Automorphic Gluing Functor.

11/16/2022

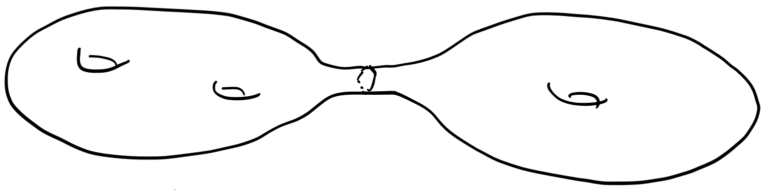
(Joint with D. Nadler)

X

$$\rightsquigarrow A(X,S) = Sh_{\mathcal{N}}(Bun_{G,N}(X,S))$$

$$\rightsquigarrow \mathcal{B}(X,S) = IndCoh_{\mathcal{N}}(Loc_{G^\vee,B^\vee}(X,S))$$

Spectral Gluing Equivalence (Ben-Zvi, Nadler, Preygel)

$X =$ 

$X_- \quad \# \quad X_+$

$p_- \quad p_+$

$X_- \qquad X_+$

$$\mathcal{B}(X_-,p_-) \underset{\underline{\underline{\mathcal{H}_{aff}}}}{\otimes} \mathcal{B}(X_+,p_+) \xrightarrow{\sim} \mathcal{B}(X)$$

Expect:

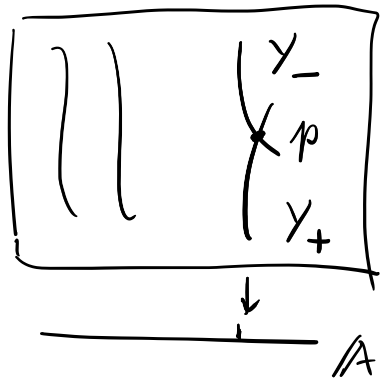
$$A(X_-,p_-) \underset{\mathcal{H}_{aff}}{\otimes} A(X_+,p_+) \xrightarrow{\sim} \underline{\underline{A(X)}}.$$

Today: give functor $\longrightarrow$

$\mathcal{Y}$

$\downarrow$ proj. family curves.

$\mathbb{A}^1$



Sm. surface

$$\begin{tikzcd}
A(Y_-, p) \otimes A(Y_+, p) \arrow[rr] \arrow[dr, "\text{glue}"] & & A(\mathcal{Y}_t) \\
& Sh_{\mathcal{N}}\left(\mathrm{Bun}_G\left(\mathcal{Y}|_{\mathbb{R}_{>0}}/\mathbb{R}_{>0}\right)\right) \arrow[ur, "i_t^*"'] &
\end{tikzcd}$$

H_{aff}

$t>0$.

auto. gluing functor.

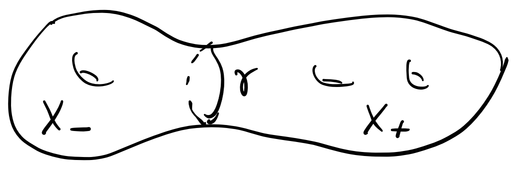
universal nilp. cone.

$\downarrow$ $\mathbb{R}_{>0}$

Conj: i_t^* is an equivalence.

Properties 1) "glue": $Eis^{\lambda_+} \otimes Eis^{\lambda_-} \longmapsto Eis^{\lambda_+ + \lambda_-}$

2) glue takes $Wh_+ \otimes Wh_- \longmapsto \underline{\underline{Wh}} \otimes_{\underline{\underline{\mathcal{O}(H^\vee/\!/W)}}} \mathcal{O}(H^\vee)$

