some } C$$

is bounded below*.*

In other words, this theorem says that the matrix coefficient functions $\langle v^*, \exp(t \cdot a) \cdot v\rangle$ cannot decay faster than all exponents.

14.2.8. Let us deduce Theorem 14.2.3 from Theorem 14.2.7. We will need the following assertion of independent interest, proved below.

Proposition 14.2.9. *Any admissible finitely generated* $(\mathfrak{g}, K)$*-module is finitely generated over* $U(\mathfrak{n})$*.*

Proof of Theorem 14.2.2. Pick $a \in \mathfrak{a}$ to be regular anti-dominant. We pick a realization V of $\mathcal{M}$. Fix $v^* \in (V^*)^{K\text{-fin}}$. For $v \in V^{K\text{-fin}}$ such that $\langle v^*, \exp(t \cdot a) \cdot v\rangle$ is not identically equal to zero, let λ_v be the infimum of those real numbers that

$$|\langle v^*, \exp(t \cdot a) \cdot v\rangle| \leq C \cdot \exp(t \cdot \lambda) \text{ for some } C.$$

By Lemma 14.2.5 and Theorem 14.2.7, this infimum is well-defined.

We claim that if $v = u \cdot v'$ for u an element of $U(\mathfrak{n})$ with eigenvalue μ with respect to a (remember, it is negative), then

$$\lambda_v = \lambda_{v'} + \mu.$$

Indeed, this is just the fact that

$$\exp(t \cdot a) \cdot v = \exp(t \cdot \mu) \cdot \exp(t \cdot a) \cdot v'.$$

Hence, since $\mathcal{M}$ is finitely generated over $U(\mathfrak{n})$, the function

$$v \mapsto \lambda_v$$

attains a maximum. Let $v_0 \in \mathcal{M}$ be a vector on which this maximum is attained. By the above, it cannot be of the form $u \cdot v'$ for u in the augmentation ideal of $U(\mathfrak{n})$. I.e., v_0 projects to non-zero in $\mathcal{M}_{\mathfrak{n}}$.

□

Proof of Proposition 14.2.9. It is enough to show that any object of $(\mathfrak{g}, K)$-mod of the form

$$U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} \rho,$$

where ρ is a finite-dimensional representation of K, is finitely generated over $U(\mathfrak{n}) \otimes Z(\mathfrak{g})$.

I.e., it suffices to show that $U(\mathfrak{g})$ is finitely generated as a module over $U(\mathfrak{n}) \otimes U(\mathfrak{k})^{\mathrm{op}} \otimes Z(\mathfrak{g})$. The latter is suffices to do at the associated graded level. I.e., it suffices to show that $\mathrm{Sym}(\mathfrak{g})$ is finitely generated as a module over $\mathrm{Sym}(\mathfrak{n}) \otimes \mathrm{Sym}(\mathfrak{k}) \otimes \mathrm{Sym}(\mathfrak{g})^G$.

Since everything is positively graded, it suffices to show that $\mathrm{Sym}(\mathfrak{g}/\mathfrak{n})$ is finitely generated over $\mathrm{Sym}(\mathfrak{k}) \otimes \mathrm{Sym}(\mathfrak{g})^G$. However, the action of $\mathrm{Sym}(\mathfrak{g})^G$ on $\mathrm{Sym}(\mathfrak{g}/\mathfrak{n})$ factors through $\mathrm{Sym}(\mathfrak{a})$ (which is finite as a $Z(\mathfrak{g})$-module), and the assertion follows.

□

14.3. **Further realization results.**

14.3.1. We will now consider the category $(\mathfrak{q}, K_M)$-mod. It contains as a full subcategory $(\mathfrak{m}, K_M)$-mod; it consists of those objects $\mathcal{L}$ on which $\mathfrak{n}$ acts trivially.

We have the pair of adjoint functors

$$\mathrm{Res}^{(\mathfrak{g},K)}_{(\mathfrak{q},K_M)} : (\mathfrak{g}, K)\text{-mod} \rightleftarrows (\mathfrak{q}, K_M)\text{-mod} : i^G_Q.$$

The precomposition of i^G_Q with $(\mathfrak{m}, K_M)\text{-mod} \hookrightarrow (\mathfrak{q}, K_M)$-mod is the functor that we earlier denoted i^G_Q. The proof of Proposition 12.2.3 applies and we have an isomorphism

$$\mathrm{Res}^{(\mathfrak{g},K)}_K \circ i^G_Q \simeq \mathrm{Ind}^K_{K_M} \circ \mathrm{Res}^{(\mathfrak{q},K_M)}_{K_M}.$$

14.3.2. We have a similar situation at the level of group representations:

$$\mathrm{Res}^G_Q : \mathrm{Rep}(G) \rightleftarrows \mathrm{Rep}(P) : I^G_Q.$$

As in Proposition 12.2.6, for a K_M-admissible $(\mathfrak{q}, K_M)$-module W, we have

$$(I^G_Q(W))^{K\text{-fin}} \simeq i^G_Q(W^{K_M\text{-fin}}).$$

14.3.3. Inside the category $(\mathfrak{q}, K_M)$-mod we single out the full subcategory $(\mathfrak{q}, K_M)\text{-mod}^{\mathfrak{n}\text{-nilp}}$ consisting of objects on which the Lie algebra $\mathfrak{n}$ acts locally nilpotently.

Writing $\mathfrak{m}$ as $\mathfrak{k}_M \oplus \mathfrak{a}$, we obtain that an object of $(\mathfrak{q}, K_M)\text{-mod}^{\mathfrak{n}\text{-nilp}}$ can be thought of as an algebraic representation of the group

$$\ker(Q \to M \to T_A),$$

equipped with an action of $\mathfrak{a}$ that commutes with K_M and interacts with the N-action according to the adjoint action of $\mathfrak{a}$ on N.

It is easy to see that any finite-dimensional object of $(\mathfrak{q}, K_M)$-mod belongs in fact to $(\mathfrak{q}, K_M)\text{-mod}^{\mathfrak{n}\text{-nilp}}$. Indeed, the action of $\mathfrak{n}$ is necessarily nilpotent because $\mathfrak{a}$ has non-trivial adjoint eigenvalues on $\mathfrak{n}$.

Moreover, it is easy to see that restriction defines an equivalence from the category of finite-dimensional representations of the (real Lie group) Q to that of finite-dimensional objects in $(\mathfrak{q}, K_M)$-mod.

14.3.4. We will prove:

Theorem 14.3.5. *Let $\mathcal{M}$ be a $(\mathfrak{g}, K)$-module of finite length. Then there exists a finite-dimensional object $\mathcal{L} \in (\mathfrak{q}, K_M)$-mod such that $\mathcal{M}$ embeds into $i_Q^G(\mathcal{L})$.*

As a consequence, as in we obtain (as in the case of Theorem 13.2.3):

Theorem 14.3.6. *And $(\mathfrak{g}, K)$-module of finite length can be realized as K-finite vectors in a Banach (or even Hilbert) representation of G.*

14.4. **Proof of Theorem 14.3.5.** We will replace K by its connected component–this does not change the results of the preceding sections.

14.4.1. For an integer k, consider $\mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}$ as a $(\mathfrak{q}, K_M \cap K_0)$-module. According to Proposition 14.2.9, it is finite-dimensional. We will prove that there exists an integer k, such that the map

$$\mathcal{M} \to i_Q^G(\mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}),$$

(obtained by adjunction from the tautological map $\text{Res}^{(\mathfrak{g},K_0)}_{(\mathfrak{q},K_M \cap K_0)}(\mathcal{M}) \to \mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}$) is injective.

14.4.2. Let $\mathfrak{n}$ be a nilpotent Lie algebra. We consider the following functor on the category of $\mathfrak{n}$-modules:

$$\mathcal{M} \mapsto \widehat{\mathcal{M}} := \lim_{\substack{\leftarrow \\ k}} \mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}.$$

By induction on the degree of nilpotence, from the Artin-Rees lemma we deduce:

Lemma 14.4.3. *The above functor is exact when restricted to the subcategory of finitely generated $\mathfrak{n}$-modules.*

Let $\mathcal{M}$ be a $(\mathfrak{g}, K_0)$-module of finite length. We claim:

Corollary 14.4.4. *The map $\mathcal{M} \mapsto \widehat{\mathcal{M}}$ is injective.*

Proof. Lemma 14.4.3 and Proposition 14.2.9 reduce the assertion to the case when $\mathcal{M}$ is irreducible, which is what we will now assume.

We now note that $\widehat{\mathcal{M}}$ also naturally acquires an $\mathfrak{g}$-action so that the natural map $\mathcal{M} \to \widehat{\mathcal{M}}$ is a map of $\mathfrak{g}$-modules (this is true for any $\mathfrak{g}$-module $\mathcal{M}$). This follows from the fact that there exists an integer k such that for any $\xi \in \mathfrak{g}$, if we take k times its bracket with elements from $\mathfrak{n}$, we get zero. For such k, the action of $\mathfrak{g}$ is well-defined as a map

$$\mathcal{M}/\mathfrak{n}^{k'+k} \cdot \mathcal{M} \to \mathcal{M}/\mathfrak{n}^{k'} \cdot \mathcal{M}.$$

Recall also that if $\mathcal{M}$ is irreducible as a $(\mathfrak{h}, K_0)$-module, then it is such as a $\mathfrak{g}$-module. Hence, it suffices to show that the map $\mathcal{M} \to \widehat{\mathcal{M}}$ is non-zero. However, we know that the composition

$$\mathcal{M} \to \widehat{\mathcal{M}} \to \mathcal{M}/\mathfrak{n} \cdot \mathcal{M}$$

is non-zero. Indeed, this is equivalent to the statement of Theorem 14.2.2.

□

14.4.5. Consider now the map of $(\mathfrak{g}, K_0)$-modules

$$\mathcal{M} \to \lim_{\substack{\leftarrow \\ k}} i_Q^G(\mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}).$$

It is injective because its composition with

$$\lim_{\substack{\leftarrow \\ k}} i_Q^G(\mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}) \to \lim_{\substack{\leftarrow \\ k}} \mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M} = \widehat{\mathcal{M}}$$

is injective by Corollary 14.4.4.

Let $\mathcal{M}'_k$ denote the kernel of the map $\mathcal{M} \to i_Q^G(\mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M})$. We have

$$\mathcal{M}'_{k+1} \subset \mathcal{M}'_k \subset ...$$

Since $\mathcal{M}$ was assumed of finite length, this chain stabilizes. Hence its stable value $\mathcal{M}'$ lies in the kernel of all the maps $\mathcal{M} \to i_Q^G(\mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M})$. But then $\mathcal{M}'$ lies in the kernel of the map

$$\mathcal{M} \to \lim_{\substack{\leftarrow \\ k}} i_Q^G(\mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}).$$

Hence $\mathcal{M}' = 0$.

15. Week 9, Days 1 and (Tue., March 28 and Thurs., March 30)

15.1. **Asymptotics of matrix cofficients.**

15.1.1. Let V be an admissible representation, and v (resp., v^*) be a K-finite vector (resp., covector). We will be interested in the asympotic behavior of the function

$$\mathrm{MC}_{v^*,v}(g) = \langle v^*, g \cdot v \rangle,$$

where we will take g to be $\exp(a)$ for a regular anti-dominant element $a \in \mathfrak{a}$.

We will prove:

Theorem 15.1.2. *For every real number R there exists a finite collection of characters λ_n of $\mathfrak{a}$ and polynomial functions p_n on $\mathfrak{a}$ such that the function*

$$\mathrm{MC}_{v^*,v}(\exp(a)) - \sum_n \exp(\lambda_n(a)) \cdot p_n(a)$$

tends to zero faster than $\exp(R \cdot \min_\alpha(\alpha(a)))$. The characters λ_n appear as eigenvalues of $\mathfrak{a}$ on $\mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}$ for a sufficiently large integer k (here $\mathcal{M} = V^{K\text{-fin}}$).

The rest of this subsection is devoted to the proof of this theorem.

