# Automorphic Gluing Functor.

11/16/2022

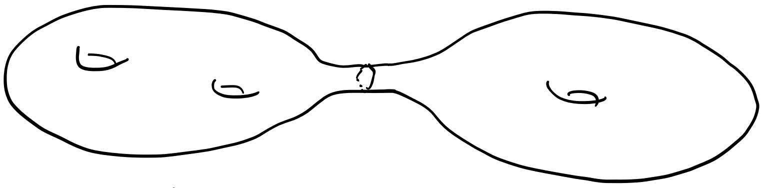
(Joint with D. Nadler)

X

$\rightsquigarrow \mathcal{A}(X,S) = Sh_{\mathcal{N}}(Bun_{G,N}(X,S))$

$\rightsquigarrow \mathcal{B}(X,S) = IndCoh_{\mathcal{N}}(Loc_{G^\vee,B^\vee}(X,S))$

Spectral Gluing Equivalence (Ben-Zvi, Nadler, Preygel)

$X =$ 

$X_- \quad \# \quad X_+$

$p_- \quad p_+$

$X_- \qquad X_+$

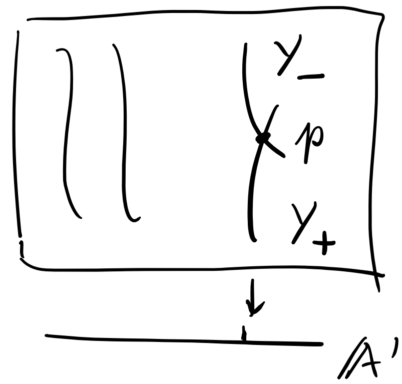
$$\mathcal{B}(X_-,p_-) \underset{\underline{\underline{\mathcal{H}_{aff}}}}{\otimes} \mathcal{B}(X_+,p_+) \xrightarrow{\sim} \mathcal{B}(X)$$

Expect:

$$\mathcal{A}(X_-,p_-) \underset{\mathcal{H}_{aff}}{\otimes} \mathcal{A}(X_+,p_+) \xrightarrow{\sim} \underline{\underline{\mathcal{A}(X)}}.$$

Today: give functor $\longrightarrow$

$$\begin{array}{c} \mathcal{Y} \\ \downarrow \text{ proj. family curves.} \\ \mathbb{A}^1 \end{array}$$




Sm. surface

$$\begin{tikzcd} \mathcal{A}(Y_-, p) \otimes \mathcal{A}(Y_+, p) \arrow[dr, "\text{glue}", "H_{aff}"'] \arrow[rr] & & \mathcal{A}(Y_t) \quad t>0. \\ & Sh_{\mathcal{N}}\left(\underbrace{Bun_G(\mathcal{Y}|_{\mathbb{R}_{>0}}/\mathbb{R}_{>0})}\right) \arrow[ur, "i_t^*"] \arrow[d] & \\ & \mathbb{R}_{>0} & \end{tikzcd}$$

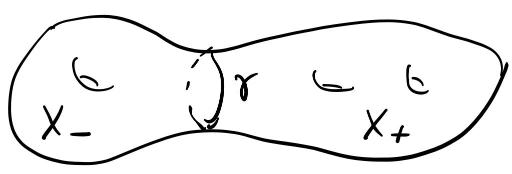
auto. gluing functor.

universal nilp. cone.

<u>Conj</u>: $i_t^*$ is an equivalence.

<u>Properties</u>. 1) "glue": $Eis^{\lambda_+} \otimes Eis^{\lambda_-} \longmapsto Eis^{\lambda_+ + \lambda_-}$

2) glue takes $Wh_+ \otimes Wh_- \longmapsto \underline{\underline{Wh}} \otimes \mathcal{O}(H^\vee)$ $\underline{\underline{\mathcal{O}(H^\vee /\!/ W)}}$

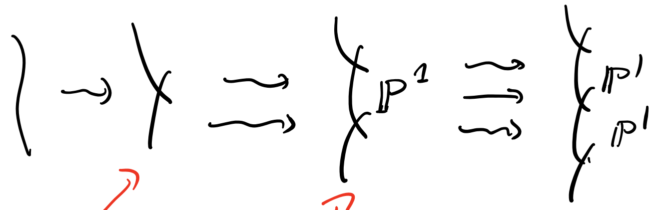



$$Loc_{G^\vee}(X) \longrightarrow G^\vee/G^\vee$$

monodromy along $\gamma$

1. Univ. nilp. cone

2. Bubbling of curves

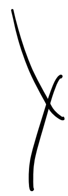
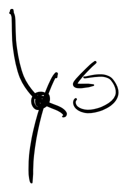


$$A_- \otimes_H A_+$$

$$A(Y_t) \leftarrow A_- \otimes A_+ \leftleftarrows A_- \otimes H \otimes A_+ \leftleftarrows \cdots$$

arrows: ${}^L\Psi$.