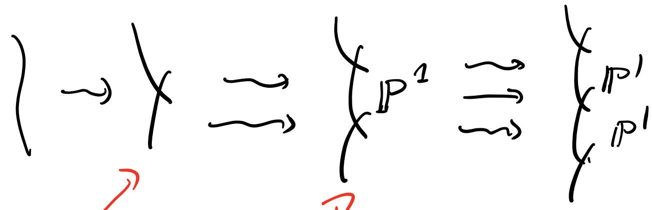


$$Loc_{G^\vee}(X) \longrightarrow G^\vee/G^\vee$$

$\uparrow$ monodromy along γ

① Univ. nilp. cone

② Bubbling of curves

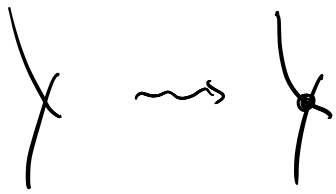


$A_- \underset{H}{\otimes} A_+$

$A(Y_t) \leftarrow A_- \otimes A_+ \leftleftarrows A_- \otimes H \otimes A_+ \leftleftarrows \cdots$

arrows: $^L\Psi$.

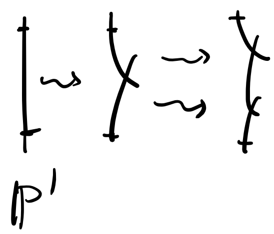
③ Twisting :



$\circlearrowleft \mu_k \quad \eta \in \mathrm{Hom}(\mu_k, G)/\sim.$

$= \quad \mathrm{Bun}_{G,N}(X_-, p) \underset{H}{\times} \mathrm{Bun}_{G,N}(X_+, p)$

④



$\rightsquigarrow$ bubbling Hecke category

$\mathcal{H}^{bub.} \cong \mathcal{H}_{aff}.$

⑤ Commutativity of Ψ. over higher dim base

Thom-Nadler criterion.

① Universal nilp. cone.

$$\begin{array}{cl} \mathcal{X} & \text{smooth} \\ \pi \downarrow & \text{proj. family of curves} \\ S & = \text{smooth.} \end{array}$$

$$\begin{tikzcd} \mathrm{Bun}_G(\mathcal{X}/S) = \mathrm{Bun}_G(\pi) \arrow[d] & \supset & \mathrm{Bun}_G(\mathcal{X}_s) \arrow[d] \\ S & \ni & s \end{tikzcd}$$

$$\mathcal{N}_s \subset T^*_{rel}\big|_s = T^*\mathrm{Bun}_G(\mathcal{X}_s).$$

$$\mathcal{N}_{rel} \subset T^*_{rel} \qquad \text{relative nilp cone}$$

Q. Lift $\mathcal{N}_{rel}$ to $T^*\mathrm{Bun}_G(\mathcal{X}/S)$.
(as a conic Lagrangian)

Construction. $\mathrm{Bun}_B(\mathcal{X}/S) \xrightarrow{p} \mathrm{Bun}_G(\mathcal{X}/S)$

Singular cone of $p : X \to Y$.

$$\begin{tikzcd} & (T^*Y) \times_Y X \arrow[dl] \arrow[dr] & \\ T^*X & & T^*Y. \end{tikzcd}$$

$$0_X \longmapsto \text{singular cone of } p.$$

Def. $\mathcal{N}_{G,\mathcal{X}/S}$ = Singular Cone of p. above.

Th 1) $\mathcal{N}_{G,\mathfrak{X}/S}$ is a closed conic Lag in $T^*\mathrm{Bun}_G(\mathfrak{X}/S)$.

2) $\mathcal{N}_{G,\mathfrak{X}/S} \longrightarrow \mathcal{N}_{rel}$ is a bijection.

3) Formation of $\mathcal{N}_{G,\mathfrak{X}/S}$ commutes with $\underbrace{S' \longrightarrow S}_{\text{base change}}$.

$$\left(\mathrm{Bun}_G(\mathfrak{X}'/S') \longrightarrow \mathrm{Bun}_G(\mathfrak{X}/S) \right).$$

$S = pt.$

Ginzburg: 0-fiber of Hitchin map

$$\begin{array}{ccc} \text{0-fiber of Hitchin map} & & \\ \| & \cup & \\ \text{Singular cone of } \mathrm{Bun}_B(X) \longrightarrow \mathrm{Bun}_G(X). & & \end{array}$$

$(\mathcal{E}, \varphi)$ nilp. Higgs field.

at generic pt η of X. $\exists$ B-red of $\mathcal{E}_\eta$ $\mathcal{E}_{B,\eta}$

s.t. $\varphi_\eta \in \mathcal{E}_{B,\eta} \overset{B}{\times} \underline{\underline{\mathfrak{n}}}$.