15.1.3. Let us assume that V is a Banach representation of G. Since the operators $T_{\exp(a)}$ are bounded, we can find a real number R_1 such that

$$||T_{\exp(a)}|| < \exp(-R' \cdot \rho(a))$$

for $a \in \mathfrak{a}$ anti-dominant.

Let m be such that $m \cdot \min_{\alpha}(\alpha(a)) < \rho(a)$. Let the integer k be such that

$$k > R + m \cdot R'.$$

Consider the quotient

$$\mathcal{M} \xrightarrow{\pi} \mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}.$$

This is a finite-dimensional representation of the Lie algebra $\mathfrak{a}$; hence it integrates to a representation of the Lie group A.

Let $v_1, ..., v_n$ be vectors in $\mathcal{M}$ that project to a basis of $\mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}$. Then there exists polynomial functions p_n on $\mathfrak{a}$ such that the action of A on $\pi(v)$ is given by

$$\sum_n \exp(\lambda_n(a)) \cdot p_n(a) \cdot \pi(v_n),$$

where λ_n are among the eigenvalues of $\mathfrak{a}$ on $\mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}$.

We will show that

$$\mathrm{MC}_{v^*,v}(\exp(a)) - \sum_n \exp(\lambda_n(a)) \cdot p_n(a) \cdot \langle v^*, v_n \rangle \tag{15.1}$$

decreases faster than $\exp(R \cdot \rho(a))$.

15.1.4. First, we claim:

Lemma 15.1.5. *The map $i : \mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M} \to V^\infty/\mathfrak{n}^k \cdot V^\infty$ intertwines the action of A on $\mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}$ (obtained by integrating the Lie algebra action) and the action of A on $V^\infty/\mathfrak{n}^k \cdot V^\infty$, coming from the action of $A \subset G$ on V.*

Proof. For every $\overline{v} \in \mathcal{M}/\mathfrak{n}^k \cdot \mathcal{M}$, the $V^\infty/\mathfrak{n}^k \cdot V^\infty$-valued functions on $\mathfrak{a}$, given by

$$i(\exp(a) \cdot \overline{v}) \text{ and } \exp(a) \cdot i(\overline{v})$$

take the same value at $a = 0$ and satisfy the same differential equation.

□

From this lemma, we obtain that the function (15.1) equals $\mathrm{MC}_{v^*,v'}(\exp(a))$, where $v' \in \mathfrak{n}^k \cdot V^\infty$. Write v' as a sum of vectors of the form $\xi_1 \cdot ... \cdot \xi_k \cdot v''$ with $\xi_i \in \mathfrak{n}_{\alpha_i}$ and $v'' \in V^\infty$. We have

$$\mathrm{MC}_{v^*,v'}(\exp(a)) = \prod_i \exp(\alpha_i(a)) \cdot \mathrm{MC}_{v^*,v''}(\exp(a)).$$

Here $\mathrm{MC}_{v^*,v''}(\exp(a))$ increases slower than $\exp(-m \cdot R' \cdot \min_{\alpha}(\alpha(a)))$, while $\prod_i \exp(\alpha_i(a))$ decreases faster that $\exp(k \cdot \min_{\alpha}(\alpha(a)))$.

□

15.2. **Growth conditions.**

15.2.1. Let λ be an element of $\mathfrak{a}^*$. We shall say that the exponents of an admissible $(\mathfrak{g}, K)$-module are $\geq \lambda$ if for every v, v^*, the characters λ' that appear Theorem 15.1.2 satisfy:

$$\mathrm{Re}(\lambda') - \lambda \in \mathrm{Span}_{\mathbb{R}\geq 0}(\text{positive roots}).$$

The proof of Theorem 15.1.2 actually shows:

Proposition 15.2.2. *Let V be an admissible representation and let $\mathcal{M}$ be the $(\mathfrak{g}, K)$-module of its K-finite vectors. Then for a given $\lambda \in \mathfrak{a}^*$, the following conditions are equivalent:*

(i) *The exponents of V are $\geq \lambda$;*

(ii) *For all the eigenvalues of $\mathfrak{a}$ on $\mathcal{M}/\mathfrak{n} \cdot \mathcal{M}$, we have*

$$\mathrm{Re}(\lambda') - \lambda \in \mathrm{Span}_{\mathbb{R}\geq 0}(\textit{positive roots}).$$

Here is a direct argument:

Proof. Assume first that the condition on the eigenvalues of $\mathfrak{a}$ on $\mathcal{M}/\mathfrak{n} \cdot \mathcal{M}$ is satisfied. Fix a regular anti-dominant $a \in \mathfrak{a}^*$. Let $v \in \mathcal{M}$ be such that the corresponding function

$$f(t) := \mathrm{MC}_{v^*,v}(\exp(t \cdot a))$$

has the asymptotic expansion

$$f(t) = \sum_c \exp(t \cdot c) \cdot p_c(t) \tag{15.2}$$

with the maximal $\mathrm{Re}(c)$. It is easy to see that we can assume that v projects to a generalized $\mathfrak{a}$-eigenvector in $\mathcal{M}/\mathfrak{n} \cdot \mathcal{M}$. To simplify the argument, we will assume that it projects to an eigenvector (in general, the idea is the same); let λ' denote the corresponding eigenvalue.

Consider the function $f'(t)$. On the one hand, it equals $\mathrm{MC}_{v^*,a\cdot v}(\exp(t \cdot a))$, and hence

$$\lambda'(a) \cdot \mathrm{MC}_{v^*,v}(\exp(t \cdot a)) + \mathrm{MC}_{v^*,v'}(\exp(t \cdot a)), \quad v' \in \mathfrak{n} \cdot \mathcal{M}.$$

In particular, the maximal real part of the exponent in the asymptotic expansion for $\mathrm{MC}_{v^*,v'}(\exp(t \cdot a))$ is strictly $<$ than that of $f(t)$. Hence,

$$f'(t) - \lambda'(a) \cdot f(t)$$

has the maximal real part of the exponent in its asymptotic expansion strictly $<$ than that of $f(t)$.

On the other hand,

$$f'(t) - \lambda'(a) \cdot f(t) = \sum_c \exp(t \cdot c) \cdot ((c - \lambda'(a)) \cdot p_c - p'_c)$$

If c_0 denotes the c with the maximal real part, we obtain that

$$\mathrm{Re}(c_0) = \lambda'(a),$$

and hence

$$\mathrm{Re}(c_0) \geq \lambda(a),$$

as required.

Vice versa, assume that the exponents are $\geq \lambda$, and let λ' be an eigenvalue of $\mathfrak{a}$ on $\mathcal{M}/\mathfrak{n} \cdot \mathcal{M}$. Let $v \in \mathcal{M}$ project to an eigenvector with the this eigenvalue. By the same logic as above, the maximal real part of the exponent in the asymptotic expansion of

$$f'(t) - \lambda'(a) \cdot f(t)$$

is strictly $<$ than that of $f(t)$. And if c_0 denotes the maximal part of the latter, we again obtain that $\mathrm{Re}(c_0) = \lambda'(a)$, i.e.,

$$\lambda'(a) \geq \lambda(a).$$

Since this inequality is valid for any regular anti-dominant a, we obtain that

$$\mathrm{Re}(\lambda') - \lambda \in \mathrm{Span}_{\mathbb{R}^{\geq 0}}(\text{positive roots}),$$

as required.

□

Remark 15.2.3. Suppose that the exponents of V are $\leq \lambda$ for some λ. Note that this forces that the split part of Z_G should act on V by a unitary character. Indeed, the condition implies that for every exponent, the real part of its restriction to $\mathfrak{z}_G$ is trivial.

15.3. **Square-integrable representations.**

15.3.1. An admissible $(\mathfrak{g}, K)$-module is said to be *square-integrable* if its exponents are

$$\geq (1 + \epsilon) \cdot \rho$$

for some $\epsilon \in \mathbb{R}^{>0}$.

15.3.2. We have:

Theorem 15.3.3. *An admissible $(\mathfrak{g}, K)$-module is square-integrable if and only of all its matrix coefficients lie in $L_2(G)$.*

To simplify the exposition, we will prove this theorem in the case of $G = SL_2(\mathbb{R})$.

Remark 15.3.4. The general case is not much more difficult; the main difference is that one needs to control the behavior of matrix coefficients near the walls of the anti-dominant cone; for that one proves (in essentially) the same way a version of Theorem 15.1.2 for an arbitrary parabolic (rather than the minimal parabolic).

Proof. Consider the map

$$K \times \exp(t \cdot a, t > R) \times K \to G.$$

This is an open embedding, the complement to whose image is compact. Let $f(g)$ me a matrix coefficient of a $(\mathfrak{g}, K)$-module. We need to study the integrability of $|f(g)|^2$ times the the pullback of the invariant top differential form under this map. Since f is K-finite on both sides, the question of integrability only depends on the further restriction of $f(g)$ to $t \cdot a, t > R$ along the exponential map. Now, the required assertion follows from the next geometric assertion:

Lemma 15.3.5. *The pullback of the invariant top differential form on G to $\mathbb{R}$ along the exponential map $t \mapsto \exp(t \cdot a)$ behaves like $\exp(-2t \cdot \rho(a))$ times a function that tends to 1 at $t \to \infty$.*

Proof. We identify the tangent space to G at the point $\exp(t \cdot a)$ with $\mathfrak{g}$ via right translations. We need to calculate the determinant of the differential

$$\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{k} \to \mathfrak{g}$$

at the point $t \cdot a$. The differential in question acts as the embedding on the first two factors and the map

$$\mathrm{Ad}_{\exp(-t \cdot a)} |_{\mathfrak{k}} : \mathfrak{k} \to \mathfrak{g}$$

along the third factor.

Consider another map $\mathfrak{k} \to \mathfrak{g}$ equal to the composition

$$\mathfrak{k} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{n} \overset{\mathrm{Ad}_{\exp(-t\cdot a)}}{\longrightarrow} \mathfrak{n} \hookrightarrow \mathfrak{g},$$

where the second arrow $\mathfrak{g} \twoheadrightarrow \mathfrak{n}$ is the orthogonal projection. However, it is easy to see that the two maps differ by a map $\mathfrak{k} \to \mathfrak{g}$ that tends to 1 as $t \to \infty$. Hence, it suffices to calculate the determinant of the map

$$\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \to \mathfrak{g},$$

which equals to the identity on the first two factors and $\mathrm{Ad}_{\exp(-t\cdot a)}$ on $\mathfrak{n}$. However, the latter equals $\exp(-2t \cdot \rho(a))$, as required.

□

□

15.3.6. Let V be a square-integrable representation, and let $\mathcal{M} := V^{K\text{-fin}}$. Consider the matrix coefficient map

$$\mathcal{M}^{*,\mathrm{alg}} \otimes \mathcal{M} \to C^\infty(G).$$

By Theorem 15.3.3, its image consists of functions that belong to $L_2(G)$. Hence, we can introduce an inner form on $\mathcal{M} \otimes \mathcal{M}^{*,\mathrm{alg}}$ by

$$(v_1^* \otimes v_1, v_2^* \otimes v_2) := \int\limits_G \langle v_1^*, g\cdot v_1\rangle \cdot \overline{\langle v_2^*, g \cdot v_2\rangle} \cdot \mu_{\mathrm{Haar}_G}. \tag{15.3}$$

We claim:

Proposition 15.3.7. *The above inner form on* $\mathcal{M}^{*,\mathrm{alg}} \otimes \mathcal{M}$ *is* $(\mathfrak{g} \oplus \mathfrak{g}, K \times K)$*-invariant.*

Proof. The $K \times K$-invariance follows immediately from the invariance of the Haar measure on G. To prove the $(\mathfrak{g} \oplus \mathfrak{g})$-invariance, we note that for, say a vector $(0, \xi) \in \mathfrak{g} \oplus \mathfrak{g}$, we have:

$$(v_1^* \otimes \xi \cdot v_1, v_2^* \otimes v_2) + (v_1^* \otimes v_1, v_2^* \otimes \xi \cdot v_2) = \int\limits_G \mathrm{Lie}_\xi(\langle v_1^*, g \cdot v_1\rangle \cdot \overline{\langle v_2^*, g\cdot v_2\rangle}) \cdot \mu_{\mathrm{Haar}_G},$$

where Lie_ξ denotes the Lie derivative with respect to the corresponding left-invariant vector field.

