## MATH 224, PROBLEM SET 3.

**1.** In this problem we will prove, following Sasha's suggestion that the group $SO(n)$ is connected. Consider the standard action of $SO(n)$ on $\mathbb{A}^n$, and let $S^{n-1} \subset \mathbb{A}^n$ be the subvariety consisting of elements of length one (with respect to the inner form). Show that:

(a) The action of $SO(n)$ on $S^{n-1}$ is transitive;

(b) The variety $S^{n-1}$ is irreducible;

(c) The stabilizer of the unit vector $e_1 \in S^{n-1}$ is isomorphic to $SO(n-1)$.

(d) Show by induction that (a), (b), (c) imply that $SO(n)$ is connected.

**1'.** Adapt the above proof to show that $Sp(2n)$ is connected by considering its action on $\mathbb{A}^{2n} - 0$. (Note that the stabilizer of a non-zero vector is no longer $Sp(2(n-1))$, but a group that projects onto it with a connected kernel.)

**2.** Let $G$ be an algebraic group over an algebraically closed field $k$. Let $H \subset G(k)$ be a subgroup, and let $Y \subset G$ be the Zariski closure of $H$. Show that $Y$ is a subgroup.

**3.** Let $\phi : G_1 \to G_2$ be a homomorphism of algebraic groups such that $\ker(\phi)$ is finite. Show that $\phi$ is finite as a map of schemes. Hint: use the fact that if $\phi : X_1 \to X_2$ is a map between schemes with finite fibers, then it is finite over a non-empty open subscheme of $X_2$.

**4.** Let $G$ act on $X$ and assume that $X$ is reduce and the action of $G(k)$ on $X(k)$ is transitive. Show that the morphism

$$G \times X \to X \times X, \quad (g, x) \mapsto (g \cdot x, x)$$

is faithfully flat.

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