A geometric approach to a description of the spectrum of the Hecke algebra on the subspace generated by unramified Eisenstein series.

David Kazhdan (joint with Andrei Okounkov)

Hebrew University

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Introduction

I present (a joint work with Andrei Okounkov) a description of the spectrum of the algebra of Hecke operators on the closure of the space of unramified pseudo-Eisenstein series E_ϕ. As it is now our analysis is applicable only to unramified case.

In my talk I consider only the case when $\phi = \phi(t)$ are functions on the split torus depending only on the norm of t. We expect our approach to work for unramified tempered cuspidal representations π of any Levi subgroup when we have a control on poles and zeros of the L-function $L(\pi, \mathfrak{u}^\vee)$ where $\mathfrak{u}^\vee \subset \mathfrak{g}^\vee$ is the nilpotent radical of the corresponding parabolic Lie subalgebra. In particular we expect to be able describe the spectrum of the algebra of Hecke operators on the closure of space of unramified pseudo-Eisenstein series coming from tempered cuspidal representations of Levi subgroups for global fields on positive characteristic.

Groups

We fix a split simple group $G \supset B = TU$ denote by $G^\vee \supset T^\vee U^\vee$ be the complex dual group of G, by $\mathfrak{g}^\vee$ the Lie algebra of $G^\vee$ and by $\mathcal{B}^\vee$ the flag variety of $G^\vee$. We denote by $R = R_+ \cup R_-$ the set of roots of $G^\vee$, by $T^\vee_{comp} \subset T^\vee$ the maximal compact subgroup, by $T^\vee_+ \subset T^\vee(\mathbb{R})$ the positive Weyl chamber and by W the Weyl group of G. Let Λ be the lattice of characters of $T^\vee$ = the lattice of cocharacters of T.

Let $S := G^\vee/Ad(G^\vee)$ be the adjoint quotient and $r : G^\vee \to S$ be the projection. The natural map $T^\vee/W \to S$ is an ismorphism.

Example If $G = Sl_2$, then $G^\vee = PGL_2$ $R_+ = \{\alpha\}$ where α defines an isomorphism $T^\vee \to \mathbb{C}^\star$, $S = \mathbb{C}$ and the map $r : G^\vee \to S$ is given by $r(g) = tr^2(g)/det(g)$.

Fields

Let F be a global field. That is either F is a number field or F is the field of rational functions on a smooth irreducible complete curve $\mathcal{C}$ is over $\mathbb{F}_q$. In the main part of the talk I assume that $F = \mathbb{F}_q(\mathcal{C})$ and comment on the case of number fields only in the very end.

Let $\xi_F(t) = t^{1-g}\zeta_F(t)$ be the completed ζ-function of $\mathcal{C}$ where g is the genus of $\mathcal{C}$. $\xi_F(t)$ satifies the functional equation $\xi_F(1-t) = \xi_F(t)$.

Let $\mathbb{A}$ be the ring of adeles of $F, \mathcal{O} \subset \mathbb{A}$ be the subring of integers adeles and $\mathbb{G}_m(\mathbb{A}) \to \mathbb{Z}$ be the norm map. This norm map induces a surjection $\|\| : T(\mathbb{A}) \to \Lambda$.

Vector spaces

- $\Phi = \mathbb{C}[\Lambda]$ with the natural scalar product $(\cdot, \cdot)$.
- $H = \mathbb{R}[\Lambda]^W$. The algebra H acts on Φ by multiplication.

The Mellin transform $\phi \to \phi^\vee$ identifies Φ with the space of regular functions on $T^\vee$ and H with the algebra of real-valued regular functions on $S(\mathbb{R})$.

The space Bun

Let $Bun_G(\mathcal{C})$ be the stack of G-bundles on $\mathcal{C}$. We can identify the groupoid $\mathrm{Bun} := \mathrm{Bun_G}(\mathcal{C})(\mathbb{F}_q)$ with the double quotient $G(\mathcal{O})\backslash G(\mathbb{A})/G(F)$.