$$\mathcal{E}_{B,\eta} \overset{\text{satnate}}{\rightsquigarrow} \underline{\underline{\mathcal{E}_B}}.$$

$$Sh_{\mathcal{N}}\left(\mathrm{Bun}_G(\mathfrak{X}/S)\right).$$

- stable under base change $S' \longrightarrow S$. (pull back)
- <u>Hecke property</u>:

$K \in \mathcal{H}_{sph}$

$$\begin{tikzcd} & \mathcal{X} \arrow[d] \\ S' \arrow[ur, bend left, "\sigma"] \arrow[r] & S \end{tikzcd}
\qquad
\begin{tikzcd} Hk^K_\sigma : Sh(Bun_G(\mathcal{X}/S)) \arrow[d] \\ Sh(Bun_G(\mathcal{X}'/S')) \end{tikzcd}$$

Th. Hk^K_σ sends $Sh_{\mathcal{N}}$ to $Sh_{\mathcal{N}}$.

Th $\Rightarrow$ Hecke local constancy [NY].

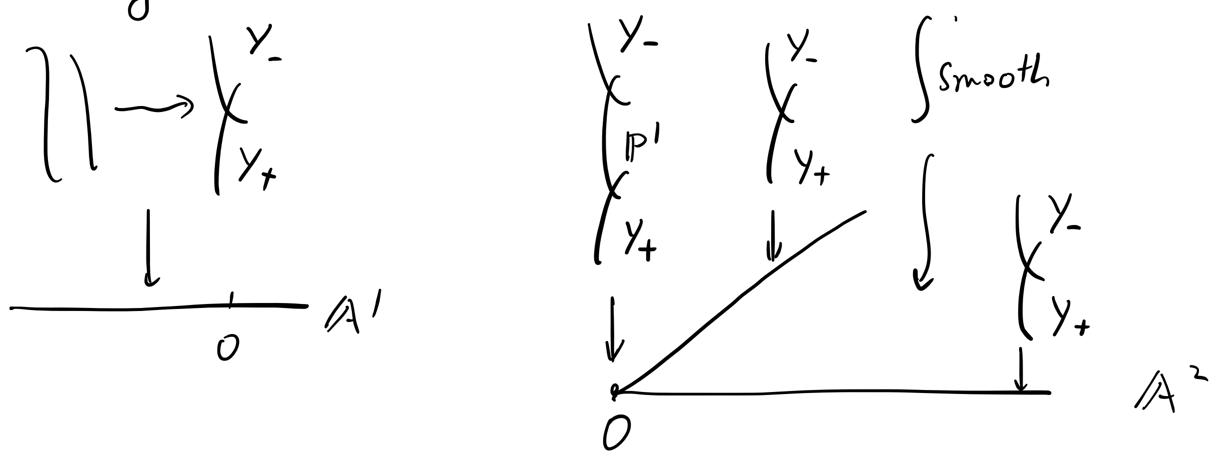
$$\begin{tikzcd} & X \arrow[d] \\ X \arrow[ur, equal, "\sigma = id"] \arrow[r] & pt \end{tikzcd}
\qquad
\begin{tikzcd} Sh_{\mathcal{N}} \arrow[d] \subset & Hk^K_\sigma : Sh(Bun_G(X)) \arrow[d] \\ Sh_{\mathcal{N}\times O_X} \subset & Sh(Bun_G(X)\times X) \end{tikzcd}$$

Conj
$$\begin{tikzcd} \mathcal{X} \arrow[d] \\ S \end{tikzcd} = \text{contractible}$$

$$Sh_{\mathcal{N}}(Bun_G(\mathcal{X}/S)) \xrightarrow[\sim]{i_s^*} Sh_{\mathcal{N}}(Bun_G(X_s)).$$

② Bubbling of curves

y:



next: Y_-, $\mathbb{P}^1$, $\mathbb{P}^1$, Y_+ central fiber over $\mathbb{A}^3$.

