Coherent sheaves on the stack of Langlands parameters

G connected reductive grp / F global field

$$\left\{\begin{matrix}\text{Automorphic forms} \\ \text{of } G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Langlands parameters} \\ \rho: \mathcal{L}_F \to {}^L G\end{matrix}\right\}$$

Automorphic forms: Bernstein center of real group?

(1) Complex theory
(2) p-adic theory
} analytic nature (Isolation of cuspidal spectrum, ... conj about this)

(3) Algebraic automorphic forms, ← algebro-geometric

(3) = (1) ∩ (2) cohomology of modular varieties, ... Arithmetic nature

1. The stack of Langlands parameters

- $F = \mathbb{F}_q(X)$, X smooth proj curve $/\mathbb{F}_q$, char $F = p$.
- S non-empty finite set of places of X, $U = X - S$
- $W_{F,S} = \mathrm{Weil}(U)$ $\qquad 1 \to \pi_1^{\mathrm{geom}}(U) \to W_{F,S} \to \mathbb{Z} \to 1$
- $E/\mathbb{Q}_\ell$ finite ext. $\ni \sqrt{q}$, $\quad \ell \neq p$.
- G split reductive gp $/F$

<u>Def</u>. ${}^{cl}\mathrm{Loc}^{\square}_{\hat{G},F,S} : \mathcal{O}_E\text{-alg} \longrightarrow \mathrm{Sets}$

$$A \longmapsto \left\{\begin{matrix}\text{continuous homomorphisms} \\ \rho: W_{F,S} \to \hat{G}(A)\end{matrix}\right\}$$

$${}^{cl}\mathrm{Loc}_{\hat{G},F,S} = {}^{cl}\mathrm{Loc}^{\square}_{\hat{G},F,S} / \hat{G}. \qquad \square$$

Topology: Usually topology on $W_{F,S}$

every $\mathcal{O}_E$-mod

$$M = \varinjlim_i M_i, \quad M_i \text{ f.g.} \quad M_i = \varprojlim_n M_i/\varpi^n$$

$\rightsquigarrow$ ind-ϖ-adic top.

- A $\mathcal{O}_E$-alg $\rightsquigarrow$ equipped with ind-ϖ-adic top.

$\Rightarrow$ $\cdot$ $A/\mathcal{O}_E$ finite, A/E finite usual ϖ-adic top.

• $A/\mathcal{O}_E/\varpi^n$ discrete top.

• $A = \overline{E}$ stronger than the usual ϖ-adic top.

<u>Thm</u>. $(\ell>2)$ ${}^{cl}\mathrm{Loc}^{\square}_{\hat{G},F,S}$ is repr. by disjoint union of affine schemes of finite type $/\mathcal{O}_E$.

<u>Key ingredient</u>: Let $\bar\rho: W_{F,S} \to \hat{G}(\kappa)$ be a continuous residual repn, κ/κ_E finite

$\rightsquigarrow (R^{\square}_{\bar\rho}, \mathfrak{m}, \kappa)$ universal framed deformation ring of $\bar\rho$

$$\rho: W_{F,S} \longrightarrow \hat{G}(R^{\square}_{\bar\rho})$$

ρ a priori is continuous w.r.t. $\mathfrak{m}$-adic top. on $R^{\square}_{\bar\rho}$

<u>Claim</u>. ρ is continuous w.r.t. ind-ϖ-adic top.

What does this mean? could still be

$R^{\square}_{\bar\rho}/\varpi$ is discrete $(R^{\square}_{\bar\rho} \simeq \kappa[[t_1, \ldots]])$

Claim $\Rightarrow \rho(\pi_1^{geom}(U)) \subset \hat{G}(R^{\square}_{\bar\rho}/\varpi)$ is finite

$\Leftarrow$ de Jong's conjecture proved by Dennis $(\ell>2)$

<u>Another ingredient</u>: Deformation theory is controlled by

$C^*_{cts}(W_{F,S}, \mathrm{Ad}_\rho)[1]$. Better to formulate derived moduli problem $\rightsquigarrow \mathrm{Loc}_{\hat{G},F,S}$.

<u>Rmk</u> 1. Similar construction works for G/F local field residue char $p \neq \ell$.

$\rightsquigarrow \mathrm{Loc}_{\hat{G},F}/\mathcal{O}_E$.

simpler: • No need something like de Jong's conj

• no – · non-trivial str.