Using the Iwasawa decomposition $G(\mathbb{A}) = G(\mathcal{O})T(\mathbb{A})U(\mathbb{A})$ we define the map $a : G(\mathbb{A}) \to \Lambda$ by $a(g) = \|t\|$ for $g = ktu$ where $k \in G(\mathcal{O}), t \in T(\mathbb{A}), u \in U(\mathbb{A})$.

Pseudo-Eisenstein series

For $\phi \in \Phi$ we denote by E_ϕ the pseudo-Eisenstein series which is a compactly supported function on $\mathrm{Bun} = \mathrm{G}(\mathcal{O})\backslash \mathrm{G}(\mathbb{A})/\mathrm{G}(\mathrm{F})$ defined as the series

$$E_\phi(g) := \sum_{\gamma \in G(F)/B(F)} \phi(a(g\gamma))$$

Let $\mathcal{H}_{Eis} \subset L^2(G(\mathbb{A})/G(F), dg)$ be the closure of space $\{E_\phi\}$.

The action of the algebra H on Φ extends to an action on $\mathcal{H}_{Eis}$ by bounded self-adjoint operarors.

It is easy to see that the spectrum of $H \subset End(\mathcal{H}_{Eis})$ is simple and we may consider it as a subset X_F of $S(\mathbb{R})$.

To describe $X_F \subset S(\mathbb{R})$ I introduce some notations.

Notations

Let $\mathcal{U} \subset G^\vee$ be the subset of unipotent elements and $[\mathcal{U}]$ be the set of conjugacy classes of $\mathcal{U}$. For $[e] \in [\mathcal{U}]$ we choose a homomorphism (defined uniquely up to a conjugation) $\sigma_e : Sl_2 \to G^\vee$ such that $\sigma(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) \in [e]$, define $h_{e,q} := \sigma_e(\begin{bmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{bmatrix})$ and denote by $C_{e,comp}$ a maximal compact subgroup of $C_e := Cent_{G^\vee}(e, h_{e,q})$. Let

$$Y_{e,q} := h_{e,q} C_{e,comp} \subset G^\vee, \quad S^q_{[e]} := r(Y_{e,q}) \subset S(\mathbb{R}),$$

where $r : G^\vee \to S = G^\vee / Ad(G^\vee)$ is the projection. The subsets $S^q_{[e]} \subset S(\mathbb{R})$ depend only on the conjugacy class $[e]$ of e and are disjoint.

The Main theorem

Theorem

1. $X_F = \cup_{[e]} S^q_{[e]}$.
   This decomposition induces the direct sum decomposition $\mathcal{H}_{Eis} = \bigoplus_{[e]\in[\mathcal{U}]} \mathcal{H}^{[e]}_{Eis}$. *Let* $p_{[e]} : \mathcal{H}_{Eis} \to \mathcal{H}^{[e]}_{Eis}, [e] \in [\mathcal{U}]$ *be the orthogonal projection.*
2. $\mathcal{H}^{[e]}_{Eis} \simeq L^2(S^q_{[e]}, \mu_{[e]})$, *where* $\mu_{[e]}$ *is a Lebesgue measure.*

Corollary

The discrete spectrum of the action of H *on* $\mathcal{H}_{Eis}$ *is parametrized by conjugacy classes of pairs* (e, γ), *where* $e \in G^\vee$ *is a distinguished unipotent and* $\gamma \in C_{e,comp}$.

Remark

Functions $p_{[e]}(E_\phi)$ are non necessarily of compact support.

The fundamental equality

All known proofs of the Main theorem (even for $G = Sl_2$) are based of the following equality valued for all global fields.

Claim

The exists $c_F > 0$ such that
$c_F < \phi, \phi >_{Eis} = \sum_{w \in W} < \phi, \phi >_w$ where

$$< \phi, \phi >_w = \int_{\theta \in T_\epsilon} d\theta \phi^\vee(\theta)\, \tilde{\phi}^\vee(w\theta^{-1}) \prod_{\substack{\alpha > 0 \\ w \cdot \alpha < 0}} \frac{\xi_F(\alpha(\theta))}{\xi_F(q^{-1}\alpha(\theta))},$$

for any ϵ deep in the chamber $T_+^\vee(\mathbb{R})$, where $T_\epsilon := \epsilon T_{comp}^\vee$, $\alpha \in R$ and $\tilde{\phi}(\theta) := \bar{\phi}(\bar{\theta})$ (here $\theta \to \bar{\theta}$ is the complex conjugation).