We need show that the above integral vanishes. Note that this would be automatic if the support of the function

$$f(g) = \langle v_1^*, g\cdot v_1\rangle \cdot \overline{\langle v_2^*, g\cdot v_2\rangle}$$

was compact (which it never is unless G is compact). The issue is to show that $f(g)$ decays fast enough so that the vanishing still holds. Since we only proved Theorem 15.3.3 in the case of $SL_2(\mathbb{R})$, we will be able to prove the assertion at hand only in this case. The assertion following from the next geometric claim:

Lemma 15.3.8. *Let f be a function on G that is $K \times K$-finite, and assume that its restriction to $\mathbb{R}^{\geq 0}$ along*

$$t \mapsto \exp(t \cdot a)$$

(here a is an anti-dominant generator of $\mathfrak{a}$) decays faster than $\exp(-2(1+\epsilon)\cdot t)$ *for some* $\epsilon \in \mathbb{R}^{>0}$*. Then*

$$\int\limits_G \mathrm{Lie}_\xi(f) \cdot \mu_{\mathrm{Haar}_G} = 0.$$

Proof. We represent the integral μ_{Haar_G} as the limit of integrals over closed subsets of G equal to the images of

$$K \times \exp(0 \leq t \leq R) \times K.$$

For $R > 0$, this is a manifold with boundary, and we calculate the integral via the Stokes theorem. I.e., it equals the integral over the boundary, i.e., the subset $K \times \exp(R) \times K$ of the sub-top differential form

$$i_\xi(f \cdot \mu_{\text{Haar}_G}).$$

This sub-top differential form is $K \times K$-finite. So, it suffices to show that its value at the point $\exp(R)$ tends to 0 as $t \to \infty$. However, this follows from the assumption on the decay of f. □

□

15.3.9. Picking an arbitrary non-zero vector in $v^* \in \mathcal{M}^{*,\text{alg}}$, and restricting the above inner form to $v^* \otimes \mathcal{M}$, we obtain that $\mathcal{M}$ acquires a $(\mathfrak{g}, K)$-invariant inner form. Applying Theorem 9.2.6, we obtain:

Theorem 15.3.10. *A square-integrable $(\mathfrak{g}, K)$-module can be realized as the space of K-finite vectors in a unitary representation.*

15.4. **Tempered representations.** An admissible $(\mathfrak{g}, K)$-module is said to be *tempered* if it has exponents $\geq \rho$.

15.4.1. We have:

Theorem 15.4.2. *The normalized induction of a tempered $(\mathfrak{g}, K)$-module is tempered.*

We will only give a proof in the case of a split group G (e.g., $G = SL_2(\mathbb{R})$) and the minimal parabolic (which in this case is the Borel subgroup). In this case, we start with a *unitary* chracater λ of T (see Remark 15.3.4) and we would like to see that the eigenvalues of the action of $\mathfrak{a} \simeq \mathfrak{t}$ on

$$r_B^{n,G} \circ i_B^{n,G}(\lambda)$$

have real part ≥ 0.

We will prove a more precise result:

Theorem 15.4.3. *The eigenvalues of $\mathfrak{a} \simeq \mathfrak{t}$ on $r_B^{n,G} \circ i_B^{n,G}(\lambda)$ are of the form $w(\lambda)$ for $w \in W$.*

15.4.4. To prove Theorem 15.4.3 we will have to revisit the Harish-Chandra homomorphism. Recall that the action of $Z(\mathfrak{g})$ acts on the module

$$U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} U(\mathfrak{t})$$

via a homomorphism $\phi : Z(\mathfrak{g}) \to Z(\mathfrak{t}) \simeq U(\mathfrak{t}) \simeq \text{Sym}(\mathfrak{t})$.

We now introduce the *normalized* Harish-Chandra homomorphism

$$\phi^n : Z(\mathfrak{g}) \to U(\mathfrak{t})$$

by composing ϕ with the automorphism of $U(\mathfrak{t}) \simeq \text{Sym}(\mathfrak{t})$ induced by the map

$$t \mapsto t - \rho(t).$$

I.e., for $\lambda \in \mathfrak{t}^*$, we have

$$\lambda(\phi^n(z)) = (\lambda - \rho)(\phi(z)).$$

We have:

Theorem 15.4.5. *The map ϕ^n maps $Z(\mathfrak{g})$ into the subalgebra of W-invariant functions in* $\mathrm{Sym}(\mathfrak{t})$. *The resulting map*

$$Z(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{t})^W$$

is an isomorphism.

Proof. To show that the image of ϕ^n consists of W-invariant elements, it is enough to check this for every simple reflection s_i.

Let P be a standard parabolic with Levi quotient M (in practice, we will take the sub-minimal parabolic corresponding to the simple root α_i). Consider the corresponding Harish-Chandra homomorphism

$$\phi_P : Z(\mathfrak{g}) \to Z(\mathfrak{m}) \simeq Z(\mathfrak{m}') \otimes \mathrm{Sym}(\mathfrak{z}_M).$$

We introduced the normalized version ϕ^n_P of ϕ_P by composing it with the automorphism of $\mathrm{Sym}(\mathfrak{z}_M)$, induced by the map

$$t \mapsto t - \rho_P(t),$$

where ρ_P is the character of $\mathfrak{z}_M$ equal to half the trace of its adjoint action on $\mathfrak{n}(P)$. It follows from the constuction, that

$$\phi^n = \phi^n_M \circ \phi^n_P,$$

where ϕ^n_M is the normalized Harish-Chandra homomorphism for the reductive group M.

This reduces the verification of s_i-invariance of the image of ϕ^n to the case when G is of semi-simple rank 1. Splitting of the center, we can assume that $G = SL_2$. In this case, the assertion is equivalent to the fact that for $\lambda \in \mathfrak{t}^*$, the action of $Z(\mathfrak{g})$ on the modules

$$M^\lambda := U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}^\lambda$$

and $M^{-\lambda-2\rho}$ is given by the same character. Moreover, since expression corresponds to the evaluation of a polynomial function at λ, we can assume that λ is a positive integer. However, in this case, the assertion follows from the fact that we have a non-zero map

$$M^{-\lambda-2\rho} \to M^\lambda;$$

in fact we have a short exact sequence

$$0 \to M^{-\lambda-2\rho} \to M^\lambda \to V^\lambda \to 0,$$

where V^λ is the irreducible SL_2-module of highest weight λ.

Thus, ϕ^n is a homomorphism

$$Z(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{t})^W,$$

and it remains to show that it is an isomorphism. The latter is enough to do at the associated graded level. However, the maps $\mathrm{gr}(\phi)$ and $\mathrm{gr}(\phi^n)$

$$\mathrm{Sym}(\mathfrak{g})^G \to \mathrm{Sym}(\mathfrak{t})^W$$

agree, and both correspond to restricting G-invariant functions on $\mathfrak{g}$ to W-invariant functions on $\mathfrak{t}$. The injectivity of this map is clear. The surjectivity follows from the fact that W-invariant functions on $\mathfrak{t}$ are generated by characters of irreducible representations.

□

15.4.6. Let us now prove Theorem 15.4.3:

Proof. Let $\mathcal{M}$ be a $(\mathfrak{g}, K)$-module. By the construction of the normalized Harish-Chandra homomorphism, the action of $Z(\mathfrak{g})$ on $r_B^{n,G}(\mathcal{M})$ (coming from the $Z(\mathfrak{g})$-action on $\mathcal{M}$) equals that obtained from ϕ^n and the $\mathrm{Sym}(\mathfrak{t})$-action on $r_B^{n,G}(\mathcal{M})$.

By adjunction, for an $\mathfrak{t}$-module $\mathcal{N}$, the action of $Z(\mathfrak{g})$ on $i_B^{n,G}(\mathcal{N})$ equals that obtained from ϕ^n and the $\mathrm{Sym}(\mathfrak{t})$-action on $\mathcal{N}$.

Combining, we obtain that if $\mathrm{Sym}(\mathfrak{t})$ acts on $\mathcal{N}$ by a character λ, then the eigenvalues λ' of $\mathrm{Sym}(\mathfrak{t})$ on $r_B^{n,G} \circ i_B^{n,G}(\mathcal{N})$ are such that they have the same projection as λ under the map

$$\mathrm{Spec}(\mathrm{Sym}(\mathfrak{t})) \to \mathrm{Spec}(Z(\mathfrak{g})).$$

However, this means that $\lambda' = w(\lambda)$ for $w \in W$.

□

16. Week 10, Day 1 (Tue, April 4)

16.1. **Tempered vs square-integrable.**

16.1.1. We now claim:

Theorem 16.1.2. *Any irreducible tempered $(\mathfrak{g}, K)$-module can be realized as a direct summand of a normalized induction of a square-integrable representation of the Levi quotient of a parabolic.*

Combining with Theorem 15.3.10 and Sect. 14.1.10, we obtain:

Corollary 16.1.3. *An irreducible tempered $(\mathfrak{g}, K)$-module can be realized as the space of K-finite vectors in a unitary representation.*

In addition, combining with Theorem 15.4.2 and Proposition 14.1.6, we obtain:

Corollary 16.1.4. *The dual of a tempered representation is tempered.*

16.1.5. *Proof of Theorem 16.1.2.* To simplify the exposition, we will assume that G is split.

Let $\mathcal{M}$ be an irreducible tempered $(\mathfrak{g}, K)$-module. It suffices to find a non-zero map

$$\mathcal{M} \to i_P^{n,G}(\mathcal{N}),$$

where $\mathcal{N}$ is a square-integrable $(\mathfrak{m}, K_M)$-module for some parabolic P with Levi quotient M. (Indeed, the induction is unitary, so any submodule splits off as a direct summand.)

Consider $r_Q^{n,G}(\mathcal{M})$ and consider the eigenvalues of $\mathfrak{t}$ on it. For each such eigenvalue λ, write

$$\mathrm{Re}(\lambda) = \Sigma\, n_i \cdot \alpha_i,$$

where α_i's are positive simple roots and n_i are non-negative.

Let λ_0 be such that the number of i's, for which the coefficient n_i is non-zero, is *minimal*. Let $\mathfrak{p} \supset \mathfrak{b}$ be the parabolic subalgebra, where we add to $\mathfrak{b}$ all the negative root spaces corresponding to i for which $n_i \neq 0$ in the expansion of λ_0.

Consider $r_P^{n,G}(\mathcal{M})$, and let $\widetilde{\mathcal{N}}$ be its direct summand on which $\mathfrak{z}_M$ acts with generalized eigenvalues equal to $\lambda_0|_{Z_M}$. Let $\mathcal{N}$ be some irreducible quotient of $\widetilde{\mathcal{N}}$. By construction, we have a surjective (in particular, non-zero) map

$$r_P^{n,G}(\mathcal{M}) \to \mathcal{N},$$

and hence a map $\mathcal{M} \to i^{n,G}_{QP}(\mathcal{N})$.

It suffices to show that $\mathcal{N}$ is square-integrable. Let λ be an eigenvalue of $\mathfrak{t}$ on $r^{n,M}_{B\cap M}(\mathcal{N})$. By construction,

$$\mathrm{Re}(\lambda) = \lambda_0 + \Sigma\, m_i \cdot \alpha_i = \Sigma\, (m_i + n_i) \cdot \alpha_i,$$

where α_i's are roots of M. Now, none of the $m_i + n_i$ are zero, because it would contradict the minimality assumption on λ_0. However, this by definition means that $\mathcal{N}$ is square-integrable.

□

16.2. **The Langlands classification.** Finally, here is the Langlands classification of all irreducible $(\mathfrak{g}, K)$-module in terms of the tempered ones. We will only state it in the case when G is split.

16.2.1. Let P be a standard parabolic of G with Levi quotient M. Let $\mathcal{N}$ be an irreducible tempered $(\mathfrak{m}, K_M)$-module.