• Can define $/\mathbb{Z}[\frac{1}{p}]$.

wild inertia, inertia

$P_F \hookrightarrow I_F \hookrightarrow W_F \qquad W_F^t = W_F/P_F$

$1 \to I_F^t \to W_F^t \to \mathbb{Z} \to 1$

$\mathrm{Loc}_F \supset \mathrm{Loc}_F^{tame} \supset \mathrm{Loc}_F^{unr}$

$\prod \mathbb{Z}_\ell(1)$

$\ell \neq p$ $\qquad \{\rho : W_F^t \to \hat{G}\}/\hat{G} \quad \{\rho : \mathbb{Z} \to \hat{G}\}/\hat{G}$

$\mathrm{Loc}_F^{\mathrm{tame}} \subset \mathrm{Loc}_F$ is open and closed.

$$\begin{tikzcd}
\mathrm{Loc}_{\hat{G},F,S} \arrow[r, "\pi"] \arrow[d] & \prod_{v\in S} \mathrm{Loc}_v \times \mathrm{Loc}_w^{\mathrm{unr}} \arrow[d] \\
\mathrm{Loc}_{\hat{G},F,S\cup\{w\}} \arrow[r] & \prod_{v\in S} \mathrm{Loc}_v \times \mathrm{Loc}_w
\end{tikzcd}$$

2. $S \neq \emptyset \rightsquigarrow C_{cts}(W_{F,S}, -)[1] \simeq R\Gamma_{et}(U, -)$

not quasi-smooth

$$\begin{tikzcd}
\mathrm{Loc}_{\hat{G}}(X) \arrow[r] \arrow[d] & \mathrm{Loc}_v^{\mathrm{unr}} \arrow[d] \\
\mathrm{Loc}_{\hat{G},F,\{v\}} \arrow[r] & \mathrm{Loc}_v
\end{tikzcd}$$

everywhere unramified.

quasi-smooth

e.g. $X = \mathbb{P}^1$.

$$\begin{tikzcd}
& \mathrm{Loc}_{\hat{G}}(X) \arrow[r] \arrow[d] & \mathrm{Loc}_\infty^{\mathrm{unr}} \arrow[d] \\
\mathrm{Loc}_0^{\mathrm{unr}} \simeq & \mathrm{Loc}_{\hat{G},\infty}^{\mathrm{tame}} \arrow[r] & \mathrm{Loc}_\infty^{\mathrm{tame}}
\end{tikzcd}$$

3. F # field. then

$$\rho : \Gamma_{F,S} = \pi_1(\mathcal{O}_{F,S}) \to \hat{G}(R_{\bar{\rho}}^{\square})$$

is in general <u>not</u> continuous w.r.t. the ind-ϖ-adic top.

(Joint in progress with Emerton).

(i) $R_{\bar{\rho}}^{\square} \twoheadrightarrow R_{\bar{\rho}}^{\square, \mathrm{cris}, \underline{k}}$ crystalline deformations.

Then conjecture:

$$\rho : \Gamma_{F,S} \to \hat{G}(R_{\bar{\rho}}^{\square, \mathrm{cris}, \underline{k}})$$

$S \supseteq$ all places above ℓ.

is continuous w.r.t. ind-ϖ-adic top.

$\rightsquigarrow$ should exist alg stack $/\mathcal{O}_E$ classifying Galois repns coming from algebraic geometry

(ii) $(G = GL_n)$ $\pi : \mathrm{Loc}_{\hat{G}, F, S} \longrightarrow \prod_{v \in S} \mathrm{Loc}_v$ $\left(\begin{array}{l} v \mid \ell \ \ \mathrm{Loc}_v \\ \text{EG stack} \end{array} \right)$

Conj $\Rightarrow$ π is relatively representable.

2. Cohomology of moduli of Shtukas

G semisimple (for simplicity) $/ F = \mathbb{F}_q(X)$.

$\Lambda = E, \mathcal{O}_E, k_E, \ldots$

$K \subset G(\mathbb{A})$ open compact, K_w hyperspecial outside S.

I finite set

$\rightsquigarrow \mathrm{Sht}_I \longrightarrow \eta^I = \eta(X^I)$ moduli of Shtukas.

$V \in \mathrm{Rep}(\hat{G}^I)$ Λ-coefficient.

$$\rightsquigarrow H_I(V) = R\Gamma_c\left(\mathrm{Sht}_I \big|_{\bar{\eta}^I}, \mathrm{Sat}(V) \right)$$

$$\mathcal{H}_K = C_c(K \backslash G(\mathbb{A}) / K) \circlearrowright \qquad \circlearrowright \mathrm{FWeil}(\eta^I, \bar{\eta}^I) \twoheadrightarrow W_{F,S}^I$$

E.g. $V = \mathbb{1}$ trivial representation, $\Lambda = E$.