The summation over W comes from the Bruhat decomposition of the group $G(F)$ and the ratios of ξ_F-functions appear as spherical matrix elements of intertwining operators.

$G = Sl_2$

In this case it is easy to derive the description of the spectrum X_F from the fundamental equality.

In this case
$\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_{reg}, S_1^q = [0,4], S_{reg}^q = \{2+q+q^{-1}\}, T_\epsilon = \{t | |t| = \epsilon\}$
and
$< \phi, \phi >_{Eis} =$
$\int_{|t|=\epsilon} \phi^\vee(t)\,\tilde{\phi}^\vee(t^{-1}) + \phi^\vee(t)\,\tilde{\phi}^\vee(t)\xi_F(t)/\xi_F(q^{-1}t)dt, \epsilon << 1$
Since the functions $\phi^\vee(t)$, and $\frac{(1-qt)\xi_F(t)}{\xi_F(q^{-1}t)}$ are holomorphic for $|t| \leq 1$ we have $< \phi, \phi >_{Eis} = < \phi, \phi >_1 + < \phi, \phi >_{reg}$, where

$$< \phi, \phi >_1 = \int_{\theta \in T_{comp}} \|\phi^\vee(\theta) + \phi^\vee(\theta^{-1})\|^2 d\theta$$

and $< \phi, \phi >_{reg} = \|\phi^\vee(1/q)\|^2$.
So $\mathcal{H}_{Eis} = \mathcal{H}_{Eis}^1 \bigoplus \mathcal{H}_{Eis}^{reg}$, where $\mathcal{H}_{Eis}^1 = L^2(S_{[1]}^q, \mu_1)$ and $\mathcal{H}_{Eis}^{reg} \subset L^2(G(\mathbb{A})/G(F), dg)$ is the subspace of constants.

Groups of higher ranks

For groups of higher ranks the individual terms $<\phi,\phi>_w$ of the Fundamental equality have much worse analytic behavior then the sum $<\phi,\phi>_{Eis}$.

The situation is similar to the case of the Atiyah's localization formula when the total sum is regular while individual terms have poles.

In spite of these complications Langlands analyzed the case of groups of rank ≤ 2, C. Moeglin and J.-L. Waldspurger the case of classical groups and recently V. Heiermann, M. de Martino and E. Opdam, using the information on the structure of unitary representations of groups over local fields were able to deal with the general case.

I now present our approach to a description of X_F.

A generalization

In the proof of Main Theorem for $G = Sl_2$ we used only very basic properties of functions ξ_F. So it is natural to extend the Main Theorem to a broader class of functions.

q-admissible functions

Given $q > 1$ we say that a meromorphic function ξ on $\mathbb{C}^\star$ is *q-admissible* if

- ξ is regular outside 1 and $1/q$.
- ξ has simple poles at 1 and $1/q$.
- ξ satifies the functional equation $\xi(1/qt) = \xi(t)$.
- The restriction of ξ on the interval $(0, 1/q)$ is positive.
- All zeros of ξ are in the ring $1/q < \|z\| < 1$.

For a q-admissible function ξ we define a Hermitian form $<,>_\xi$ on Φ by $< \phi, \phi >_\xi =$

$$= \int_{\theta \in C_\epsilon} d\theta \sum_{w \in W} \phi^\vee(\theta)\, \tilde{\phi}^\vee(w\theta^{-1}) \prod_{\substack{\alpha > 0 \\ w \cdot \alpha < 0}} \frac{\xi(\alpha(\theta))}{\xi(q^{-1}\alpha(\theta))},$$

More notations

Let $e \in G^\vee$ be a unipotent element and $\sigma_e : Sl_2 \to G^\vee$ be a morphism such that $e := \sigma\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)$.