Let λ be a *real* character of $\mathfrak{m}/[\mathfrak{m},\mathfrak{m}]$, which is *regular anti-dominant.* By definition, this means the following:

Note that we have a surjective map

$$\mathfrak{t} \to \mathfrak{m} \to \mathfrak{m}/[\mathfrak{m},\mathfrak{m}].$$

We require that the evaluation of λ on every positive coroot of G, which is not a coroot of M, to be strictly negative.

16.2.2. The Langlands classification theorem says:

Theorem 16.2.3. *The induced $(\mathfrak{g}, K)$-module $i^{n,G}_P(\mathcal{N}\otimes\mathbb{C}^\lambda)$ has a unique irreducible submodule. Every irreducible $(\mathfrak{g}, K)$-module arises in this way for a uniquely defined triple $(P, \mathcal{N}, \lambda)$.*

The key point is the following assertion:

Theorem 16.2.4. *Let P^- be the parabolic opposite to P and identify the Levi quotients of P and P^- via*

$$P/N(P) \simeq P \cap P^- \simeq P^-/N(P^-).$$

Consider the corresponding induced $(\mathfrak{g}, K)$-module $i^{n,G}_{P^-}(\mathcal{N} \otimes \mathbb{C}^\lambda)$. Then

$$\mathrm{Hom}(i^{n,G}_{P_2^-}(\mathcal{N}_2 \otimes \mathbb{C}^{\lambda_2}), i^{n,G}_{P_1}(\mathcal{N}_1 \otimes \mathbb{C}^{\lambda_1}))$$

is non-zero only if $P_1 = P_2$, $\mathcal{N}_1 \simeq \mathcal{N}_2$ and $\lambda_1 = \lambda_2$. In the latter case, the above Hom is at most one-dimensional.

Let us deduce Theorem 16.2.3 from Theorem 16.2.4. We will also need the following proposition, proved below:

Proposition 16.2.5. *For a $(\mathfrak{g}, K)$-module $\mathcal{M}$ there* exists *a parabolic P, an irreducible tempered $(\mathfrak{m}, K_M)$-module $\mathcal{N}$ and a regular anti-dominant character λ of $\mathfrak{m}/[\mathfrak{m},\mathfrak{m}]$ such that $\mathcal{M}$ admits a non-zero map to $i^{n,G}_P(\mathcal{N} \otimes \mathbb{C}^\lambda)$.*

Proof of Theorem 16.2.3. First off, Proposition 16.2.5 implies that every irreducible $(\mathfrak{g}, K)$-module $\mathcal{M}$ can be realized as a submodule in some $i^{n,G}_{P_1}(\mathcal{N}_1 \otimes \mathbb{C}^{\lambda_1})$ for some triple $(P_1, \mathcal{M}_1, \lambda_1)$.

Swapping the roles of P and P^-, and using the duality property of normalized induction, i.e., Proposition 14.1.6 (and Corollary 16.1.4), we obtain that $\mathcal{M}$ can also be realized as a quotient of some $i^{n,G}_{P_2^-}(\mathcal{N}_2 \otimes \mathbb{C}^{\lambda_2})$ for some triple $(P_2, \mathcal{M}_2, \lambda_2)$.

Consider the composite map

$$i^{n,G}_{P_2^-}(\mathcal{N}_2 \otimes \mathbb{C}^{\lambda_2}) \twoheadrightarrow \mathcal{M} \hookrightarrow i^{n,G}_{P_1}(\mathcal{N}_1 \otimes \mathbb{C}^{\lambda_1}).$$

From Theorem 16.2.4, we obtain that $P_1 = P_2$, $\mathcal{N}_1 \simeq \mathcal{N}_2$ and $\lambda_1 = \lambda_2$.

Suppose that $\mathcal{M}$ could be realized as a submodule of some potentially different $i^{n,G}_{P_1'}(\mathcal{N}_1' \otimes \mathbb{C}^{\lambda_1'})$. Then we would still have a diagram

$$i^{n,G}_{P_2^-}(\mathcal{N}_2 \otimes \mathbb{C}^{\lambda_2}) \twoheadrightarrow \mathcal{M} \hookrightarrow i^{n,G}_{P_1'}(\mathcal{N}_1' \otimes \mathbb{C}^{\lambda_1'}).$$

By Theorem 16.2.4, this implies $P_1' = P_2 = P_1$, $\mathcal{N}_1 \simeq \mathcal{N}_2 \simeq \mathcal{N}_1'$ and $\lambda_1 = \lambda_2 = \lambda_1'$.

Finally, suppose that a given $i^{n,G}_P(\mathcal{N} \otimes \mathbb{C}^\lambda)$ has two different irreducible submodules, i.e., we have an embedding

$$\mathcal{M}_1 \oplus \mathcal{M}_2 \hookrightarrow i^{n,G}_P(\mathcal{N} \otimes \mathbb{C}^\lambda).$$

Consider the corresponding surjections

$$i^{n,G}_{P_1^-}(\mathcal{N}_1 \otimes \mathbb{C}^{\lambda_1}) \twoheadrightarrow \mathcal{M}_1 \text{ and } i^{n,G}_{P_2^-}(\mathcal{N}_2 \otimes \mathbb{C}^{\lambda_2}) \twoheadrightarrow \mathcal{M}_2.$$

We obtain the non-zero maps

$$i^{n,G}_{P_1^-}(\mathcal{N}_1 \otimes \mathbb{C}^{\lambda_1}) \to i^{n,G}_P(\mathcal{N} \otimes \mathbb{C}^\lambda) \text{ and } i^{n,G}_{P_2^-}(\mathcal{N}_2 \otimes \mathbb{C}^{\lambda_1}) \to i^{n,G}_P(\mathcal{N} \otimes \mathbb{C}^\lambda). \tag{16.1}$$

By Theorem 16.2.4, this implies that $P_1 = P = P_2$, $\mathcal{N}_1 \simeq \mathcal{N} \simeq \mathcal{N}_2$ and $\lambda_1 = \lambda = \lambda_2$. Moreover, since in this case, the Hom in question is at most 1-dimensional, the maps (16.1) have the same image. However, the image of the first was $\mathcal{M}_1$, and the image of the second was $\mathcal{M}_2$, a contradiction.

□

Proof of Proposition 16.2.5. We will use the following combinatorial assertion (Langlands lemma):

Lemma 16.2.6. *Let λ be an element of (the real) $\mathfrak{t}^*$. Then there exists a unique subset I' of the Dynkin diagram and a unique way to write λ as $\lambda_1 + \lambda_2$, where:*

(i) *λ_1 is a character of $\mathfrak{m}/[\mathfrak{m},\mathfrak{m}]$ (where $\mathfrak{m}$ is the Levi of the standard parabolic corresponding to $I - I'$), and is regular anti-dominant as such. I.e., $\check{\alpha}_i(\lambda_1) < 0$ for $i \in I'$ and $\check{\alpha}_i(\lambda_1) = 0$ for $i \notin I'$;*

(ii) $\lambda_2 = \sum\limits_{i \notin I'} n_i \cdot \alpha_i, \quad n_i \geq 0.$

The proof of the lemma is obtained by applying the orthogonal projection from $\mathfrak{t}^*$ to the anti-dominant chamber, viewed as a convex subset of $\mathfrak{t}^*$.

Let $\mathcal{M}$ be a $(\mathfrak{g}, K)$-module. Consider $r^{n,G}_B(\mathcal{M})$, and consider the eigenvalues of $\mathfrak{t}$ on it. Let λ be such that the corresponding $\mathrm{Re}(\lambda)$ is minimal in the order relation

$$\lambda' \geq \lambda'' \Leftrightarrow \lambda' - \lambda'' \in \mathrm{Span}_{\mathbb{R}^{\geq 0}}(\alpha_i).$$

Consider the corresponding decomposition

$$\mathrm{Re}(\lambda) = \lambda_1 + \lambda_2.$$

Let P be the parabolic, whose simple roots correspond to the subset of the Dynkin diagram given by $I - I'$.

Consider $r^{n,G}_P(\mathcal{M})$, and let $\widetilde{\mathcal{N}}$ be its direct summand corresponding to the eigenvalues of $\mathfrak{z}_M$ equal to the restriction of λ. Let $\mathcal{N}'$ be some irreducible quotient of $\widetilde{\mathcal{N}}$. By construction, we have a non-zero map

$$\mathcal{M} \to i^{n,G}_P(\mathcal{N}').$$

By construction, the real part of the character of $\mathcal{N}'$ with respect to $\mathfrak{z}_M$ is λ_1, and so it is anti-dominant. $\mathcal{N} := \mathcal{N}' \otimes \mathbb{C}^{-\lambda_1}$. The eigenvalues of $\mathfrak{t}$ on $r^{n,M}_{B\cap M}(\mathcal{N}')$ are of the form

$$\lambda + \sum_{i \notin I'} m_i \cdot \alpha_i.$$

Moreover, $\mathrm{Re}(m_i) \geq 0$ by the minimality assumption on λ. Hence, the eigenvalues of $\mathfrak{t}$ on $r^{n,M}_{B\cap M}(\mathcal{N}')$ are of the form

$$\lambda_2 + \sum_{i \notin I'} m_i \cdot \alpha_i, \quad \mathrm{Re}(m_i) \geq 0.$$

This means $r^{n,M}_{B\cap M}(\mathcal{N} \otimes \mathbb{C}^{-\lambda_1})$ is tempered, as required.

□

Proof of Theorem 16.2.4. We will only give the proof when P_1 and P_2 are either all of G or the Borel subgroup (for the same reasons as in Theorem 15.4.2).

By symmetry, it is enough to consider the following cases:

Case 1: $P_2 = G$ and $P_1 = B$. In the case, by adjunction, the Hom in question equals

$$\mathrm{Hom}(r^{n,G}_B(\mathcal{M}_2), \mathbb{C}^{\mu_1} \otimes \mathbb{C}^{\lambda_1}),$$

where μ_1 is imaginary.

By assumption, the eigenvalues of $\mathfrak{t}$ on $r^{n,G}_B(\mathcal{M}_2)$ have real part ≥ 0, whereas λ_1 is strictly anti-dominant.

Case 2: $P_2 = B$ and $P_1 = B$. By adjunction, the Hom in question equals

$$\mathrm{Hom}(r^{n,G}_B \circ i^{n,G}_{B^-}(\mathbb{C}^{\mu_2} \otimes \mathbb{C}^{\lambda_2}), \mathbb{C}^{\mu_1} \otimes \mathbb{C}^{\lambda_1}),$$

where μ_1 and μ_2 are imaginary.

By Theorem 15.4.3, the eigenvalues of $\mathfrak{t}$ on $r^{n,G}_B \circ r^{n,G}_{B^-}(\mathbb{C}^{\mu_2} \otimes \mathbb{C}^{\lambda_2})$ are of the form $w(\mu_2 + \lambda_2)$ for $w \in W$. Since both λ_1 and λ_2 are regular anti-dominant, $w(\mu_2 + \lambda_2)$ can equal $\mu_1 + \lambda_1$ only for $w = 1$, $\mu_2 = \mu_1$ and $\lambda_1 = \lambda_2$

Thus, it remains to show that

$$\mathrm{Hom}(i^{n,G}_{B^-}(\mathbb{C}^{\mu} \otimes \mathbb{C}^{\lambda}), i^{n,G}_B(\mathbb{C}^{\mu} \otimes \mathbb{C}^{\lambda}))$$

is at most 1-dimensional. I don't have a palatable argument at the moment...

□

16.3. **Characters of representations.**

16.3.1. Let V be an irreducible admissible representation of G on a Hilbert space. Recall the notion of *operator of trace class*, see Sect. 2.6.3.

Our current goal is to prove the following theorem:

Theorem 16.3.2. *For $f \in C^\infty_c(G)$, the corresponding operator $T_f \in \mathrm{End}(V)$ is of trace class. The functional*

$$f \mapsto \mathrm{Tr}(T_f, V), \quad C^\infty_c(G) \to \mathbb{C}$$

is continuous, and only depends on the $(\mathfrak{g}, K)$-module $V^{K\text{-fin}}$.