Standard situation

$$\boxed{\begin{array}{cc} \mathbb{A}^2 & (x,y) \\ \downarrow & \downarrow \\ \mathbb{A}^1 & t = xy \end{array}}$$

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\text{see below}"] \arrow[dr] & {[\mathbb{A}^2/\mathbb{G}_m]} \arrow[d] \\ & t \in \mathbb{A}^1 \end{tikzcd}$$

$x: \quad \mathcal{Y} \longrightarrow [\mathbb{A}^1/\mathbb{G}_m] \iff (\mathcal{O}(\underline{Y_+}), 1)$

$y: \quad \mathcal{Y} \longrightarrow [\mathbb{A}^1/\mathbb{G}_m] \iff (\mathcal{O}(Y_-), 1)$

$$\mathcal{O}(Y_+) \otimes \mathcal{O}(Y_-) = \mathcal{O}(\pi^{-1}(0)) \overset{t}{\simeq} \mathcal{O}$$

$(x,y): \mathcal{Y} \longrightarrow [\mathbb{A}^2/\mathbb{G}_m]$ — hyp. action on $\mathbb{A}^2$

Upshot: to bubble $\mathcal{Y}$, only need to bubble the standard model $\mathbb{G}_m$-equivtly

$$\begin{array}{c} \mathbb{A}^2 \\ t=xy \downarrow \\ \mathbb{A}^1 \end{array}$$

to get n-dim'l family

Gieseker:

$$\begin{tikzcd} W_n \arrow[r, "\text{birat'l}"] & \blacksquare \arrow[r] \arrow[d] & \mathbb{A}^2 \arrow[d, "t=xy"] \\ & \mathbb{A}^n \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

$(t_1, \dots, t_n) \longmapsto t_1 \cdots t_n$

bubbling of standard model (toric variety patched using explicit coord. charts).

③ Twisting.

For standard model

hyp. action.

$$\begin{tikzcd}
\mathbb{A}^2 \arrow[d] & (x,y) \arrow[d] & {[\widetilde{\mathbb{A}}^2/\mu_k]} \arrow[ll, "((-)^k,(-)^k)"'] \arrow[d] & (u,v) \arrow[d, mapsto] \\
\mathbb{A}^1 & t = xy & \widetilde{\mathbb{A}}^1 = \mathbb{A}^1_\varepsilon \arrow[ll, "(-)^k"] & uv = \varepsilon,
\end{tikzcd}$$

$u = x^{1/k}$

$v = y^{1/k}$

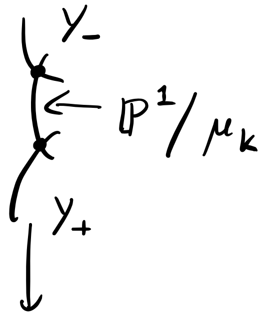
$\varepsilon = t^{1/k}$

$$\begin{tikzcd}
Y \arrow[d] & \widetilde{Y} \arrow[l] \arrow[d] \\
\mathbb{A}^1 & \widetilde{\mathbb{A}}^1 \arrow[l]
\end{tikzcd}$$

After twisting, get $\mathcal{X}_n$

$$\begin{tikzcd}
\mathcal{X}_n \arrow[d] \\
\widetilde{\mathbb{A}}^n
\end{tikzcd} \ni 0$$

(affine space)



Bundles on orbifold curves

X = smooth, unique orbifold pt $p \circlearrowleft \mu_k$.

$$\mathrm{Bun}_G(X) \longrightarrow \coprod_{\eta \in \mathrm{Bun}_G(B\mu_k)/\sim} \mathrm{Bun}_G(X)_\eta$$

$$\mathrm{Bun}_G(B\mu_k)/\sim \;=\; \left(\tfrac{1}{k}\Lambda/\Lambda\right)/W. \;=\; \left(\tfrac{1}{k}\Lambda\right)/\widetilde{W}.$$

<u>Fact</u>. $\mathrm{Bun}_G(X)_\eta \simeq \mathrm{Bun}_{G,\widetilde{P}_\eta}(X_{coarse}, p)$

local:

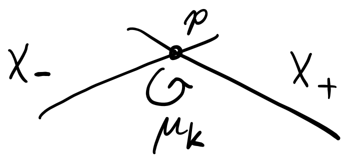
$$\left(G((t^{1/k})) / G[[t^{1/k}]] \right)^{\mu_k}$$

$$\|$$

$$\coprod_{\eta \in (\frac{1}{k}\Lambda)/\tilde{W}} G((t))/\tilde{P}_\eta$$

$\uparrow$ parahoric (when G is s.c.)