The representation $Ad \circ \sigma_e$ of Sl_2 on $\mathfrak{g}^\vee$ defines a direct sum decomposition $\mathfrak{g}^\vee = \oplus_i (V_i \otimes \mathfrak{g}_i^e)$ where V_i is the i-dimensional irreducible representation of Sl_2. The group C_e acts naturally on the spaces $\mathfrak{g}_i^e$.

We denote by $\xi^1_{\mathfrak{g}^\vee}(\theta)$ the function on $T^\vee$ equal to $\prod_{\alpha \in R} \xi^1(\alpha(\theta))$ where $\xi^1(t) := \xi(t)(1-t)(1-qt)$. Using the isomorphism $T^\vee/W \to S = G^\vee//G^\vee$ we consider the W-invariant function $\xi^1_{\mathfrak{g}^\vee}$ as an Ad-invariant function on $G^\vee$.

More notations

We define functions $\mathcal{P}_{\pm}$ on $T^{\vee}$ and functions Ψ_e on $h_{e,q}C_e$ by

$\mathcal{P}_{\pm}(\theta) = \prod_{\alpha \in R^{\pm}} \frac{1-q\alpha(\theta)}{1-\alpha(\theta^{-1})} \xi^1(\alpha(\theta))$

and

$\Psi_e(h_{e,q}(c))$

$$= q^{-\frac{\dim \mathfrak{g} + \dim \mathfrak{g}^e}{2}} \frac{(1-q)^{2r}}{\xi^1_{\mathfrak{g}^{\vee}}(h_{e,q}c-1)} \frac{\prod_{i>0} \det_{\mathfrak{g}_i^e}(1 - q^{-i/2} h_{e,q} c)}{\prod_{i \geq 0} \det_{\mathfrak{g}_i^e}(1 - q^{i/2+1} h_{e,q} c^{-1})},$$

Forms $<,>_{\xi}^{[e]}$ and the Main Equality

For $[e] \in \mathcal{U}$ we define the Hermitian form $<,>_{\xi}^{[e]}$ on Φ by

$$< \phi, \phi >_{\xi}^{[e]} = \int_{c \in Ce,comp} \Pi \mathcal{P}_{+} \phi(h_{e,q}c) \overline{\Pi \mathcal{P}_{-} \phi}(h_{e,q}c) \Psi_e(h_{e,q}c) dc$$

where $\Pi := \sum_{w \in W} w$ is the map from functions on $T^{\vee}$ to functions on $T^{\vee}/W$, (which by the Chevalley theorem are Ad-invariant regular functions on $G^{\vee}$) and dc is the Haar measure on C_e.

Theorem

$<,>_{\xi} = \sum_{[e]} <,>_{\xi}^{[e]}$ *for any q-admissible function* ξ.

The space $\mathcal{H}_\xi$

Corollary

1. *The form* $<,>_\xi$ *is semi-definite.*
   We denote by $\mathcal{H}_\xi$ *the completion of the space* Φ *in respect to* $<,>_\xi$.
2. *The action of* H *on* Φ *extends to an action on* $\mathcal{H}_\xi$ *by self-adjoint operarors.*
   We denote by $X_\xi \subset S(\mathbb{R})$ *the spectrum of* $H \subset End(\mathcal{H}_\xi)$.
3. $X_\xi = \cup_{[e]\in[\mathcal{U}]} S^q_{[e]}$.
   This equality implies the direct sum decomposition $\mathcal{H}_\xi = \bigoplus_{[e]} \mathcal{H}_\xi^{[e]}$.
4. $\mathcal{H}_\xi^{[e]} \simeq L^2(S^q_{[e]}, \mu_{[e]})$ *as an* H*-module.*

The last equality implies the Main Theorem.

