Coherent sheaves on the stack of Langlands parameters

G connected reductive grp $/F$ global field

$$\left\{\begin{array}{c}\text{Automorphic forms}\\ \text{of } G\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Langlands parameters}\\ \rho: \mathcal{L}_F \to {}^L G\end{array}\right\}$$

Automorphic forms:

(1) Complex theory } analytic nature ← Bernstein center of real group? (Isolation of cuspidal spectrum, ... conj about this)

(2) p-adic theory

(3) Algebraic automorphic forms, cohomology of modular varieties, ... ← algebro-geometric Arithmetic nature

(3) = (1) ∩ (2)

1. The stack of Langlands parameters

- $F = \mathbb{F}_q(X)$, X smooth proj curve $/\mathbb{F}_q$, char $F = p$.
- S non-empty finite set of places of X, $U = X - S$
- $W_{F,S} = \mathrm{Weil}(U)$ $\quad 1 \to \pi_1^{\mathrm{geom}}(U) \to W_{F,S} \to \mathbb{Z} \to 1$
- $E/\mathbb{Q}_\ell$ finite ext. $\ni \sqrt{q}$, $\quad \ell \neq p$.
- G split reductive gp $/F$

<u>Def</u>. ${}^{cl}\mathrm{Loc}^{\square}_{\hat{G},F,S} : \mathcal{O}_E\text{-alg} \longrightarrow \mathrm{Sets}$

$$A \longmapsto \left\{\begin{array}{c}\text{continuous homomorphisms}\\ \rho: W_{F,S} \to \hat{G}(A)\end{array}\right\}$$

$${}^{cl}\mathrm{Loc}_{\hat{G},F,S} = {}^{cl}\mathrm{Loc}^{\square}_{\hat{G},F,S} / \hat{G}. \quad \square$$

Topology: Usually topology on $W_{F,S}$

every $\mathcal{O}_E$-mod

$$M = \varinjlim_i M_i, \quad M_i \text{ f.g.} \quad M_i = \varprojlim_n M_i/\varpi^n$$

$\rightsquigarrow$ ind-ϖ-adic top.

- A $\mathcal{O}_E$-alg $\rightsquigarrow$ equipped with ind-ϖ-adic top.

$\Rightarrow$ $A/\mathcal{O}_E$ finite, A/E finite usual ϖ-adic top.

- $A/\mathcal{O}_E/\varpi^n$ discrete top.

- $A = \bar{E}$ stronger than the usual ϖ-adic top.

Thm ($l>2$). ${}^{cl}\mathrm{Loc}^{\square}_{\hat{G},F,S}$ is repr. by disjoint union of affine schemes of finite type $/\mathcal{O}_E$.

Key ingredient: Let $\bar{\rho}: W_{F,S} \to \hat{G}(k)$ be a continuous residual repn, k/k_E finite

$\rightsquigarrow (R^{\square}_{\bar{\rho}}, m, k)$ universal framed deformation ring of $\bar{\rho}$

$$\rho: W_{F,S} \longrightarrow \hat{G}(R^{\square}_{\bar{\rho}})$$

ρ a priori is continuous w.r.t. m-adic top. on $R^{\square}_{\bar{\rho}}$

Claim. ρ is continuous w.r.t. ind-ϖ-adic top.

What does this mean? could still be

$R^{\square}_{\bar{\rho}}/\varpi$ is discrete ($R^{\square}_{\bar{\rho}} \simeq k[[t_1, \dots]]$)

Claim $\Rightarrow \rho(\pi_1^{geom}(U)) \subset \hat{G}(R^{\square}_{\bar{\rho}}/\varpi)$ is finite

$\Leftarrow$ de Jong's conjecture proved by Dennis ($l>2$)

Another ingredient: Deformation theory is controlled by

$C^*_{cts}(W_{F,S}, \mathrm{Ad}_\rho)[1]$. Better to formulate derived moduli problem $\rightsquigarrow \mathrm{Loc}_{\hat{G},F,S}$.

Rmk 1. Similar construction works for $G/_F$ local field residue char $p \neq l$.

$\rightsquigarrow \mathrm{Loc}_{\hat{G},F} / \mathcal{O}_E$.

simpler: · No need something like de Jong's conj

· no – · non-trivial str.