$$H_I(\mathbb{1}) = C_c\left(G(F) \backslash G(\mathbb{A}) / K, E \right)$$

Thm. $\Lambda = E$. For every i, $\exists$ a quasi-coherent sheaf $\mathcal{A}_K^i$ on ${}^{cl}\mathrm{Loc}_{\hat{G}, F, S}$, with $\mathcal{H}_K$-action, s.t.

$$(\ast) \quad H_I^i(V) \simeq \Gamma\left({}^{cl}\mathrm{Loc}_{\hat{G}, F, S}, \mathcal{A}_K^i \otimes \tilde{V} \right).$$

$\mathcal{H}_K \circlearrowright H_I^i(V) \circlearrowleft \mathrm{FWeil}(\eta^I, \bar{\eta}^I)$; $\mathcal{H}_K \circlearrowright \mathcal{A}_K^i$; $W_{F,S}^I \circlearrowright \tilde{V}$

$\tilde{V}$ is the "tautological/evaluation" bundle on

$\mathrm{Loc}_{\hat{G},F,S}$ $\quad (\text{"}\mathrm{Loc}^{\wedge}_{G,F,S} \to \mathbb{B}G \to \mathbb{B}G \;)$

<u>Rmk</u>. 1. AGKRRV explains how to get such formula by categorical trace construction ($S=$ empty)

2. Such be true for $\Lambda = \mathcal{O}_E$ (Work in progress by Hemo).

Example (w/ V. Lafforgue)

<u>Def</u>. A Langlands parameter $\rho : W_{F,S} \to \hat{G}(\bar{E})$ is called elliptic if

$$\mathfrak{S}_\rho = S_\rho / Z_{\hat{G}} \quad \text{is finite}$$

$$S_\rho = Z_{\hat{G}}(\rho).$$

<u>Ex</u>. If $G = GL_n$, elliptic = irreducible

<u>Lemma</u>. Elliptic Langlands are isolated smooth pts in $\mathrm{Loc}_{\hat{G},F,S} \hookleftarrow */S_\rho$.

<u>Lemma</u> A^i_ρ is a f.d. vector space $\circlearrowleft \mathfrak{S}_\rho$.

$$H^i_I(V)_\rho \simeq (A^i_\rho \otimes V_\rho)^{S_\rho} = (A^i_\rho \otimes V_\rho)^{\mathfrak{S}_\rho}$$

localized at ρ $\qquad \mathcal{H}_K \circlearrowright A^i_\rho, \quad W^I_{F,S} \circlearrowright V_\rho$

In particular $H_I(\mathbb{1})_\rho = (A^0_\rho)^{\mathfrak{S}_\rho}$

If $G = GL_n$, $\mathfrak{S}_\rho = \mathrm{triv}$. $A^0_\rho = \pi^K$ $\qquad \pi \leftrightarrow \rho$ LC + multiplicity one

$$H^0_I(V)_\rho = A^0_\rho \otimes V_\rho = \pi^K \otimes V_\rho.$$

<u>Def</u>. The parameter ρ is called stable if $\mathfrak{S}_\rho = \{1\}$.

Cor. ρ stable

$$\Rightarrow \quad H^0_I(V)_\rho = H_I(\mathbb{1})_\rho \otimes V_\rho .$$

In general, $\mathcal{H}_K \circlearrowright \underline{A^i_\rho} \circlearrowleft \mathfrak{S}_\rho$

$$A^i_\rho = \bigoplus_\pi \pi^K \otimes \rho^i_\pi \quad \left(\begin{array}{c}\text{Up to Hecke} \\ \text{semisimplification}\end{array}\right)$$

$\pi^K \circlearrowleft \mathcal{H}_K$, $\rho^i_\pi \circlearrowleft \mathfrak{S}_\rho$

$$H^0_I(V)_\rho = \bigoplus \pi^K \otimes \operatorname{Hom}_{\mathfrak{S}_\rho}(\rho^0_\pi, V_\rho)$$

as $\mathcal{H}_K$-mod.

On the other hand, $W_{F,S} \circlearrowright V_\rho \circlearrowleft S_\rho$

$$A^i_\rho = \bigoplus_\chi W(\chi) \otimes \chi$$

$W(\chi) \circlearrowleft \mathcal{H}_K$, $\chi \circlearrowleft \mathfrak{S}_\rho$

$$H^0_I(V)_\rho = \bigoplus_\chi W(\chi) \otimes \operatorname{Hom}_{\mathfrak{S}_\rho}(\chi, V_\rho)$$

$\operatorname{Hom}_{\mathfrak{S}_\rho}(\chi, V_\rho) \circlearrowleft$ Irr. $W_{F,S}$-mod.

$\rightsquigarrow$ Semisimplicity of Galois repn.