The idea of a proof the Main Equality

For a complex reductive group L we denote by $\mathrm{Rep(L)}$ the ring of representatons of L. Then $\mathrm{Rep(L)}_{\mathbb{C}} := \mathrm{Rep(L)} \otimes \mathbb{C}$ is isomorphic to the ring of regular *Ad*-invariant functions on L. Linear functionals on $\mathrm{Rep(L)}_{\mathbb{C}}$ we call *distributions on* L.

Since $\Phi = \mathrm{Rep}_{\mathbb{C}}(\mathrm{T}^{\vee})$ and we may consider $<,>_{\xi}$ as a linear functional on $Rep_{\mathbb{C}}(T^{\vee}) \otimes Rep_{\mathbb{C}}(T^{\vee})$ that is as a distribution Ψ_{ξ} on $T^{\vee} \times T^{\vee}$. For a proof of Main Theorem we provide a geometric interpretation of the distribution Ψ_{ξ} in terms of an equivariant genus defined by ξ.

A construction of distributions

Let M be a compact algebraic variety with an action of the group L and $\mathcal{F}$ be a L-equivariant vector bundle. We denote by $\kappa_{\mathcal{F}}$ the distribution on L given by $\kappa_{\mathcal{F}}(\pi) := dim(RHom_H(\pi, R\Gamma(M, \mathcal{F}))$. The distribution $\kappa_{\mathcal{F}}$ depends only on the class $[\mathcal{F}]$ of $\mathcal{F}$ in the equivariant K-theory $K_L(M)$ on M.

The definition of distributions $\kappa_{\mathcal{F}}$ can be extended from compact L-varieties to the class of *cohomologically proper* L-varieties.

Cohomologically proper varieties

Definition

An L-variety M is *cohomologically proper* if for any coherent L-equivariant sheaf $\mathcal{F}$ on M and a finite-dimensional representation ρ of L the spaces $Ext^i_L(\rho, H^j(M, \mathcal{F}))$ are finite-dimensional.

Example

$\mathbb{A}^n$ as a $\mathbb{C}^\star$-variety under the action by homothety is cohomologically proper.

If $L = R \times \mathbb{C}^*$, M is cohomologically L-proper and $\mathcal{F}$ is a coherent L-equivariant sheaf we can consider $\kappa_{\mathcal{F}}$ as a distribution on $\mathbb{C}^*$ with values in distributions on R. In the case when there exists a $\mathbb{C}^*$- *attracting* stratification of M we may consider $\kappa_{\mathcal{F}}$ as an analytic function of $\mathbb{C}^*$ with values in distributions on R defined for $|q| > 1$.

Integrands

It is straightforward to extend these definitions to the case when M is a derived stack and $\mathcal{F} \in K_L(M)_{\text{analytic}}$ (that is, analytic function on the spectrum of $K_L(M)$).

For a Laurent polynomial ψ such that $\psi(1) = 1$ and an L-equivariant line bundle $\mathcal{L}$ on M we denote by $\psi(\mathcal{L})$ the corresponding element of $K_L(M)$. One can extend this construction and define $\psi(\mathcal{F}) \in K_L(M)$ for L-equivariant vector bundles on M which takes $\oplus$ to $\otimes$.

In the case when ψ is an analytic function on $\mathbb{C}^*$ one can define $\psi(\mathcal{F}) \in K_L(M)_{\text{analytic}}$.

If $\mathcal{F} \in K_L(M)$ is not a vector bundle then zeros of ψ could create poles. We say that $\mathcal{F}$ is ψ-admissible if $\kappa_{\psi(\mathcal{F})}$ is well-defined.

Scissor relatons

The distributions $\kappa_{\hat{\psi}(\mathcal{F})}$ factor through the relation

$$[M] = [M \setminus M'] + [\mathrm{Thom}(M' \to M)]\,, \qquad (1)$$

where $M' \subset M$ and the pushforward from the Thom space is computed by local cohomology of $\mathcal{F}$.

Cohomological properness of any two terms in (1) implies cohomological properness for the third. A good condition to impose is that M' is smooth and the q-weights in $N_{M/M'}$ are negative. For instance, if M' is the orbit of a nilpotent element fixed by $q^{-1}q^{h/2}$, then the weights on the slice to this orbit are negative.