Fact



$$\mathrm{Bun}_G(X) = \coprod_\eta \mathrm{Bun}_G(X)_\eta$$

$$\mathrm{Bun}_G(X)_\eta \simeq \mathrm{Bun}_{G,\tilde{P}_\eta}(\overset{\text{coarse}}{X_-}, p) \underset{\mathbb{B}\tilde{L}_\eta}{\times} \mathrm{Bun}_{G,\tilde{P}_{-\eta}}(X_+, p)$$

η generic. $\tilde{P}_\eta = I$.

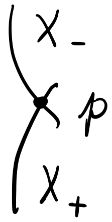
$$\mathrm{RHS} = \mathrm{Bun}_{G,B}(X_-, p) \underset{\mathbb{B}H.}{\times} \mathrm{Bun}_{G,B}(X_+, p)$$

$$= \mathrm{Bun}_{G,N}(X_-, p) \overset{H}{\times} \mathrm{Bun}_{G,N}(X_+, p).$$

Dennis: What should be the spectral side for $\mathrm{Bun}_G(X)$ (X: orbifold curve)?

Drinfeld: In what way is the construction of the family of Bun_G dependent on k, η? Is there a construction without choosing η?

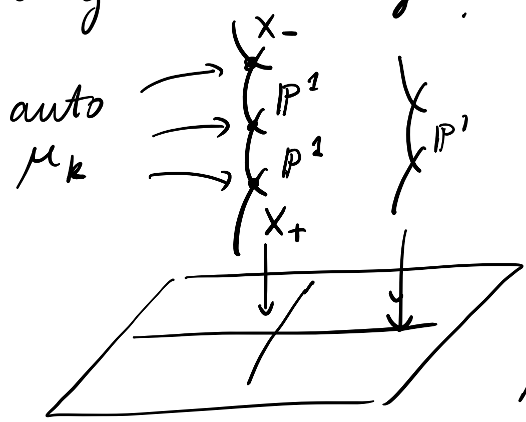
11/23. Automorphic Gluing (Cont'd)

Goal. $X \rightsquigarrow X_0 =$ 

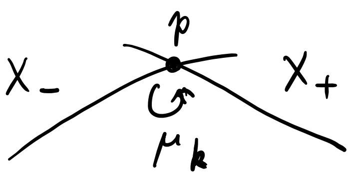
$$\mathcal{A}(X_-, p) \underset{\mathcal{H}_{\mathrm{aff}}}{\otimes} \mathcal{A}(X_+, p) \xrightarrow{?} \mathcal{A}(X)$$

$$\mathcal{A}(X_-, p) = Sh_{\mathcal{N}}\left(\mathrm{Bun}_{G,N}(X_-, p)\right)$$

Bubbling + Twisting :



$\mathbb{A}^n$ $n \geq 1$.

Recall: 

$= Y$.

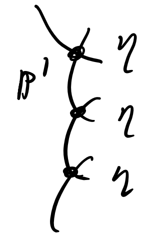
$$\mathrm{Bun}_G(Y) = \coprod_{\eta \in \frac{1}{k}\Lambda/\widetilde{W}} \mathrm{Bun}_G(Y)_\eta$$

Take generic η.

$$\mathrm{Bun}_G(Y)_\eta \simeq \mathrm{Bun}_{G,N}(\underline{X_-}, p) \overset{H}{\times} \mathrm{Bun}_{G,N}(\underline{X_+}, p)$$

coarse curves.

$$A(Y)_\eta \simeq A(\underline{X}_-, p) \underset{Sh_0(H)}{\otimes} A(\underline{X}_+, p)$$



$X(n)$ = special fib of the $\mathbb{A}^n$-family.