· Can define $/\mathbb{Z}[\frac{1}{p}]$.

$\underset{\text{wild inertia}}{P_F} \hookrightarrow \underset{\text{inertia}}{I_F} \hookrightarrow W_F \qquad W_F^t = W_F/P_F$

$1 \to I_F^t \to W_F^t \to \mathbb{Z} \to 1$

$\Pi \mathbb{Z}_l(1) \qquad \mathrm{Loc}_F \supset \mathrm{Loc}_F^{tame} \supset \mathrm{Loc}_F^{unr}$

$\ell \neq p$ $\qquad \{\rho : W_F^t \to \hat{G}\}/\hat{G} \qquad \{\rho : \mathbb{Z} \to \hat{G}\}/\hat{G}$

$\mathrm{Loc}_F^{\mathrm{tame}} \subset \mathrm{Loc}_F$ is open and closed.

$$\begin{tikzcd}
\pi : \mathrm{Loc}_{\hat{G},F,S} \arrow[r] \arrow[d] & \prod_{v\in S} \mathrm{Loc}_v \times \mathrm{Loc}_w^{\mathrm{unr}} \arrow[d] \\
\mathrm{Loc}_{\hat{G},F,S\cup\{w\}} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \prod_{v\in S} \mathrm{Loc}_v \times \mathrm{Loc}_w
\end{tikzcd}$$

2. $S \neq \emptyset \rightsquigarrow C_{cts}(W_{F,S}, -)[1] \simeq R\Gamma_{et}(U, -)$

not quasi-smooth

$$\begin{tikzcd}
\mathrm{Loc}_{\hat{G}}(X) \arrow[r] \arrow[d] & \mathrm{Loc}_v^{\mathrm{unr}} \arrow[d] \\
\mathrm{Loc}_{\hat{G},F,\{v\}} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \mathrm{Loc}_v
\end{tikzcd}$$

everywhere unramified

quasi-smooth

e.g. $X = \mathbb{P}^1$.

$$\begin{tikzcd}
& \mathrm{Loc}_{\hat{G}}(X) \arrow[r] \arrow[d] & \mathrm{Loc}_\infty^{\mathrm{unr}} \arrow[d] \\
\mathrm{Loc}_0^{\mathrm{unr}} \simeq & \mathrm{Loc}_{\hat{G},\infty}^{\mathrm{tame}} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \mathrm{Loc}_\infty^{\mathrm{tame}}
\end{tikzcd}$$

3. F # field. then

$$\rho : \Gamma_{F,S} = \pi_1(\mathcal{O}_{F,S}) \to \hat{G}(R_{\bar{\rho}}^{\square})$$

is in general <u>not</u> continuous w.r.t. the ind-ϖ-adic top.

(Joint in progress with Emerton).

(i) $R_{\bar{\rho}}^{\square} \twoheadrightarrow R_{\bar{\rho}}^{\square, \mathrm{cris}, \underline{k}}$ crystalline deformations.

Then conjecture:

$$\rho : \Gamma_{F,S} \to \hat{G}(R_{\bar{\rho}}^{\square,\mathrm{cris},\underline{k}}) \qquad S \supseteq \text{all places above } \ell.$$

is continuous w.r.t. ind-ϖ-adic top.

$\rightsquigarrow$ should exist alg stack $/\mathcal{O}_E$ classifying Galois repns coming from algebraic geometry $(G = GL_n)$

(ii) $\pi : \mathrm{Loc}_{\widehat{G}, F, S} \longrightarrow \prod_{v \in S} \mathrm{Loc}_v$ $\left(\begin{array}{l} v \mid \ell \quad \mathrm{Loc}_v \\ \mathrm{EG\ stack} \end{array} \right)$

Conj $\Rightarrow$ π is relatively representable.

2. Cohomology of moduli of Shtukas

G semisimple (for simplicity) $/ F = \mathbb{F}_q(X)$.

$\Lambda = E, \mathcal{O}_E, \kappa_E, \cdots$

$K \subset G(\mathbb{A})$ open compact, K_w hyperspecial outside S.