An interpretation of the distribution Ψ_ξ

Let ψ_ξ be a holomorphic function on $\mathbb{C}^\star$ with only zeros in the ring $1/q < \|z\| < 1$ and such that $\xi(t) = \frac{\psi_\xi(t)\psi_\xi(1/qt)}{(1-t)(1-1/qt)}$.

We denote by $\mathcal{T}$ the $T^\vee \times T^\vee \times \mathbb{C}^*$-equivariant derived stack which is the cotangent bundle to the $T^\vee \times T^\vee$-equivariant stack $B^\vee \backslash G^\vee / B^\vee$. Then the tangent complex $T(\mathcal{T})$ is $T^\vee \times T^\vee \times \mathbb{C}^*$-equivariant.

Claim

1. The tangent "bundle" $T\mathcal{T}$ is ψ_ξ-admissible.
2. The pushforward of $\psi_\xi(T\mathcal{T})$ to $T^\vee \times T^\vee$ is equal to the distribution Ψ_ξ.

Remark

1. The second part of Claim follows from the localization formula when terms $<\phi,\phi>_w$ correspond to $T^\vee$-fixed points on $\mathcal{B}^\vee$.
2. Since $\mathcal{T}$ is a stack it tangent "bundle" $T\mathcal{T}$ is an element of the equivariant K-theory of $\mathcal{T}$ but not a vector bundle. Our assumption on the location of zeros ξ are crutial for the possibility to define $\kappa_{\psi(T\mathcal{T})}$

Decomposition of $\mathcal{T}$

With a unipotent element $e \in G^\vee$ we associate

1. the *Springer fiber* $\mathcal{B}^e \subset \mathcal{B}^\vee$ which is the fixed locus of e acting on $\mathcal{B}^\vee$ which, as the zero locus of a vector field, has a natural derived scheme structure with the virtual tangent bundle and
2. the Slodowy slice $Slice_e \subset G^\vee$ to the orbit $[e]$ through e.

Let $Thom(e \xrightarrow{f} Slice_e)$ be the difference $[Slice_e] - [Slice_e \setminus e]$ and

$$Thom(e) := \frac{1}{[q^{-1}]^{2s}} \left([\mathcal{B}^e] \times [\mathcal{B}^e] \times Thom(0 \to Slice_e \mathfrak{g})\right) / C_e$$

where s is the rank of G.

The completion of the proof

The Geometric invariant theory implies the decomposition $[\mathcal{T}] = \coprod_e [Thom(e)]$.

Proposition

1. The stacks $Thom(e)$ are ψ_ξ-admissible.
2. The Hermitian form corresponding to the distribution $\tau_{\psi_\xi}(Thom(e))$ is equal to $<,>_\xi^{[e]}$ up to a scalar.

Remark

The computation of distributions $\tau_{\psi_\xi}(Thom(e))$ is algorithmic.

This Proposition and the scissor relations imply the Main Equality.

Global fields of characteristic zero

The case of global fields of characteristic zero is analogous. Let $\mathfrak{g}^\vee := Lie(G^\vee), \bar{S} = \mathfrak{g}^\vee / G^\vee$.
In this case we replace:

1. $\Phi = K_{T^\vee}(\star)$ by $H^\star_{T^\vee}(\star) = Sym(\mathfrak{g})$
2. $h_{e,q}$ by $h_e := \sigma_e \left(\begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix} \right)$,
3. $Y_{e,q} := h_{e,q} C_{e,comp} \subset G^\vee$ by $Y_e = h_e + Lie(C_{e,comp}) \subset \mathfrak{g}^\vee$ and
4. $S^q_{[e]} := r(Y_{e,q})$ by $\bar{r}(Y_e) \subset \bar{S}$, where $\bar{r} : \mathfrak{g}^\vee \to \bar{S}$ is the projection.

As in the case of fields of finite characteristic the proof of the Langlands conjecture is reduced to a proof of equality between two explicit expressions involving integrals. This equality could be derived from the previous result as the limit for $q \to 1$.