3. Twisting:

 $\rightsquigarrow$  $\circlearrowleft \mu_k$ $\quad \eta \in \mathrm{Hom}(\mu_k, G)/\sim$.

$$= \mathrm{Bun}_{G,N}(X_-, p) \times_H \mathrm{Bun}_{G,N}(X_+, p)$$

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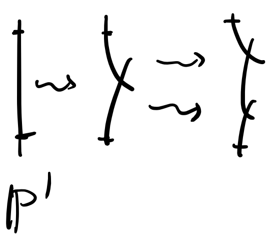
4.  $\rightsquigarrow$ bubbling Hecke category

$$H^{bub.}_{\neq} \cong H_{aff}.$$

5. Commutativity of $\Psi$. over higher dim base

Thom-Nadler criterion.

① Universal nilp. cone.

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \\ S \end{tikzcd}$$

smooth proj. family of curves

$S$ = smooth.

$$\begin{tikzcd} \mathrm{Bun}_G(\mathcal{X}/S) = \mathrm{Bun}_G(\pi) \arrow[d] & \supset & \mathrm{Bun}_G(\mathcal{X}_s) \arrow[d] \\ S & \ni & s \end{tikzcd}$$

$$\mathcal{N}_s \subset T^*_{rel}\big|_s = T^*\mathrm{Bun}_G(\mathcal{X}_s).$$

$\mathcal{N}_{rel} \subset T^*_{rel}$ relative nilp cone

<u>Q</u>. Lift $\mathcal{N}_{rel}$ to $T^*\mathrm{Bun}_G(\mathcal{X}/S)$.

(as a conic Lagrangian)

<u>Construction</u>. $\mathrm{Bun}_B(\mathcal{X}/S) \xrightarrow{p} \mathrm{Bun}_G(\mathcal{X}/S)$

Singular cone of $p : X \longrightarrow Y$.

$$\begin{tikzcd} & (T^*Y) \times_Y X \arrow[dl] \arrow[dr] & \\ T^*X & & T^*Y \end{tikzcd}$$

$0_X \longmapsto$ singular cone of $p$.

<u>Def</u>. $\mathcal{N}_{G, \mathcal{X}/S}$ = Singular cone of $p$. above.

**Th.** 1) $\mathcal{N}_{G,\mathfrak{X}/S}$ is a closed conic Lag in $T^*\mathrm{Bun}_G(\mathfrak{X}/S)$.

2) $\mathcal{N}_{G,\mathfrak{X}/S} \longrightarrow \mathcal{N}_{rel}$ is a bijection.

3) Formation of $\mathcal{N}_{G,\mathfrak{X}/S}$ commutes with $\underset{\text{base change}}{\widehat{\ }}$ $S' \longrightarrow S$

$$\left( \mathrm{Bun}_G(\mathfrak{X}'/S') \longrightarrow \mathrm{Bun}_G(\mathfrak{X}/S) \right).$$

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$S = pt.$

Ginzburg: $\quad$ 0-fiber of Hitchin map

$$\begin{matrix} \text{0-fiber of Hitchin map} \\ \| \quad \cup \\ \text{Singular cone of } \mathrm{Bun}_B(X) \longrightarrow \mathrm{Bun}_G(X). \end{matrix}$$

$(\mathcal{E}, \varphi)$ nilp. Higgs field.

at generic pt $\eta$ of $X$. $\exists$ $B$-red of $\mathcal{E}_\eta$ $\mathcal{E}_{B,\eta}$

s.t. $\varphi_\eta \in \mathcal{E}_{B,\eta} \overset{B}{\times} \underline{\mathfrak{n}}$.

$$\mathcal{E}_{B,\eta} \overset{\text{saturate}}{\rightsquigarrow} \underline{\underline{\mathcal{E}_B}}.$$

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$$Sh_{\mathcal{N}}\left(\mathrm{Bun}_G(\mathfrak{X}/S)\right).$$

- stable under base change $S' \longrightarrow S$.
  (pull back)

- Hecke property:

$K \in \mathcal{H}$

$K \in \mathcal{H}_{sph}$

$$\begin{tikzcd} & \mathcal{X} \arrow[d] \\ S' \arrow[ur, "\sigma"] \arrow[r] & S \end{tikzcd}
\qquad
\begin{tikzcd} Hk_\sigma^K : Sh(Bun_G(\mathcal{X}/S)) \arrow[d] \\ Sh(Bun_G(\mathcal{X}'/S')) \end{tikzcd}$$

<u>Th</u>. $Hk_\sigma^K$ sends $Sh_{\mathcal{N}}$ to $Sh_{\mathcal{N}}$.