I finite set

$\rightsquigarrow \mathrm{Sht}_I \longrightarrow \eta^I = \eta(X^I)$ moduli of Shtukas.

$V \in \mathrm{Rep}(\widehat{G}^I)$ Λ-coefficient.

$$\rightsquigarrow \quad H_I(V) = R\Gamma_c\left(\mathrm{Sht}_I \big|_{\bar{\eta}^I}, \mathrm{Sat}(V) \right)$$

$$\mathcal{H}_K = C_c\left(K \backslash G(\mathbb{A}) / K \right) \circlearrowright \qquad \circlearrowleft \mathrm{FWeil}(\eta^I, \bar{\eta}^I) \twoheadrightarrow W_{F,S}^I$$

E.g. $V = \mathbb{1}$ trivial representation, $\Lambda = E$.

$$H_I(\mathbb{1}) = C_c\left(G(F) \backslash G(\mathbb{A}) / K, E \right)$$

Thm. $\Lambda = E$. For every i, $\exists$ a quasi-coherent sheaf $\underline{\underline{\mathcal{A}_K^i}}$ on ${}^{cl}\mathrm{Loc}_{\widehat{G}, F, S}$, with $\mathcal{H}_K$-action s.t.

$$(\ast) \quad H_I^i(V) \simeq \Gamma\left({}^{cl}\mathrm{Loc}_{\widehat{G}, F, S}, \mathcal{A}_K^i \otimes \widetilde{V} \right).$$

$$H_I^i(V):\ \mathcal{H}_K \circlearrowright,\ \circlearrowleft \mathrm{FWeil}(\eta^I, \bar{\eta}^I) \qquad \mathcal{A}_K^i:\ \circlearrowleft \mathcal{H}_K \qquad \widetilde{V}:\ \circlearrowleft W_{F,S}^I$$

$\widetilde{V}$ is the "tautological / evaluation" bundle on ${}^{cl}\ldots$

$\mathrm{Loc}_{\hat{G},F,S}$ $\quad$ ($``\mathrm{Loc}_{\hat{G},F,S} \to \mathbb{B}G \to \mathbb{B}G$)

Rmk. 1. AGKRRV explains how to get such formula by categorical trace construction (S = empty)

2. Such be true for $\Lambda = \mathcal{O}_E$ (Work in progress by Hemo).

Example (w/ V. Lafforgue)

Def. A Langlands parameter $\rho : W_{F,S} \to \hat{G}(\bar{E})$ is called elliptic if it is semisimple & $\mathfrak{S}_\rho = S_\rho / Z_{\hat{G}}$ is finite

$$S_\rho = Z_{\hat{G}}(\rho).$$

Ex. If $G = GL_n$, elliptic = irreducible

Lemma. Elliptic Langlands are isolated smooth pts in $\mathrm{Loc}_{\hat{G},F,S} \hookleftarrow */S_\rho$.

Lemma A_ρ^i is a f.d. vector space $\circlearrowleft \mathfrak{S}_\rho$.

$$H_I^i(V)_\rho \simeq (A_\rho^i \otimes V_\rho)^{S_\rho} = (A_\rho^i \otimes V_\rho)^{\mathfrak{S}_\rho}$$

$A_\rho^i \circlearrowleft \mathcal{H}_K$, $\quad V_\rho \circlearrowleft W_{F,S}^J$

localized at ρ

In particular $H_I(\mathbb{1})_\rho = (A_\rho^0)^{\mathfrak{S}_\rho}$

If $G = GL_n$, $\mathfrak{S}_\rho = \mathrm{triv}$. $A_\rho^0 = \pi^K$ $\qquad$ $\pi \leftrightarrow \rho$ LC + multiplicity one

$$H_I^0(V)_\rho = A_\rho^0 \otimes V_\rho = \pi^K \otimes V_\rho.$$

Def. The parameter ρ is called stable if $\mathfrak{S}_\rho = \{1\}$.