Then

$$\underset{\substack{\eta \\ = \\ \text{generic}}}{A(X(n))_\eta} \simeq A(\underline{X}_-, p) \underset{Sh_0(H)}{\otimes} \underbrace{A(\mathbb{P}^1, 0, \infty) \otimes \cdots \otimes}_{n \text{ copies}} \underset{Sh_0(H)}{} A(X_+, p).$$

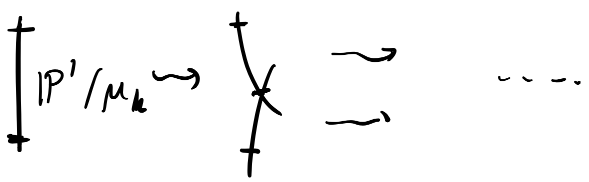
$$\mathcal{H}_{aff} \underset{\text{at } 0}{\circlearrowright} \underline{A(\mathbb{P}^1, 0, \infty)} \underset{\text{at } \infty}{\circlearrowleft} \underline{\underline{\mathcal{H}_{aff}}}$$

$$\text{SI}$$

$$Sh_{\mathcal{N}}\left(Bun_{G,1,1}([\mathbb{P}^1/\mu_k], 0, \infty)_{\eta, \eta}\right) = \mathcal{H}_{bub}.$$

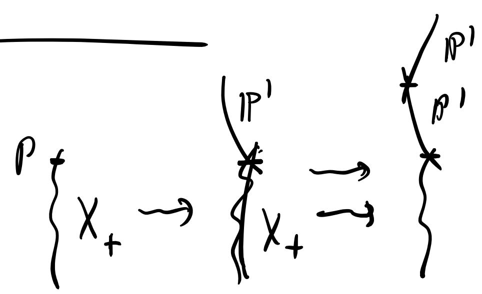
<u>Thm</u> $\mathcal{H}_{bub}$ is naturally a <u>monoidal</u> category.

and $\cong \mathcal{H}_{aff}$.

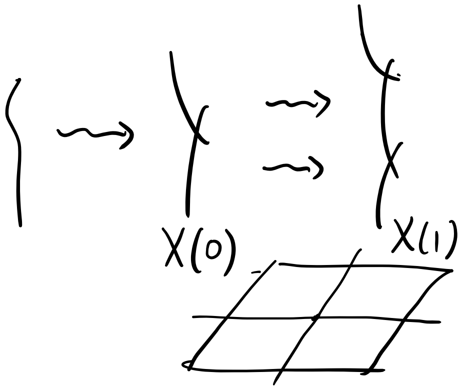


nearby cycles give

$$\mathcal{H}_{bub} \longrightarrow \mathcal{H}_{bub} \underset{Sh_0(H)}{\otimes} \mathcal{H}_{bub} \rightrightarrows \cdots$$



$A(X_+, p) \circlearrowleft \mathcal{H}_{bub.} \circlearrowright A(X_-, p).$



Take auto. cat. $\rightsquigarrow$ nearby cycles-

$$Sh_{\mathcal{N}}\left(Bun_G\left(\mathcal{X}/\mathbb{R}_{>0}\right)\right) \xrightarrow{\psi} \left[A(X_-,p) \underset{Sh_0(H)}{\otimes} A(X_+,p) \rightrightarrows A(X_-,p) \otimes H_{bub} \otimes A(X_+,p) \right.$$

└ univ. nilp cone

<u>Thm</u> 1) This is an aug. semi-cosimplicial diagram.

After passing to left adjoints

2) the non-augmented part is the bar complex calculating

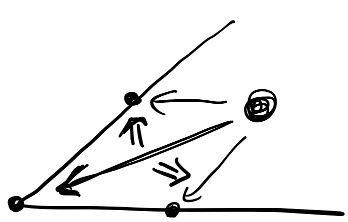
$$A(X_-,p) \underset{\mathcal{H}_{bub}}{\otimes} A(X_+,p)$$

<u>Cor</u> Passing to left adjoints, get functor.