Th $\Rightarrow$ Hecke local constancy [NY].

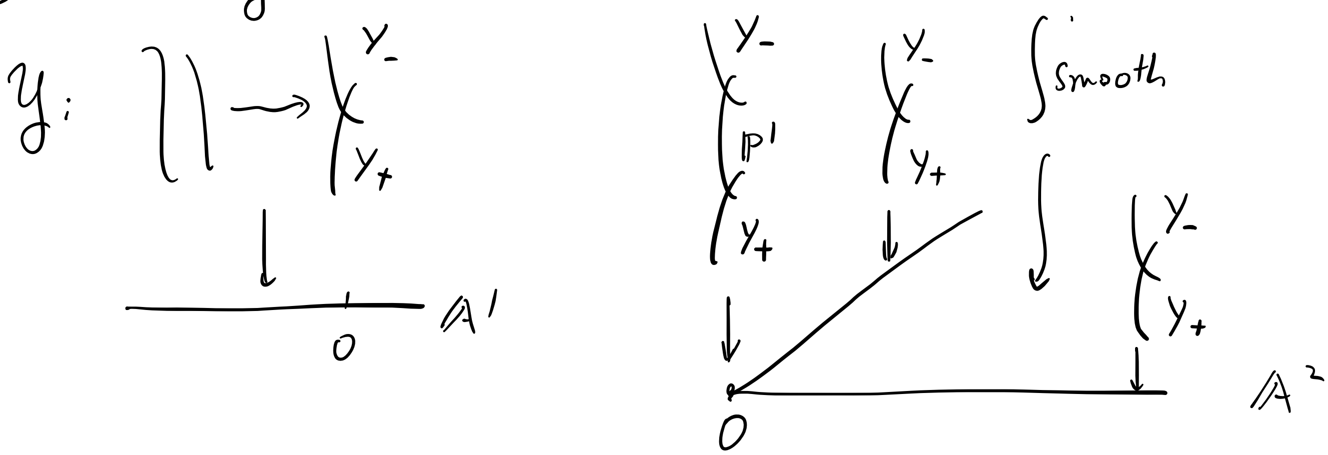
$$\begin{tikzcd} & X \arrow[d] \\ X \arrow[ur, equal, "\sigma = id"] \arrow[r] & pt \end{tikzcd}
\qquad
\begin{tikzcd} Sh_{\mathcal{N}} \arrow[d] \arrow[r, phantom, "\subset"] & Hk_\sigma^K : Sh(Bun_G(X)) \arrow[d] \\ Sh_{\mathcal{N} \times 0_X} \arrow[r, phantom, "\subset"] & Sh(Bun_G(X) \times X). \end{tikzcd}$$

<u>Conj</u>

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ S \end{array} = \text{contractible}$$

$$Sh_{\mathcal{N}}(Bun_G(\mathcal{X}/S)) \xrightarrow[\sim]{i_s^*} Sh_{\mathcal{N}}(Bun_G(X_s)).$$

(2) Bubbling of curves



next: $\begin{matrix} Y_- \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ Y_+ \end{matrix}$ central fiber over $\mathbb{A}^3$.

Standard situation
$$\boxed{\begin{array}{cc} \mathbb{A}^2 & (x,y) \\ \downarrow & \downarrow \\ \mathbb{A}^1 & t = xy \end{array}}$$

$$\begin{tikzcd} \mathcal{Y} \arrow[rr, "\text{see below}"] \arrow[dr] & & {[\mathbb{A}^2/\mathbb{G}_m]} \arrow[dl] \\ & t \in \mathbb{A}^1 & \end{tikzcd}$$

$x: \quad \mathcal{Y} \longrightarrow [\mathbb{A}^1/\mathbb{G}_m] \iff (\mathcal{O}(\underline{Y_+}), 1)$

$y: \quad \mathcal{Y} \longrightarrow [\mathbb{A}^1/\mathbb{G}_m] \iff (\mathcal{O}(Y_-), 1)$.

$$\mathcal{O}(Y_+) \otimes \mathcal{O}(Y_-) = \mathcal{O}(\pi^{-1}(0)) \overset{t}{\simeq} \mathcal{O}$$