Cor. ρ stable

$$\Rightarrow \quad H^{\circ}_{I}(V)_{\rho} = H_{I}(\mathbb{1})_{\rho} \otimes V_{\rho}.$$

In general, $\mathcal{H}_K \circlearrowright A^i_{\rho} \circlearrowleft \mathfrak{S}_{\rho}$

$$\cdot\ A^i_{\rho} = \bigoplus_{\pi} \underset{\mathcal{H}_K}{\overset{\circlearrowleft}{\pi^K}} \otimes \underset{\mathfrak{S}_{\rho}}{\overset{\circlearrowleft}{\rho^i_{\pi}}} \qquad \left(\begin{array}{c}\text{up to Hecke}\\ \text{semisimplification}\end{array}\right)$$

$$H^{\circ}_{I}(V)_{\rho} = \bigoplus \pi^K \otimes \mathrm{Hom}_{\mathfrak{S}_{\rho}}(\rho^{\circ}_{\pi}, V_{\rho})$$

as $\mathcal{H}_K$-mod.

Arthur – Kottwitz $\quad \rho^i_{\pi} = \bigotimes \rho^i_{\pi_v}$ $\quad$ π_v local component of π

$$S_{\rho_v} = Z_{\hat{G}}(\rho_v) \qquad \rho_v|_{W_{F_v}} : W_{F_v} \to \hat{G}.$$

3. Conjectures.

Assume G is adjoint (for simplicity).

Λ coefficient ring $/\mathbb{Z}[\frac{1}{p}]$. $\qquad F$ local field.

$\mathrm{Rep}_{f.g.}(G(F), \Lambda)$ category generated by under finite colimits & retracts

$$\delta_K = C_c(G(F)/K, \Lambda) \qquad K \text{ open compact}$$

Local Conj. $\exists$ a fully faithful functor

$$\mathcal{A} : \mathrm{Rep}_{f.g}(G(F), \Lambda) \to \mathrm{Coh}(\mathrm{Loc}_{\hat{G}}/\Lambda)$$

Let. $\mathcal{A}_K = \mathcal{A}(\delta_K) \circlearrowleft \mathcal{H}_K$

Then
- if K is hyperspecial, $A_K \simeq \mathcal{O}_{\mathrm{Loc}^{\mathrm{unr}}}$
- if G is quasi-split and $K = I$ Iwahori

  $A_I \simeq \mathrm{Coh}\,\mathrm{Spr}^{\mathrm{unip}}$ ← unipotent coherent springer sheaf
- if G is quasi-split & $K = I(1)$ pro-p-Iwahori

  $A_{I(1)} \simeq \mathrm{Coh}\,\mathrm{Spr}.$

$$\mathrm{Loc}^{\mathrm{tame}}_{\hat T} \xleftarrow{r} \mathrm{Loc}^{\mathrm{tame}}_{\hat B} \xrightarrow{\pi} \mathrm{Loc}^{\mathrm{tame}}_{\hat G} \overset{\text{open \& closed}}{\subset} \mathrm{Loc}_{\hat G}$$

$$\mathrm{CohSpr} := \pi_* \mathcal{O}_{\mathrm{Loc}^{\mathrm{tame}}_{\hat B}}$$

$$\mathrm{Loc}^{\mathrm{unip}}_{\hat B} := r^{-1}(\mathrm{Loc}^{\mathrm{unr}}_{\hat T})$$

$$\mathrm{CohSpr}^{\mathrm{unip}} := \pi_* \mathcal{O}_{\mathrm{Loc}^{\mathrm{unip}}_{\hat B}}. \qquad \square.$$

Rmk. G split & $\Lambda = \mathbb{C}$, Hellmann also made the similar conjecture.

F global field. $\qquad K = \prod K_v$

$$\mathrm{res} : \mathrm{Loc}_{\hat G, F, S} \longrightarrow \prod_{v \in S} \mathrm{Loc}_{\hat G, F_v}.$$

<u>Conjecture</u>: $H_I(V) = \Gamma(\mathrm{Loc}_{\hat G, F, S}, \mathrm{res}^!(\boxtimes_v \mathcal{A}_{K_v}) \otimes \tilde V)$

Coh. of moduli of Sht.

I.e. $\quad \mathcal{A}^\bullet = \mathrm{res}^!(\boxtimes \mathcal{A}_{K_v})$. $\quad \circlearrowleft \mathcal{H}_K$.