16.3.3. For the proof we will need to review a bit of theory. Let V be a Hilbert space. A continuous endomorphism S of V is said to be *Hilbert-Schmidt* if for some/any choice of orthonormal basis v_i, the sum

$$||S||^2_{\mathrm{HS}} := \sum_i ||S(v_i)||^2 \tag{16.2}$$

converges.

More generally, one defines the Hilbert-Schmidt inner form on the space of Hilbert-Schmidt operators by

$$(S_1, S_2)_{\mathrm{HS}} = \sum_i (S(v_i), S(v_i)),$$

so that $||S||^2_{\mathrm{HS}} = (S, S)_{\mathrm{HS}}$.

16.3.4. Here is a nice point of view of where this comes from. Consider the (algebraic) tensor product $V \otimes V^*$, and equip it with the inner form

$$(v_1 \otimes v_1^*, v_2 \otimes v_2^*) = (v_1, v_2) \cdot (v_1^*, v_2^*),$$

where we note that $V^* \simeq \overline{V}$ via the inner form.

Define a map

$$V \otimes V \to \mathrm{End}(V), \quad v \otimes v^* \mapsto (w \mapsto v \cdot \langle v^*, w\rangle).$$

Then this an isometric embedding

$$V \otimes V^* \to \mathrm{End}(V)_{\mathrm{HS}},$$

and one can show that the image is dense. I.e., the above embedding identifies $\mathrm{End}(V)_{\mathrm{HS}}$ with the Hilbert space completion of $V \otimes V^*$.

16.3.5. We have the following assertion:

Proposition 16.3.6.

(a) *Any Hilbert-Schmid operator is compact.*

(a') *Let v_i be an orthonormal basis of V, and let S be and endomorphism of the algebraic span of the v_i's, so that the sum* (16.2) *converges. Then S continuously extends to a Hilbert-Schmid operator on V.*

(b) *For $S \in \mathrm{End}(V)_{\mathrm{HS}}$ and S' bounded, we have $S \circ S' \in \mathrm{End}(V)_{\mathrm{HS}}$ and $||S \circ S'||_{\mathrm{HS}} \leq ||S||_{\mathrm{HS}} \cdot ||S'||$, and similarly for $S' \circ S$.*

(c) *Any trace class operator is Hilbert-Schmidt.*

(d) *If S_1, S_2 are Hilbert-Schmidt, then $S_1 \circ S_2$ is trace class and*

$$\mathrm{Tr}(S_1 \circ S_2) = (S_1, S_2^\dagger)_{\mathrm{HS}}.$$

(e) *For S is trace class and S' bounded, then $S \circ S'$ and $S' \circ S$ are trace class and* $\mathrm{Tr}(S \circ S') = \mathrm{Tr}(S' \circ S)$.

(e') *In the situation of (e), $|\,\mathrm{Tr}(S \circ S')| \leq |\,\mathrm{Tr}(S)| \cdot ||S'||$.*

16.3.7. Let us split V according to the K-isotypic components

$$V \simeq \underset{\rho}{\oplus} V^\rho,$$

and recall (see Theorem 10.4.5) that $\dim(V^\rho) \leq \dim(\rho)^2$. Let μ_ρ denote the highest weight of ρ.

Let $C_\mathfrak{k} \in U(\mathfrak{k})$ denote the Casimir operator of $\mathfrak{k}$. Note that $C_\mathfrak{k}$ acts on V^ρ by the (positive) scalar $\lambda_\rho \geq (\mu_\rho, \mu_\rho)$, where $(-,-)$ is the inner form on $\mathfrak{k}$ used to produce $C_\mathfrak{k}$. In particular, $1 + T_{C_\mathfrak{k}}$ is an invertible endomorphism of $V^{K\text{-fin}}$.

The key trick in the proof of Theorem 16.3.2 is the following:

Proposition 16.3.8. *For a sufficiently large integer n, the endomorphism $(1 + T_{C_\mathfrak{k}})^{-n}$ of $V^{K\text{-fin}}$, extends continuously to all of V, and is Hilbert-Schmidt.*

Proof. As in Sect. 16.3.10 (see below), we may assume that the inner form on V is K-invariant. In particular, the different V^ρ are pairwise orthogonal. By Proposition 16.3.6(a'), it suffices to find n so that the sum

$$\underset{\rho}{\Sigma} (1 + \lambda_\rho)^{-2n} \cdot \dim(V^\rho)$$

converges.

Recall also that $\dim(\rho)$ is bounded by $||\mu_\rho||^m$ for m large enough.

Excluding finitely many K-types, the above sum is dominated by

$$\underset{\rho}{\Sigma} ||\mu_\rho||^{-2n+m}.$$

This sum indeed converges, since it is a sum over the lattice in a positive quadrant in a Eucledian space.

□

16.3.9. *Proof of Theorem 16.3.2.* Note that as endomorphims of V^∞, we have:

$$T_{(1+T_{C_\mathfrak{k}})^n} \circ T_{(1+T_{C_\mathfrak{k}})^n} \circ T_f \simeq T_{l((1+T_{C_\mathfrak{k}})^{2n})(f)}, \quad f \in C_c^\infty(G)$$

where $l(-)$ means the differential operator obtained from the given element of $U(\mathfrak{g})$ by left translations.

Hence,

$$T_f = T_{(1+T_{C_\mathfrak{k}})^{-n}} \circ T_{(1+T_{C_\mathfrak{k}})^{-n}} \circ T_{l((1+T_{C_\mathfrak{k}})^{2n})(f)},$$

as operators on all of V.

Since $T_{(1+T_{C_\mathfrak{k}})^{-n}}$ is Hilbert-Schmidt, we obtain that $T_{(1+T_{C_\mathfrak{k}})^{-n}} \circ T_{(1+T_{C_\mathfrak{k}})^{-n}}$ is trace class, by Proposition 16.3.6(d). Since $T_{l((1+T_{C_\mathfrak{k}})^{2n})(f)}$ is continuous, we obtain that T_f is trace class, by Proposition 16.3.6(e).

Note, furthermore, that the above argument goes through not only for smooth f, but for f differentiable $4n$ times (need that $l((1+T_{C_\mathfrak{k}})^{2n})(f) \in C_c(G)$). This shows the continuity of the map

$$f \mapsto \mathrm{Tr}(T_f)$$

in view of Proposition 16.3.6(e').

16.3.10. To prove that $\mathrm{Tr}(T_f)$ only depends on $V^{K\text{-fin}}$, we argue as follows.

By averaging the given inner form on V with respect to K, we can assume that it is K-invariant. (It is easy to show that this change of inner form does not affect the fact of T_f being of trace class V nor its trace).

Then

$$\mathrm{Tr}(T_f) = \sum_\rho \xi_\rho \circ \mathrm{Tr}(T_f) \circ \xi_\rho,$$

where θ_ρ is the projection on the V^ρ factor.

However, we saw in Corollary 11.1.5 that $\xi_\rho \circ \mathrm{Tr}(T_f) \circ \xi_\rho$ only depends on the infinitesimal equivalence class of ρ.

□

17. Week 10, Day 2 (Thurs., April 6)

17.1. Linear independence of characters.

17.1.1. We will prove the following theorem:

Theorem 17.1.2. *Let V_α be a collection irreducible admissible representations, so that no two are infinitesimally equivalent. Then their characters, viewed as distributions on G, are linearly independent.*

The rest of this subsection is devoted to the proof of this theorem.

17.1.3. We argue by contradiction. Suppose there is a linear dependence. Replace the original set of α's by a minimal subset such that the corresponding characters are still linearly dependent. Let ρ be a K-type so that $V_\alpha^\rho \neq 0$ for at least one α from the above subset.

Consider the algebra

$$\xi_\rho \cdot \mathcal{H}(G) \cdot \xi_\rho \subset \mathcal{H}_G,$$

where ξ_ρ is the projector on V^ρ, see Sect. 11.1.3.

This algebra acts on V^ρ for any admissible representation ρ. We claim:

Proposition 17.1.4. *The map*

$$\xi_\rho \cdot \mathcal{H}(G) \cdot \xi_\rho \to \bigoplus_\alpha \mathrm{End}_{\mathbb{C}}(V_\alpha^\rho)$$

is surjective.

Let us assume the proposition and finish the proof of the theorem. Let α_0 be such that $V_{\alpha_0}^\rho \neq 0$. According to the above lemma, we can find an element $f \in \xi_\rho \cdot \mathcal{H}(G) \cdot \xi_\rho$ such that it acts as 1 in $V_{\alpha_0}^\rho$ and as 0 in V_α^ρ for $\alpha \neq \alpha_0$.

But then $\mathrm{Tr}(f, V_{\alpha_0}) \neq 0$ and $\mathrm{Tr}(f, V_\alpha) = 0$, which is a contradiction.

□

17.1.5. *Proof of Proposition 17.1.4.* As in Proposition 11.3.6, the functor

$$\mathcal{M} \mapsto \mathcal{M}^\rho$$

maps irreducible (pairwise distinct) $(\mathfrak{g}, K)$-modules with a non-trivial ρ-component to irreducible (pairwise distinct) $\xi_\rho \cdot \mathcal{H}(G) \cdot \xi_\rho$-modules.

In particular, for those of the α's that $V_\alpha^\rho \neq 0$, the $\xi_\rho \cdot \mathcal{H}(G) \cdot \xi_\rho$-modules V_α^ρ are pairwise non-isomorphic. Now, the assertion follows from the next lemma:

Lemma 17.1.6. *Let A be an associative algebra, and let $\mathcal{N}_\alpha$ be a finite collection of irreducible finite-dimensional A-modules. Then the map*

$$A \to \underset{\alpha}{\oplus} \operatorname{End}_{\mathbb{C}}(\mathcal{N}_\alpha)$$

is surjective.

Proof. Follows from the fact that the only A-submodules in

$$\underset{\alpha}{\oplus} \mathcal{N}_\alpha \otimes \mathcal{N}_\alpha^*$$

are of the form

$$\underset{\alpha}{\oplus} \mathcal{N}_\alpha \otimes \mathcal{M}_\alpha, \quad \mathcal{M}_\alpha \subset \mathcal{N}_\alpha^*,$$

and the identity element in $\mathcal{N}_\alpha \otimes \mathcal{N}_\alpha^*$ is not contained in any proper subspace of this form. □

□

17.2. **Regularity of characters.**

17.2.1. Our next goal is the following fundamental result of Harish-Chandra's:

Theorem 17.2.2. *The character of an irreducible admissible representation, viewed as a distribution on G, is given by integration against a locally L_1-function; moreover, this function is C^∞ on the regular semi-simple locus.*

17.2.3. The actual theorem of Harsih-Chandra's is even more precise than that. Note that the character Θ of a representation V is a distribution that is Ad_G-invariant. Furthermore, since V was assumed irreducible, by Schur's lemma, the action of $Z(\mathfrak{g})$ on V^∞ is given by a homomorphism

$$\chi : Z(\mathfrak{g}) \to \mathbb{C}.$$

Hence,

$$z(\Theta) = \chi(z) \cdot \Theta, \tag{17.1}$$

where in the left-hand side, we use the action of $U(\mathfrak{g})$ on $\operatorname{Distr}(G)$, given by the formula

$$(u(\mathfrak{d}))(f) = \mathfrak{d}(l(u)(f)).$$

Harish-Chandra's theorem says:

Theorem 17.2.4. *Any distribution Θ on G which is Ad_G-invariant and satisfies* (17.1) *for some homomorphism $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is given by integration against a locally L_1-function; moreover, this function is C^∞ on the regular semi-simple locus.*

17.2.5. In the process of proof we will also see:

Theorem 17.2.6. *The space of functions on the regular semi-simple locus that are* Ad_G*-invariant and satisfy* (17.1) *(for a given* χ*) is finite-dimensional.*

Combined with Theorem 17.1.2, this implies Theorem 10.2.6, i.e.:

Theorem 17.2.7. *The set of irreducible* $(\mathfrak{g}, K)$*-modules with a given character of* $Z(\mathfrak{g})$ *is finite.*

17.3. **More on the Harish-Chandra map.** In this subsection we let G denote the complexification of our real reductive group. In order to avoid some silly technical difficulties, we will assume that the derived group of G is simply-connected (the initial G admits a finite cover by a group like this).