$(x,y): \mathcal{Y} \longrightarrow [\mathbb{A}^2/\mathbb{G}_m]$

hyp. action on $\mathbb{A}^2$

Upshot: to bubble $\mathcal{Y}$, only need to bubble the standard model $\mathbb{G}_m$-equivtly

$$\begin{array}{c} \mathbb{A}^2 \\ t=xy \downarrow \\ \mathbb{A}^1 \end{array}$$

to get $n$-dim'l family

Gieseker:

$$\begin{tikzcd} W_n \arrow[r, "\text{birat'l}"] & \square \arrow[r] \arrow[d] & \mathbb{A}^2 \arrow[d, "t=xy"] \\ & \mathbb{A}^n \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

$(t_1, \dots, t_n) \longmapsto t_1 \cdots t_n$

bubbling of standard model (toric variety patched using explicit coord. charts).

③ Twisting.

For standard model

hyp. action.

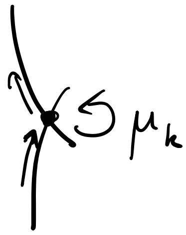
$$
\begin{tikzcd}
\mathbb{A}^2 \arrow[d] & (x,y) \arrow[d, mapsto] & \left[\widetilde{\mathbb{A}}^2/\mu_k\right] \arrow[ll, "((-)^k,(-)^k)"'] \arrow[d] & (u,v) \arrow[lll, mapsto] \arrow[d, mapsto] \\
\mathbb{A}^1 & t = xy & \widetilde{\mathbb{A}}^1 = \mathbb{A}^1_\varepsilon \arrow[ll, "(-)^k"] & uv = \varepsilon,
\end{tikzcd}
$$

$u = x^{1/k}$

$v = y^{1/k}$

$\varepsilon = t^{1/k}$

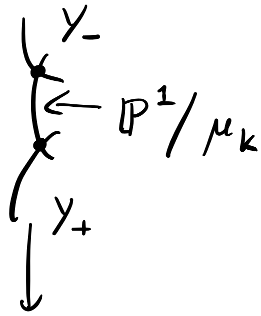
$$
\begin{tikzcd}
Y \arrow[d] & \widetilde{Y} \arrow[l] \arrow[d] \\
\mathbb{A}^1 & \widetilde{\mathbb{A}}^1 \arrow[l]
\end{tikzcd}
$$




After twisting, get

$$
\begin{tikzcd}
\mathcal{X}_n \arrow[d] \\
\widetilde{\mathbb{A}}^n
\end{tikzcd}
\ni 0
$$

(affine space)




## Bundles on orbifold curves

$X$ = smooth, unique orbifold pt $p \circlearrowleft \mu_k$.

$$
\mathrm{Bun}_G(X) \longrightarrow \coprod_{\eta \in \mathrm{Bun}_G(B\mu_k)/\sim} \mathrm{Bun}_G(X)_\eta
$$

$$
\mathrm{Bun}_G(B\mu_k)/\sim \;=\; \left(\tfrac{1}{k}\Lambda/\Lambda\right)/W \;=\; \left(\tfrac{1}{k}\Lambda\right)/\widetilde{W}.
$$

<u>Fact</u>. $\mathrm{Bun}_G(X)_\eta \simeq \mathrm{Bun}_{G,\widetilde{P}_\eta}(X_{\text{coarse}}, p)$

local:

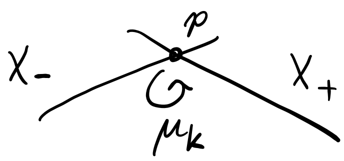
$$\left( G((t^{1/k})) \big/ G[[t^{1/k}]] \right)^{\mu_k}$$

$$\|$$

$$\coprod_{\eta \in (\frac{1}{k}\Lambda)/\tilde{W}} G((t)) \big/ \tilde{P}_\eta$$

parahoric (when $G$ is s.c.)

Fact




$$\mathrm{Bun}_G(X) = \coprod_\eta \mathrm{Bun}_G(X)_\eta$$

coarse

$$\mathrm{Bun}_G(X)_\eta \simeq \mathrm{Bun}_{G,\tilde{P}_\eta}(X_-, p) \underset{\mathbb{B}\tilde{L}_\eta}{\times} \mathrm{Bun}_{G,\tilde{P}_{-\eta}}(X_+, p)$$

$\eta$ generic. $\tilde{P}_\eta = I$.

$$\mathrm{RHS} = \mathrm{Bun}_{G,B}(X_-, p) \underset{\mathbb{B}H}{\times} \mathrm{Bun}_{G,B}(X_+, p)$$

$$= \mathrm{Bun}_{G,N}(X_-, p) \overset{H}{\times} \mathrm{Bun}_{G,N}(X_+, p).$$