ρ elliptic,

$$\rightsquigarrow \mathcal{A}^\bullet_\rho = \otimes \mathcal{A}_{K_v}\,{}^!_{\rho_v}.$$

4. Discussion of local conj. (F local).

<u>Cor of the conj</u>.

$$\mathrm{End}_{\mathrm{Loc}_{\hat G}} \mathcal{A}_K \simeq \mathrm{End}_{\mathrm{Rep}(G(F),\Lambda)} \delta_K = \mathrm{DHA}_K.$$

K hyerspecial.

$$\rightsquigarrow \mathrm{End}_{\mathrm{Loc}^{\mathrm{tame}}} \mathcal{O}_{\mathrm{Loc}^{\mathrm{unr}}} \cong \underset{\sim\sim\sim}{\mathrm{End}\, \delta_K} = H_c^\cdot(K^{G(F)}\!\!/K, \Lambda)$$

$G = T \qquad \mathrm{Loc}^{\mathrm{tame}} = \{(\sigma,\tau) \mid \sigma\tau\sigma^{-1} = \tau^q \Rightarrow \begin{array}{l}\sigma \in \hat T \\ \tau : \underset{\sim\sim}{\hat T[q-1]}\end{array}\}$

If $\mathrm{char}\,\Lambda \mid p-1$, $\quad \mathbb{O}$

$T = \mathbb{G}_m \quad \hat T[q-1] \simeq \mathrm{Spec}\, \Lambda[x]/x^{q-1}-1.$

$$\mathrm{End}_{\Lambda[x]/x^{q-1}-1} \Lambda \simeq H^\cdot(F^\times, \Lambda).$$

Even if Λ is a field of char. 0

$\Rightarrow$ $\mathrm{End}_{Loc^{tame}}\, \mathcal{O}_{Loc^{unr}}$ concentrates in deg. 0.

$K = I_w$. $\mathrm{End}_{\underline{Loc^{tame}}}\, \mathrm{CohSpr} \simeq \mathrm{DHA}_I$

(Known when $\Lambda = \overline{\mathbb{Q}}_\ell$, by Ben-Zvi-Chen-Helm-Nadler
Hemo-Z.)

We don't the ~~general~~ form of general A_K,
(For $G = GL_n$, Sasha Braverman's conjecture ...).

5. Discussion of the global Conjecture

res: $Loc_{\hat{G},F,S} \longrightarrow \prod_v Loc_{\hat{G},F_v}$

- No level structure $A = \mathrm{res}^! \boxtimes_v \mathcal{O}_{Loc_v^{unr}} = \omega_{Loc_{\hat{G},F,S}}$.
- $X = \mathbb{P}^1$

$$\begin{array}{ccc} Loc_{\hat{G}}(X) & \longrightarrow & Loc^{unr}_{\infty} \\ \downarrow & & \downarrow \\ Loc^{unr}_0 \cong Loc^{tame}_{\hat{G},\infty} & \longrightarrow & Loc^{tame}_{\infty} \end{array}$$

$$\Gamma(Loc_{\hat{G}}(X), \omega) \cong \mathrm{End}_{\mathcal{O}_{Loc^{tame}}}\, \mathcal{O}_{Loc^{unr}}$$

$\downarrow\wr$ $\qquad\qquad\qquad$ $\downarrow\wr$ local.

$$H_c^\bullet\left(G(\mathbb{O})\backslash G(\mathbb{A})/G(F), \Lambda\right) \xleftarrow{\sim} \mathrm{DHA}_K.$$

$X = \mathbb{P}^1$

(with Emerton)

<u>Ex</u>. $F = \mathbb{Q}$. $G = PGL_2$

$0 \to \mathbb{Q}_p \to H_c^1(Y_0(N), \mathbb{Q}_p) \to H^1(X_0(N), \mathbb{Q}_p) \to 0$.

Eisenstein $\qquad$ cusp.

$H_c^2(Y_0(N), \mathbb{Q}_p) = \mathbb{Q}_p(-1)$.

Eisenstein.

Stack of global Langlands parameters: $\rho: \Gamma_{\mathbb{Q},\{p,N,\infty\}} \to GL_2$.

ρ cris at p, $\rho(\text{complex conjugation}) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. $\det = \chi$.