17.3.1. In order to disambiguate the notation, if λ is a character of $\mathfrak{t}$ that integrates to a character of T, we will denote the latter by e^λ.

Let $G^{\text{reg-ss}} \subset G$ be the open subset consisting of regular semi-simple elements. Choose a Cartan subgroup $T \subset G$, and let

$$T^{\text{reg}} := T \cap G^{\text{reg-ss}} = \{t \in T,\ \alpha(t) \neq 1,\ \forall \alpha\}.$$

By the assumption that the derived group of G is simply-connected, e^ρ is a character of T. Let Δ denote the function on T equal to

$$e^\rho \cdot \prod_\alpha (1 - e^{-\alpha}),$$

where the product is taken over the set of positive roots.

Note that Δ is W-equivariant against the sign character. (check for simple reflections, using the fact that α_i is the only positive root that is flipped to negative by s_i).

Note also that Δ is invertible on T^{reg}. Hence,

$$\mathfrak{d} \mapsto \mathrm{Ad}_{\Delta^{-1}}(\mathfrak{d}) := \Delta^{-1} \cdot \mathfrak{d} \cdot \Delta$$

is a well-defined automorphism of $\mathrm{Diff}(T^{\text{reg}})$. The W-equivariance property of Δ implies that this operation preserves W-invariant elements of $\mathrm{Diff}(T^{\text{reg}})$.

17.3.2. Note also that any Ad_G-invariant differential operator on $G^{\text{reg-ss}}$ gives rise to a W-invariant differential operator on T^{reg}. Indeed, the space of the latter identifies with the space of differential operators on T^{reg}/W and restriction defines an isomorphism

$$\mathcal{O}(G^{\text{reg-ss}})^G \simeq \mathcal{O}(T^{\text{reg}})^W.$$

Recall the normalized Harish-Chandra map $\phi^n : Z(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{g})$. We are going to prove:

Proposition 17.3.3. *The following diagram commutes:*

$$\begin{array}{ccccc}
Z(\mathfrak{g}) & \longrightarrow & \mathrm{Diff}(G)^{\mathrm{Ad}_G} & \longrightarrow & \mathrm{Diff}(G^{\text{reg-ss}})^{\mathrm{Ad}_G} \\
{\scriptstyle \phi^n}\big\downarrow & & & & \big\downarrow \\
\mathrm{Sym}(\mathfrak{t})^W & & & & \mathrm{Diff}(T^{\text{reg}})^W \\
\big\downarrow & & & & \big\uparrow{\scriptstyle \mathrm{Ad}_{\Delta^{-1}}} \\
\mathrm{Diff}(T)^W & & \longrightarrow & & \mathrm{Diff}(T^{\text{reg}})^W.
\end{array}$$

Proof. Since everything in sight is affine, it is sufficient to check that the two differential operators on T^{reg} resulting from the two circuits of the diagram from a given $z \in Z(\mathfrak{g})$ act in the same way on W-invariant functions (defined on all of T).

We take our supply of functions to be given by characters of irreducible fin.dim. G-representations. Thus, f be the character of the irreducible representation with highest weight λ. On the one hand, by the definition of ϕ^n, we have

$$z(f) = \langle \lambda + \rho, \phi^n(z) \rangle \cdot f.$$

On the other hand, the Weyl character formula says

$$f|_{T^{\text{reg}}} = \Delta^{-1} \cdot \left(\underset{w \in W}{\Sigma} \operatorname{sign}(w) \cdot e^{w(\lambda+\rho)} \right).$$

Hence, for $u \in \operatorname{Sym}(\mathfrak{t})^W$, and the corresponding element $\mathfrak{D} \in \operatorname{Diff}(T^{\text{reg}})^W$, we have

$$(\operatorname{Ad}_{\Delta^{-1}}(\mathfrak{D}))(f|_{T^{\text{reg}}}) = \Delta^{-1} \cdot \langle \lambda + \rho, u \rangle \cdot \left(\underset{w \in W}{\Sigma} \operatorname{sign}(w) \cdot e^{w(\lambda+\rho)} \right) = \langle \lambda + \rho, u \rangle \cdot f|_{T^{\text{reg}}},$$

as desired.

□

17.4. Digression: conjugacy classes of tori and regular semi-simple elements.

17.4.1. We return to the case when G is a real reductive group. We know that all Cartan subgroups in $G_{\mathbb{C}}$ are conjugate. But this is not the case for G.

For example, for $G = SL_2(\mathbb{R})$, there is the split Cartan (diagonal matrices), and the compact Cartan (which equals the maximal compact).

Here is the general picture.

17.4.2. Consider the affine variety

$$G/\!/ \operatorname{Ad}_G := \operatorname{Spec}(\mathcal{O}(G)^{\operatorname{Ad}_G}).$$

Let

$$(G/\!/ \operatorname{Ad}_G)^{\text{reg}} \subset G/\!/ \operatorname{Ad}_G$$

be the regular locus.

Namely, under the complexification

$$(G/\!/ \operatorname{Ad}_G)_{\mathbb{C}} \simeq T_{\mathbb{C}}/\!/W := \operatorname{Spec}(\mathcal{O}(T_{\mathbb{C}})^W),$$

$(G/\!/ \operatorname{Ad}_G)^{\text{reg}}_{\mathbb{C}}$ corresponds to

$$T^{\text{reg}}_{\mathbb{C}}/W \simeq T^{\text{reg}}_{\mathbb{C}}/\!/W \subset T_{\mathbb{C}}/\!/W,$$

where the first isomorphism is due to the fact that the action of W on $T^{\text{reg}}_{\mathbb{C}}$ is free.

Note that $(G/\!/ \operatorname{Ad}_G)^{\text{reg}}$ is typically disconnected. For example, for $G = SL_2(\mathbb{R})$,

$$G/\!/ \operatorname{Ad}_G \simeq \mathbb{R}$$

via $g \mapsto \operatorname{Tr}(g)$, and

$$(G/\!/ \operatorname{Ad}_G)^{\text{reg}} = \mathbb{R} - \{2, -2\}.$$

17.4.3. The preimages of points under the (tautological) map

$$G^{\text{reg-ss}} \to (G/\!/ \operatorname{Ad}_G)^{\text{reg}}$$

are called *stable conjugacy classes*. In other words, $g_1, g_2 \in G^{\text{reg-ss}}$ belong to the same conjugacy class if and only if there exists an element $g \in G_{\mathbb{C}}$ such that $\operatorname{Ad}_g(g_1) = g_2$.

However, *stably conjugate* does not mean *conjugate*. For example, for an element k in the maximal compact of $SL_2(\mathbb{R})$, it is stable conjugate to k^{-1}, but not conjugate.

Lemma 17.4.4. *Every stable conjugacy class consists of finitely many conjugacy classes.*

Proof. Let g be a regular semi-simple element, and let T be its centralizer; this is a compact subgroup of G. Then the set of conjugacy classes in the stable conjugacy class of t is the quotient of the set of

$$\{g \in G_{\mathbb{C}},\ \sigma(g) \cdot g \in T_{\mathbb{C}}\}$$

by the action of $T_{\mathbb{C}}$

$$g, t \mapsto g \cdot t$$

and the action of G

$$g, g_1 \mapsto g_1 \cdot g.$$

Hence, the above quotient injects into

$$H^1(\mathbb{Z}/2\mathbb{Z}, T_{\mathbb{C}}),$$

where $\mathbb{Z}/2\mathbb{Z}$ acts on $T_{\mathbb{C}}$ by σ.

However, the latter cohomology group is finite. Indeed, writing T as

$$0 \to T^1 \to T \to T^2 \to 0,$$

with T^1 a split form and T^2 a compact form, we have $H^1(\mathbb{Z}/2\mathbb{Z}, T^1_{\mathbb{C}}) = 0$ (by Hilbert 90), where as

$$H^1(\mathbb{Z}/2\mathbb{Z}, T^2_{\mathbb{C}}) \simeq \pm 1 \underset{\mathbb{Z}}{\otimes} \Lambda,$$

where Λ is such that

$$T^2_{\mathbb{C}} = \Lambda \underset{\mathbb{Z}}{\otimes} \mathbb{G}_m.$$

□

17.4.5. For a regular semi-simple element $g \in G$, its centralizer is a Cartan subgroup in G. One can show:

Proposition 17.4.6. *Elements of the same stable conjugacy class give rise to conjugate Cartan subroups.*

Proof. Let g be a regular semi-simple element and let T be its stabilizer. Then T (viewed as a real torus), and T' be the split part of T. Replacing G by the centralizer of T', we can assume that T is compact modulo the center. In the latter case it is known that T is the only conjugacy class of Cartan subgroups that are compact modulo the center. □

17.4.7. Thus, we obtain a map

$$(G/\!/ \operatorname{Ad}_G)^{\text{reg}} \to \{\text{Conjugacy classes of Cartan subgroups}\}.$$

However, it is easy to see that this map factors via a map

$$\pi_0((G/\!/ \operatorname{Ad}_G)^{\text{reg}}) \to \{\text{Conjugacy classes of Cartan subgroups}\},$$

i.e., points in the same connected component of $(G/\!/ \operatorname{Ad}_G)^{\text{reg}}$ give rise to conjugate Cartans.

In particular, we obtain that the set of conjugacy classes of Cartan subgroups in G is *finite*.

17.4.8. Let ϵ be a conjugacy class of Cartan subgroups; fix a representative T_ϵ. Let
$$G^{\text{reg-ss}}_\epsilon \subset G^{\text{reg-ss}}$$
be the subset consisting of those elements that can be conjugated into T_ϵ.

Then $G^{\text{reg-ss}}_\epsilon$ is open and closed in $G^{\text{reg-ss}}$ (i.e., it is the union of some of the connected components). Indeed, $G^{\text{reg-ss}}_\epsilon$ equals the preimage under
$$G^{\text{reg-ss}} \to (G/\!/\operatorname{Ad}_G)^{\text{reg}}$$
of
$$(G/\!/\operatorname{Ad}_G)^{\text{reg}}_\epsilon \subset (G/\!/\operatorname{Ad}_G)^{\text{reg}},$$
the latter being the union of those connected components that give rise to ϵ.

The quotient $G^{\text{reg-ss}}_\epsilon/\operatorname{Ad}_G$ is defined as a real analytic manifold, and we have a finite covering map
$$G^{\text{reg-ss}}_\epsilon/\operatorname{Ad}_G \to (G/\!/\operatorname{Ad}_G)^{\text{reg}}_\epsilon \tag{17.2}$$
whose fibers are conjugacy classes within a given stable conjugacy class.

17.4.9. Here is what this picture looks like for $SL_2(\mathbb{R})$. As we remarked above, there are two conjugacy classes of Cartans: the split one and the compact one.

The subset
$$G^{\text{reg-ss}}_{\text{split}} \subset G^{\text{reg-ss}}$$
is that of hyperbolic elements (i.e., diagonalizable matrices with distinct eigenvalues). It consists of two connected components (the eigenvalues being positive or negative). Equivalently, they are the elements g with $\operatorname{Tr}(g) > 2$ and $\operatorname{Tr}(g) < -2$, respectively. The corresponding map (17.2) is an isomorphism.

The subset
$$G^{\text{reg-ss}}_{\text{comp}} \subset G^{\text{reg-ss}}$$
is that of elliptic elements. It is connected. It consists of elements g satisfying
$$-2 \leq \operatorname{Tr}(g) \leq 2.$$

The corresponding map (17.2) is a 2-fold covering.

17.5. **Smoothness of the distribution on the regular semi-simple locus.** In this subsection we will prove the easy part of Theorem 17.2.4, namely that for any distribution Θ on G which is Ad_G-invariant and satisfies (17.1) for some homomorphism $\chi : Z(\mathfrak{g}) \to \mathbb{C}$, its restriction to $G^{\text{reg-ss}}$ is given by an analytic function.