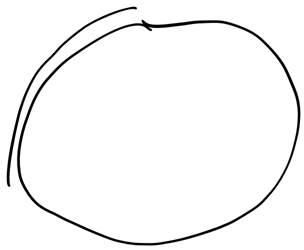
$$\begin{tikzcd} A(X_-,p) \underset{\mathcal{H}_{aff}}{\otimes} A(X_+,p) \arrow[r] \arrow[dr] & Sh_{\mathcal{N}}\left(Bun_G(\mathcal{X}/\mathbb{R}_{>0})\right) \arrow[d, "i_t^*"] \\ & A(X_t) \end{tikzcd}$$

<u>Topological Input</u>

Thom-Nadler criterion for canonicity of ψ.



$$\begin{tikzcd} A_0 \arrow[r, bend left, "L\psi"] & A_{(0,1)} \arrow[l, "\psi"] \arrow[d, "i^*"] \\ & {} \end{tikzcd}$$



$$\begin{tikzcd} \arrow[d, "c_{1/2}"] & \\ A_{\frac{1}{2}} & A_{(\frac{1}{2},1)} \arrow[l, "\psi"'] \arrow[d] \\ & A_{3/4} \end{tikzcd}$$

Thom condition:

$$\text{mfds}\left\{\begin{tikzcd} M = \bigcup_{\alpha} M_\alpha \arrow[d] \arrow[d, shift left=20ex, "\text{fibrations}"] \\ B = \bigcup_{\beta} B_\beta \end{tikzcd}\right.$$

$$\begin{tikzcd} x_0 & x_n \arrow[l] \\ M_{\alpha_0} \arrow[d, "\pi_0"] & M_{\alpha_1} \arrow[l] \arrow[d, "\pi_1"] \\ B_{\beta_0} & B_{\beta_1} \arrow[l] \end{tikzcd}$$

Suppose $\lim \ker\left(d\pi_1 | T_{x_n} M_{\alpha_1}\right) = V$

$$V \subset T_{x_0} M$$

Then $\ker(d\pi_0 | T_{x_0} M_{\alpha_0}) \subset T_{x_0} M$

Thom condition holds for $(M_{\alpha_0}, M_{\alpha_1})$
if $\ker(d\pi_0 | T_{x_0} M_{\alpha_0}) \subset V$.

Microlocal reformulation:

$$\Lambda_{\alpha_1} = T^*_{M_{\alpha_1}} M \subset T^*M$$

$$\big\downarrow p$$

$$p(\Lambda_{\alpha_1}) \subset T^*_\pi$$

$$\overline{p(\Lambda_{\alpha_1})}\Big|_{M_{\alpha_0}} = \text{rel. } T^* \text{ of } \pi^{-1}\big(\underset{\cup\,(\beta_0)}{B_{\beta_0}}\big) \longrightarrow B_{\beta_0}$$

restricted to M_{α_0}

Thom condition $\iff$ $\underbrace{\overline{p(\Lambda_{\alpha_1})}\Big|_{M_{\alpha_0}} \subset p(\Lambda_{\alpha_0})}$

Thom-Nadler criterion

$$\begin{array}{ccc} M^\times & \subset & M \\ \downarrow & & \downarrow \pi = \text{submersion} \\ \mathbb{R}^n_{>0} & \subset & B = \mathbb{R}^n_{\geq 0}. \end{array}$$

Thm (Nadler)

Given: $\Lambda^\times \subset T^*(M^\times)$ closed, conic Lag.

$$p(\Lambda^\times) \subset T^*_{rel}.$$

Condition: $\begin{cases} \overline{p(\Lambda^\times)}\Big|_0 \subset T^*M_0 \text{ is isotropic} \\ \Lambda^\times \text{ is non-char. wrt } \pi, \text{ i.e., } \Lambda^\times \cap \pi^*(T^*B) \subseteq \text{zero section} \end{cases}$

Then $\forall\ \mathcal{F} \in Sh_\Lambda(M^\times)$, the unbiased nearby cycles $\Psi(\mathcal{F})$

$$Sh(M^\times) \xrightarrow{i_0^* j_*} Sh(M_0)$$

$$(j : M^\times \hookrightarrow M^\times \cup M_0).$$

is canonically isom. to any iterated nearby cycles of $\mathcal{F}$.