$\mathbb{A}^1/\mathbb{G}_m \quad \sqcup \quad */\mu_2$. $(N=11)$.

$\{0 \to \chi \to \rho \to 1 \to 0\}$.

$\downarrow$ res.

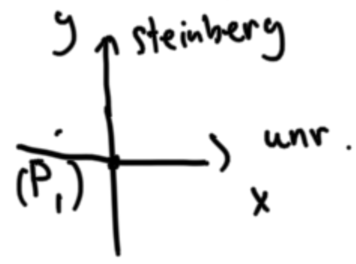
A_∞ line bundle of w.t. -1.

$\mathrm{Loc}^{\mathrm{unip}}_{\mathbb{Q}_N} \times \mathrm{Loc}_{\mathbb{Q}_\infty} = */\mathbb{G}_m$

$\mathrm{Spec}\left(\mathbb{Q}_p[x,y]/xy=0\right)/\mathbb{G}_m$

y w.t. 2

x w.t. 0.



Fact. $\mathrm{CohSpr}^{\mathrm{unip}} \simeq \mathcal{O}_{\mathrm{Loc}^{\mathrm{unip}}} \oplus \mathcal{O}_{\mathrm{Loc}^{\mathrm{unr}}}$

$*/\mu_2 \xrightarrow{\sim} \mathrm{Spec}\,\mathbb{Q}_p[y,y^{-1}]/\mathbb{G}_m \hookrightarrow \left(\mathbb{Q}_p[xy]/x\right)/\mathbb{G}_m$.

$\mathrm{res}^!(\mathrm{CohSpr}^{\mathrm{unip}} \boxtimes A_\infty)|_{*/\mu_2} \simeq \mathcal{O}$. $\mathrm{sgn} \hookrightarrow H^1(X_0(N), \mathbb{Q}_p)$

$\mathbb{A}^1/\mathbb{G}_m \xrightarrow{\sim} \mathrm{Spec}\,\mathbb{Q}_p[y]/\mathbb{G}_m \longrightarrow \left(\mathrm{Spec}\,\mathbb{Q}_p[xy]/x\right)/\mathbb{G}_m$

Eisenstein $\mathbb{Q}_p$ $\quad H^2_c(Y_0(N), \mathbb{Q}_p)$

$\mathrm{res}^!(\mathrm{CohSpr}^{\mathrm{unip}} \boxtimes A_\infty)|_{\mathbb{A}^1/\mathbb{G}_m} = \omega_{\mathbb{A}^1/\mathbb{G}_m}(\cdots) \oplus \bigoplus_{j\geq 0} \mathcal{O}_{p,y=0}[2j-1](\ldots)$

$G = D_{N,\infty}$. $\quad C_c\left(D_N^\times \backslash (D_{N,\infty} \otimes \mathbb{A})^\times / \mathcal{O}_D^\times, \mathbb{Q}_p\right)$.

$\mathbb{A}^1/\mathbb{G}_m \sqcup */\mu_2$.

$\downarrow$ res.

$G|_{\mathbb{Q}_N}$ nonsplit. $\quad \mathrm{Loc}^{\mathrm{unip}}_{\mathbb{Q}_N} \times \mathrm{Loc}_{\mathbb{Q}_\infty}$

$A_N = \omega_{\mathrm{Spec}(\mathbb{Q}_p[xy]/x)/\mathbb{G}_m}$ $\qquad A_\infty$ w.t. -2.

6. Categorical version of local Langlands.

$G/F \quad F$ local (G split.). $\qquad$ residue field of F is $\mathbb{F}_q$.

$\mathrm{Shv}\left(\frac{LG}{\mathrm{Ad}_\sigma LG}\right)$. $\qquad \frac{LG}{\mathrm{Ad}_\sigma LG^\theta}/\mathbb{F}_q$.

$B(G) = \left(\frac{LG}{\mathrm{Ad}_\sigma LG}\right)(\mathbb{F}_q) = $ Frob-conj class of $G(L)$. L/F completion of max. unr. ext of F.

$b \quad \otimes \quad i_b : {*}/J_b(F) \xhookrightarrow{\text{locally closed}} \frac{LG}{\mathrm{Ad}_\sigma LG}$.