We will also prove Theorem 17.2.6.

17.5.1. Since Θ is Ad_G-invariant we can regard it as a distribution on real analytic manifold $G^{\text{reg-ss}}/\operatorname{Ad}_G$. Since the action of $Z(\mathfrak{g})$ commutes with the Ad_G-action, we have a well-defined map
$$Z(\mathfrak{g}) \to \operatorname{Diff}(G^{\text{reg-ss}}/\operatorname{Ad}_G),$$
and out distrubition saitsfies
$$z(\Theta) = \chi(z)\cdot\Theta, \quad z \in Z(\mathfrak{g}), \tag{17.3}$$
viewed as distributions on $G^{\text{reg-ss}}/\operatorname{Ad}_G$.

The key claim is the following:

Proposition 17.5.2. *Locally on* $G^{\text{reg-ss}}/\operatorname{Ad}_G$*, the ring* $\operatorname{Diff}(G^{\text{reg-ss}}/\operatorname{Ad}_G)$ *is finitely generated over* $\mathcal{O}(G^{\text{reg-ss}}/\operatorname{Ad}_G)$ *and* $Z(\mathfrak{g})$*.*

17.5.3. Let us deduce the smoothness of Θ (and the finite-dimensionality of such Θ's) from Proposition 17.5.2.

Working locally on $G^{\text{reg-ss}}/\operatorname{Ad}_G$ (i.e., over some open), choose a set of generators $\mathfrak{d}_0, \mathfrak{d}_1, ..., \mathfrak{d}_k$ of $\operatorname{Diff}(G^{\text{reg-ss}}/\operatorname{Ad}_G)$ as a module over $\mathcal{O}(G^{\text{reg-ss}}/\operatorname{Ad}_G)$ and the image of $Z(\mathfrak{g})$ with $\mathfrak{d}_0 = 1$.

Then for a basis of vector fields $\xi_1, ..., \xi_n$, we have

$$\xi_i \cdot \mathfrak{d}_j = \sum_{j'=0,...,k} \sum_l f^l_{j'} \cdot \mathfrak{d}_{j'} \cdot z^l_{j'},$$

where l runs over some finite set depending on i, j, j', and $f^l_{j'}$ is a function and $z^l_k \in Z(\mathfrak{g})$.

Consider the system of linear first order differential equations on a tuple $(\Theta_0, ..., \Theta_k)$

$$\xi_i(\Theta_j) = \sum_{j'=0,...,k} \left(\sum_l f^l_{j'} \cdot \chi(z_{j'}) \right) \cdot \Theta_{j'}. \tag{17.4}$$

Then any distributional solution of this system is smooth. And the dimension of the space of solutions is bounded by $k+1$.

Now, if Θ is a distributional solution of (17.3), then

$$\{\Theta_i := \mathfrak{d}_i \cdot \Theta\}$$

is a solution of (17.4).

In particular, $\Theta = \Theta_0$ is smooth, and the dimension of the space of such is bounded by $k+1$. □

18. Week 11, Day 1 (Tue, April 11)

18.1. **Proof of Proposition 17.5.2.**

18.1.1. Pulling back with respect to

$$G^{\text{reg-ss}}/\operatorname{Ad}_G \to (G/\!/\operatorname{Ad}_G)^{\text{reg}}, \tag{18.1}$$

it is easy to see that the statement of the proposition for the original homomorphism

$$Z(\mathfrak{g}) \to \operatorname{Diff}(G^{\text{reg-ss}}/\operatorname{Ad}_G)$$

is equivalent to one obtained by composing with

$$\mathfrak{d} \mapsto \Delta^{-1} \cdot \mathfrak{d} \cdot \Delta.$$

Note that since (17.2) again is a covering map, we can pull back differential operators from $(G/\!/\operatorname{Ad}_G)^{\text{reg}}$ to $G^{\text{reg-ss}}$.

Now, it follows from Proposition 17.3.3 that the above homomorphism

$$Z(\mathfrak{g}) \to \operatorname{Diff}(G^{\text{reg-ss}}/\operatorname{Ad}_G)$$

(after conjugation by Δ) is obtained in this way from a homomorphism

$$Z(\mathfrak{g}) \to \operatorname{Diff}((G/\!/\operatorname{Ad}_G)^{\text{reg}}), \tag{18.2}$$

which is a real version of

$$Z(\mathfrak{g}_\mathbb{C}) \overset{\phi^n}{\simeq} \operatorname{Sym}(\mathfrak{t}_\mathbb{C})^W \to \operatorname{Diff}(T_\mathbb{C}/\!/W) \to \operatorname{Diff}(T^{\text{reg}}_\mathbb{C}/\!/W) \simeq \operatorname{Diff}((G/\!/\operatorname{Ad}_G)^{\text{reg}}_\mathbb{C}).$$

Since (18.1) is a covering map, its suffices to prove the corresponding finite generation statement for (18.2).

18.1.2. For each ϵ, the subset $\mathrm{Diff}((G/\!/\operatorname{Ad}_G)^{\mathrm{reg}})_\epsilon$ receives a finite covering map from $T^{\mathrm{reg}}_\epsilon$. Hence, it is enough to prove the finite generation assertion on $T^{\mathrm{reg}}_\epsilon$.

Since $\mathrm{Sym}(\mathfrak{t})$ is finitely generated over $\mathrm{Sym}(\mathfrak{t})^W$, it suffices to prove our assertion for the latter replaced by the former.

However, it is obvious that $\mathrm{Sym}(\mathfrak{t})$ and $\mathcal{O}(T_\epsilon)$ generate $\mathrm{Diff}(T_\epsilon)$.

□

18.2. **The local L_1-property: an introduction.** We now wish to prove the more difficult part of Theorem 17.2.4, namely that Θ is given by integration against a locally L_1 function.

This will result from the combination of the following three assertions.

18.2.1. First, we recall that since $|\Delta|$ is well-defined as a function on $G/\!/\operatorname{Ad}_G$, we can pull it back to G and obtain an Ad_G-invariant function.

We will prove:

Theorem 18.2.2. *The function $|\Theta\cdot\Delta|$, which is a C^∞ function on $G^{\text{reg-ss}}$ is* locally bounded *on G, i.e., it restriction to $U\cap G^{\text{reg-ss}}$ is bounded, whenever $U\subset G$ is compact.*

Theorem 18.2.3. *The inverse of $|\Delta|$ is locally an L_1 function, i.e., the integral*

$$f\mapsto \int\limits_{G^{\text{reg-ss}}} f\cdot|\Delta^{-1}|,\quad f\in C_c(G)$$

is absolutely convergent and defines a continuous functional $C_c(G)\to\mathbb{C}$.

Theorem 18.2.4. *If a distribution on G which is* Ad_G*-invariant and satisfies* (17.1) *vanishes when restricted to $G^{\text{reg-ss}}$, then it is zero.*

18.2.5. Let us see how these statements combined yield the desired local L_1 property of Θ.

Proof. By combining Theorems 18.2.2 and 18.2.3, we obtain that

$$\widetilde{\Theta}(f):=\int\limits_{G^{\text{reg-ss}}} f\cdot\Theta$$

is absolutely convergent and defines a continuous functional $C_c(G)\to\mathbb{C}$.

It remains to see that $\Theta(f)=\widetilde{\Theta}(f)$. However, $\Theta-\Theta'$ is a distribution that vanishes on $G^{\text{reg-ss}}$. Hence, it is zero by Theorem 18.2.4.

□

18.3. **Digression: the Grothendieck-Springer resolution and the Steinberg variety.**

18.3.1. Let G be a complex reductive group. Let X denote its flag variety, thought of as the variety of Borel subgroups in G. For $x \in X$ we let $B_x \subset G$ denote the corresponding Borel subgroup.

Recall that the Grothendieck-Springer resolution, denoted $\widetilde{G}$, is the the closed subvariety in

$$G \times X$$

consisting of pairs (g, x) such that $g \in B_x$.

We have the tautological maps

$$G \overset{q}{\longleftarrow} \widetilde{G} \overset{p}{\longrightarrow} X.$$

The map q is proper. Its fiber over a given $g \in G$ is the variety of Borel subgroups that contain g; equivalently, this is the fixed-point set X^g of $g \in G$ acting on X. Over $G^{\text{reg-ss}}$, the map is an étale Galois cover with Galois group W.

The map p is a smooth fibration. In fact that it is what is called the tautological fibration: the fiber over $x \in X$ is B_x.

We have a natural projection r from $\widetilde{G}$ to the *abstract Cartan* T.

18.3.2. The Grothendieck-Steinberg variety St_G is defined to be

$$\widetilde{G} \underset{G}{\times} \widetilde{G}.$$

In other words, this is a closed subvariety in

$$G \times X \times X$$

consisting triples (g, x_1, x_2) such that $g \in B_{x_1} \cap B_{x_2}$.

For $w \in W$, let St_G^w be the locally closed subvariety of St_G corresponding to the condition that the Borels x_1 and x_2 are in relative position w. The map

$$\mathrm{St}_G^w \to (X \times X)^w$$

is a smooth fibration, where $(X \times X)^w \subset (X \times X)$ is the corresponding G-orbit (Schubert cell), i.e., the variety of pairs (x_1, x_2) in relative position w.

Every point of St_G is contained in exactly one St_G^w. Let $\overline{\mathrm{St}}_G^w$ denote the closure of St_G^w in St_G. The subvarieties $\overline{\mathrm{St}}_G^w$ are the irreducible components of St_G^w.

18.3.3. Let now G be a real group. We consider the following hybrid, denoted ReSt_G. This is the closed subvariety in

$$G_{\mathbb{C}} \times X_{\mathbb{C}}$$

equal to

$$\widetilde{G}_{\mathbb{C}} \cap G \times \mathbb{C}.$$

Equivalently, if σ is the involution on $G_{\mathbb{C}}$ corresponding to G, then ReSt_G is the subset of

$$G_{\mathbb{C}} \times X_{\mathbb{C}} \times X_{\mathbb{C}}$$

equal to the intersection of $\mathrm{St}_{G_{\mathbb{C}}}$, $G \times X_{\mathbb{C}} \times X_{\mathbb{C}}$ and the subvariety given by the condition that $\sigma(x_1) = x_2$.

Let ReSt_G^w denote the intersection

$$\mathrm{ReSt}_G \cap \mathrm{St}_{G_{\mathbb{C}}}^w.$$

Let $\overline{\mathrm{ReSt}}_G^w$ denote the closure of ReSt_G^w.

18.3.4. Suppose that for a given $w \in W$, the variety

$$X^w_{\mathbb{C}} := \{x, \sigma(x) \overset{w}{\sim} x\}$$

is non-empty (which is a necessary condition for ReSt^w_G to be non-empty). One can show that $X^w_{\mathbb{C}}$ is the disjoint union of (finitely many) G-orbits.

In particular, we obtain that G has finitely many orbits on $X_{\mathbb{C}}$.

For a G-orbit ω on $X_{\mathbb{C}}$, let $X^\omega_{\mathbb{C}}$ be its preimage in ReSt_G, and let $\overline{\mathrm{ReSt}}^\omega_G$ denotes its closure.

18.3.5. To an orbit ω one can attach a particular real form T_ω of $T_{\mathbb{C}}$. Namely, choose a point $x \in X^\omega_{\mathbb{C}}$, and consider the group

$$B_x \cap B_{\sigma(x)}. \tag{18.3}$$

It is solvable, and its canonical toric quotient identifies with $T_{\mathbb{C}}$. Now, (18.3) has a canonical real structure, given by swapping the factors. This real structure induces one on the toric quotient.

A different point x' on the same orbit will give rise to the same real structure: if $g \cdot x = x'$, then the action of g isomorphes the two situations.

18.3.6. It follows from the construction that for a given ω, the map

$$r : \widetilde{G}_{\mathbb{C}} \to T_{\mathbb{C}}$$

restricts to a map

$$\mathrm{ReSt}^\omega_G \to T_\omega,$$

and hence also to a map

$$\overline{\mathrm{ReSt}}^\omega_G \to T_\omega.$$

Furthermore, if $(g, x) \in \mathrm{ReSt}^\omega_G$ is such that g is regular semi-simple, then the centralizer T of g on G is contained in B_x, and the the resulting map

$$T \subset B_x \cap G \to T_\omega$$

is an isomorphism.

18.3.7. Vice versa, given a point $x \in X^\omega$, the intersection $B_x \cap B_{\sigma(x)}$ is a subgroup of $G_{\mathbb{C}}$ compatible with its real structure and containing a Cartan subgroup. Hence,

$$B_x \cap B_{\sigma(x)} \cap G$$

contains a Cartan subgroup of G. This Cartan projects isomorphically to T_ω.

18.4. **Proof of Theorem 18.2.2.**

18.4.1. Let

$$\mathrm{ReSt}^{\text{reg-ss}}_G \subset \mathrm{ReSt}_G$$

be the preimage of $G^{\text{reg-ss}} \subset G$ under the map $q : \mathrm{ReSt}_G \to G$.

The map q is proper and surjective. Hence, it suffices to see that the pullback of $|\Theta \cdot \Delta|$, viewed now as a continuous function on $\mathrm{ReSt}^{\text{reg-ss}}_G$, is locally bounded.

For that, it is sufficient to show that for for every ω, the pullback of $|\Theta \cdot \Delta|$ to

$$\mathrm{ReSt}^\omega_G \cap \mathrm{ReSt}^{\text{reg-ss}}_G$$

extends to a continuous function on $\overline{\mathrm{ReSt}}^\omega_G$.

18.4.2. First, we claim that the restriction $|\Theta\cdot\Delta|$ to $\mathrm{ReSt}^{\omega}_G\cap\mathrm{ReSt}^{\text{reg-ss}}_G$ is the pullback of a function $\widetilde{\Theta}_\omega$ on T^{reg}_ω. In fact, this is the case for any Ad_G-invariant function on $\mathrm{ReSt}^{\omega}_G\cap\mathrm{ReSt}^{\text{reg-ss}}_G$:

Indeed, any two points in $\mathrm{ReSt}^{\omega}_G\cap\mathrm{ReSt}^{\text{reg-ss}}_G$ that map to the same point in T^{reg}_ω are uniquely conjugate by a unipotent element of $G_{\mathbb{C}}$, which then necessarily belongs to G.

Thus, it suffices to show that $\widetilde{\Theta}_\omega$ extends continuously to all of T_ω.

18.4.3. We claim that the function $\widetilde{\Theta}_\omega$ satisfies the differential equation

$$\phi^n(z)(\widetilde{\Theta}_\omega)=\chi(z)\cdot\widetilde{\Theta}_\omega,\quad z\in Z(\mathfrak{g}). \tag{18.4}$$

This follows via Proposition 17.3.3 from the fact that $\Theta|_{\mathrm{ReSt}^{\omega}_G\cap\mathrm{ReSt}^{\text{reg-ss}}_G}$ itself satisfies the differential equation

$$z(\widetilde{\Theta})=\chi(z)\cdot\widetilde{\Theta},\quad z\in Z(\mathfrak{g}).$$

18.4.4. Finally, we claim that any function on T^{reg}_ω that satisfies (18.4) extends continuously (in fact, as a smooth function) to all of T_ω.

Indeed, this follows as in Sect. 17.5.3: such a function can be realized as one coordinate of a solution of a vector-valued system of 1st order linear differential equations with smooth coefficients.

18.5. **Proof of the local L_1 property of $|\Delta|^{-1}$.**

18.5.1. We need to show that for every compact subset $U\subset G$, the integral of $|\Delta|^{-1}|$ over $U\cap G^{\text{reg-ss}}$ (with respect to the Haar measure on G) is convergent. Let $\widetilde{U}$ denote the preimage of U under the map $q:\mathrm{ReSt}_G\to G$.

Since the map

$$\mathrm{ReSt}^{\text{reg-ss}}_G:=\mathrm{ReSt}_G\underset{G}{\times}G^{\text{reg-ss}}\overset{q}{\to}G^{\text{reg-ss}}$$

is a finite covering map, it is enough to prove the convergence of the corresponding integral over $\widetilde{U}\cap\mathrm{ReSt}^{\text{reg-ss}}_G$, with respect to the pullback of the Haar measure (we can pullback measures along local isomorphisms).

18.5.2. Note that $\mathrm{ReSt}^{\text{reg-ss}}_G$ (unlike all of ReSt_G) is smooth, and the above measure on it is given by a top differential form, equal to the pullback of the bi-G-invariant top differential form on G.

Note that the top differential form on G is obtained by restriction from an (algebraic) top differential form ω_G on $G_{\mathbb{C}}$. Let $\omega_{\widetilde{G}}$ denote the pullback of ω_G along

$$q:\widetilde{G}_{\mathbb{C}}\to G_{\mathbb{C}}.$$

Then our top differential form on $\mathrm{ReSt}^{\text{reg-ss}}_G$ is obtained by restriction along

$$\mathrm{ReSt}^{\text{reg-ss}}_G\hookrightarrow\widetilde{G}^{\text{reg-ss}}_{\mathbb{C}}\hookrightarrow\widetilde{G}_{\mathbb{C}}.$$

18.5.3. Note that Δ is a well-defined (algebraic) function on $T_{\mathbb{C}}$. Its locus of zeros is the divisor

$$T_{\mathbb{C}} - T_{\mathbb{C}}^{\text{reg}}.$$

Note that we have a Cartesian diagram

$$\begin{array}{ccc} \widetilde{G}_{\mathbb{C}}^{\text{reg-ss}} & \longrightarrow & \widetilde{G}_{\mathbb{C}} \\ \big\downarrow & & \big\downarrow \\ T_{\mathbb{C}}^{\text{reg}} & \longrightarrow & T_{\mathbb{C}}. \end{array}$$

In particular, the complement to $\widetilde{G}_{\mathbb{C}}^{\text{reg-ss}}$ in $\widetilde{G}_{\mathbb{C}}$ is cut out by (the pullback) of Δ.

Note that the map q is ramified over $G_{\mathbb{C}} - G_{\mathbb{C}}^{\text{reg-ss}}$. Hence, $\omega_{\widetilde{G}}$ has a zero along the divisor $\widetilde{G}_{\mathbb{C}} - \widetilde{G}_{\mathbb{C}}^{\text{reg-ss}}$. Hence, we obtain that

$$\omega'_{\widetilde{G}} := \omega_{\widetilde{G}} \cdot \Delta^{-1}$$

is a well-defined algebraic differential form on $\widetilde{G}_{\mathbb{C}}$.

18.5.4. Hence, out task is to show that for any compact $\widetilde{U} \subset \text{ReSt}_G$, the integral

$$\int\limits_{\widetilde{U} \cap \text{ReSt}_G^{\text{reg-ss}}} |\omega'_{\widetilde{G}}|$$

is convergent.

However, this follows from the next general assertion:

Lemma 18.5.5. *Let Y be a real algebraic variety of dimension k. Let ω be an algebraic k-differential form on Y. Then for any compact subset $U \subset Y(\mathbb{R})$, the integral*

$$\int\limits_{U \cap Y^{\text{sm}}(\mathbb{R})} |\omega|$$

is convergent.

The proof can be obtained from the Noether normalization lemma.

18.6. **Proof of the uniqueness of extension (for $SL_2(\mathbb{R})$).** Our current task is to prove Theorem 18.2.4. We will only do this in the case of $G = SL_2(\mathbb{R})$.

18.6.1. We first consider the open subset

$$G^{\text{reg}} = G - \{\pm 1\}.$$

It contains as an open $G^{\text{reg-ss}}$, and the complement

$$G^{\text{reg}} - G^{\text{reg-ss}}$$

is the union of two G-orbits: one is the orbit of the non-trivial unipotent element, and the other its translate by -1.

We will first show that if Θ is an Ad_G-invariant distribution on G^{reg}, supported on the closed subset $G^{\text{reg}} - G^{\text{reg-ss}}$, and satisfying

$$z(\Theta) = \chi(z) \cdot \Theta,$$

then $\Theta = 0$.

We will consider the case when Θ is supported on the regular unipotent orbit. The other case is similar.

18.6.2. The key observation is that $G^{\mathrm{reg}}/\operatorname{Ad}_G$ is still a well-defined differentialble manifold. The distribution Θ descend to a well-defined distribution, denoted $\widetilde{\Theta}$ on $G^{\mathrm{reg}}/\operatorname{Ad}_G$.

The regular unipotent orbit corresponds to a point $x \in G^{\mathrm{reg}}/\operatorname{Ad}_G$; let t denote a local coordinate around this point so that $t(x) = 0$. Being a distribution supported at one point, $\widetilde{\Theta}$ has the form

$$\sum_{k\geq 0} a_k \cdot \partial_t^k \cdot \delta_x. \tag{18.5}$$

An element $z \in Z(\mathfrak{g})$ gives rise to a differential operator on $G^{\mathrm{reg}}/\operatorname{Ad}_G$. Take $z = C_{\mathfrak{g}}$, the Casimir element. The corresponding differential operator on $G^{\mathrm{reg}}/\operatorname{Ad}_G$ is of order 2; write it as

$$\mathfrak{d} := (b_0 \cdot \partial_t^2 + b_1 \cdot t \cdot \partial_t^2) + (c_0 \cdot \partial_t) + \text{ terms with higher order of vanishing with respect to } t.$$

Lemma 18.6.3. *We have: $b_0 = 0$, $b_1 = 2$, $c_0 = 3$.*

Let us assume the lemma and finish the proof of the theorem.

18.6.4. Let K be the largest integer such that $a_K \neq 0$ in (18.6). When we apply to it an operator

$$(b_1 \cdot t \cdot \partial_t^2) + (c_0 \cdot \partial_t) + \text{ terms with higher order of vanishing with respect to } t$$

we get a distribution

$$\sum_{K+1\geq k\geq 0} a'_k \cdot \partial_t^k \cdot \delta_x, \tag{18.6}$$

where

$$a'_{K+1} = a_K \cdot (c_0 - (K+1) \cdot b_1).$$

In particular, (18.6) cannot be eigen-distribution for the above operator unless

$$c_0 = (K+1) \cdot b_1.$$

However, in our case $c_0 = 3$ and $b_1 = 2$ and no such K exists, since 2 is not divisible by 3.

18.6.5. *Proof of Lemma 18.6.3.* We will determine the coefficients by acting by $C_{\mathfrak{g}}$ on some particular Ad_G-invariant functions. Namely, let f be such a function with the first order of vanishing at x. We need to show:

$$(\mathfrak{d}(f^2))(x) = 0, \quad (\mathfrak{d}(f)((x) = 3, \quad \frac{\mathfrak{d}(f^2)}{f}(x) = 10. \tag{18.7}$$

We take

$$f(g) = \widetilde{f}(g) - 2, \quad \widetilde{f}(g) = \operatorname{Tr}(g).$$

Recall that

$$C_{\mathfrak{g}} = E \cdot F + F \cdot E + \frac{1}{2} \cdot H^2.$$

Hence,

$$C_{\mathfrak{g}}(\widetilde{f}) = \frac{3}{2} \cdot \widetilde{f}, \quad C_{\mathfrak{g}}(\widetilde{f}^2 - 1) = 4 \cdot (\widetilde{f}^2 - 1).$$

This implies (18.7).

□

18.6.6. Finally, we are left to deal with the case that Θ is a distribution supported at $\pm 1 \subset G$. We claim that no such distribution cab be eigen for $C_{\mathfrak{g}}$. We will deal with the case of $1 \in G$; the case of -1 is similar.

The space of distributions supported at 1 is isomorphic to $U(\mathfrak{g})$, and the action of $Z(\mathfrak{g})$ corresponds to multiplication. Now the assertion follows from the fact that $U(\mathfrak{g})$ has no zero divisors.