J_b : Frobenius-twisted centeralizer of b in G.

In partical, $b=1$. $J_b = G$.

b basic element $\quad J_b$ inner form of G.

$\mathrm{Rep}(J_b(F),\Lambda) \simeq \mathrm{Shv}({*}/J_b(F),\Lambda) \hookrightarrow \mathrm{Shv}\left(\frac{LG}{\mathrm{Ad}_\sigma LG}\right)$.

Conj. $\quad \mathrm{Shv}\left(\frac{LG}{\mathrm{Ad}_\sigma LG}\right) \xrightarrow{\sim} \mathrm{Ind}\left(\mathrm{Coh}_{\mathrm{Nilp.}}\left(\mathrm{Loc}_{\hat{G},F}\right)\right)$.

$$\begin{array}{ccc} \mathrm{Shv}\left(\frac{LG}{\mathrm{Ad}_\sigma LG}\right) & \xrightarrow{\sim} & \mathrm{Ind}\left(\mathrm{Coh}_{\mathrm{Nilp.}}\left(\mathrm{Loc}_{\hat{G},F}\right)\right) \\ \uparrow i_{b,*} & & \uparrow \\ \mathrm{Rep}_{f.g.}(G(F),\Lambda) & \xhookrightarrow{\mathbb{A}} & \mathrm{Coh}(\mathrm{Loc}_{\hat{G},F}) \end{array}$$

Suppose: $f: X \to Y$ proper surj. / of finite type.

$\cdots \ \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \ X \times_Y X \times_Y X \ \substack{\rightarrow\\\rightarrow\\\rightarrow} \ X \times_Y X \rightrightarrows X$

$\mathrm{Shv}(Y) = \underset{\substack{\longrightarrow \\ \text{!-push forward}}}{\mathrm{colim}} \mathrm{Shv}(X_Y^\bullet)$.

$Y = \frac{LG}{\mathrm{Ad}_\sigma LG}$. $\quad X = \frac{LG}{\mathrm{Ad}_\sigma I}$. $\leftarrow$ moduli of local shtukas with Iwahori level.

Fact (w/ Hemo). $\mathrm{Shv}\left(\frac{LG}{\mathrm{Ad}_\sigma LG}\right)$ has semi-orthogonal decomposition by $\mathrm{Rep}(J_b(F),\Lambda)$, Verdier duality, ...

$$\begin{array}{c} H_I(V) \\ \circlearrowright \\ \mathrm{End}(\mathbb{A}\otimes\tilde{V}) \\ \uparrow \\ \mathrm{End}(\mathbb{A}_{K_v}\otimes\tilde{V}) \end{array}$$

$\mathbb{A}_{K_v} = \mathbb{A}(\delta_{K_v})$

$\mathrm{End}(\delta_{K_v} \otimes V)$.

$$\pi : \frac{LG}{\mathrm{Ad}_\sigma I} \to \frac{LG}{\mathrm{Ad}_\sigma LG}$$

$$\underline{A_{K_v} \otimes \tilde{V}} \cong \underbrace{\pi_! \left(\text{Nearby cycle of } \cancel{\text{Stake}} \text{ sheaf } \otimes \delta_{K_v} \right)}$$

$$\mathrm{End}(\mathcal{F}) \circlearrowleft H_I(V)$$

$$\mathrm{Shv}^{\mathrm{Ren}}\left(\frac{LG}{\mathrm{Ad}_\sigma LG}\right) \to \mathrm{Ind}\, \underbrace{\mathrm{Shv}_c\left(\frac{LG}{\mathrm{Ad}_\sigma LG}\right)}.$$

$$\mathrm{Shv}_c\left(\frac{LG}{\mathrm{Ad}_\sigma LG}\right) \simeq \operatorname{colim} \mathrm{Shv}_c\left(X^\circ_Y \right).$$

$$\underbrace{\left(\frac{LG}{\mathrm{Ad}_\sigma LG}\right)^\diamond} \dashrightarrow \mathrm{Bun}_G \quad {}^{\text{FS}}.$$

$$\int \left(\mathbb{A}^1/\mathbb{F}_p\right)^\diamond$$