Lecture 3: Recall 1) $LG \simeq \operatorname{colim}_\alpha X_\alpha$

$X_\alpha \subseteq LG$ is $I \times I$-invariant closed

2) $\forall \alpha, n$, $\pi_{\alpha,n} : \mathrm{Shv}^{\Gamma}(LG) \to \mathrm{Shv}^{\Gamma}(LG)$

$\pi_{\alpha,n}(\mathcal{F}) = (\mathcal{F} \overset{!}{\otimes} \omega_{X_\alpha}) * 1_{I_n} = (\mathcal{F} * 1_{I_n}) \overset{!}{\otimes} \omega_{X_\alpha}$.

3) $\widetilde{LG} := LG \overset{I}{\times} I \xrightarrow{p} LG \quad p(g,x) = gxg^{-1}$

$S := p_! \omega_{\widetilde{LG}} \in \mathrm{Shv}(LG)$

$\Big\downarrow$ upgraded

$S_{\widetilde{W}} \in \mathrm{Shv}^{\widetilde{W}}(LG)$

Thm: $\forall \alpha, n \quad \pi_{\alpha,n}(S_{\widetilde{W}}) \in \mathrm{Shv}^{\widetilde{W}}(LG)_{\mathrm{cons}}$.

Cor: $\forall \gamma \in LG_{rss}$, $H_i(Fl_\gamma)$ is a f.g. $\widetilde{W}$-module.

Rem: $\exists$ global pf based on Thm of Yun.

Lemma: $\forall \gamma\ \exists n$ s.t. $(S_{\widetilde{W}} * 1_{I_n})|_\gamma = S_{\widetilde{W}}|_\gamma$

perf $\widetilde{W}$-mod // Tum $\qquad$!-stalk

pf of lemma: (i) S is $\mathrm{Ad}\,LG$-equiv
$\Rightarrow$ $\mathrm{Ad}\,I$-equivariant

(ii) $\forall \gamma\ \exists m$ $\widetilde{LG} \to LG$ is constant family
$\widetilde{LG} \cong LG \times Fl$

on $\underline{L(G_\gamma) \cap (\gamma I_m) = U_{m,\gamma}}$; $[\forall \gamma_1, \gamma_2 \in U_{m,\gamma}$
$Fl_{\gamma_1} = Fl_{\gamma_2} \subseteq Fl]$

(iii) $(\mathrm{Ad}\, I)(\underline{U_{m,\gamma}}) \ni \gamma I_n$ for some n. $\boxtimes$

Cor 2: $\forall \tau \in \mathrm{Rep}_{fd}(\widetilde{W})$, $S_\tau := S \otimes_{\widetilde{W}} \tau$ sat
$\pi_{a,n}(S_\tau) \in \mathrm{Shv}(LG)_{\mathrm{cons.}}$ $(\forall a, n)$

Case 1: $\tau = 1$ $\quad S_\tau = \omega_{LG_c}$, $LG_c = \mathrm{Ad}\,LG(I) \subseteq LG$
bounded elements

analog: $S^W_{\mathrm{Fin}} = \mathbb{Q}_\ell$ | in part $S_\tau \otimes \omega_{X_a}$ - compact,

Case 2: $\tau = \mathrm{sgn}$, $(S_{\mathrm{sgn}}|_{LG_{tu}})^* 1_{I_n} \in \mathrm{Shv}(LG)_c$
$\forall n$.

$LG_{tu} = \mathrm{Ad}\,G(I^+) \subseteq LG$ top

pf of Thm: $(G = G^{sc})$

Recall $\mathcal{Y} = \{ Y \subseteq \widetilde{W} \mid \text{finite closed} \}$

$$\updownarrow$$

$Y \subseteq \widetilde{W}$ - loc closed $\qquad Fl^Y \subseteq Fl$

Step 1: let $\widetilde{LG}^Y = LG^Y \times^I I = \widetilde{LG} \cap (Fl^Y \times LG)$

let $p^Y : \widetilde{LG}^Y \longrightarrow LG$, $\quad S^Y := (p^Y)_* \, \omega_{\widetilde{LG}^Y}$

fp-morphism

Claim: $\forall Y \in \mathcal{Y}, \ \forall J \in Par$

$$S^{YW_J} \circlearrowleft W_J \rightsquigarrow S^Y_{W_J} \in Shv^{W_J}(LG)$$

$\uparrow$ upgrade (functor in Y)

$\exists$ nat action

Pf:

$$\begin{tikzcd}
\widetilde{LG}^{YW_J} \arrow[r] \arrow[d, "pr_2"'] \arrow[dr, phantom, "\square"] & \widetilde{LG}_J^{YW_J} \arrow[r] \arrow[d, "pr_2"] \arrow[dr, phantom, "\rotatebox{-45}{$\sim$}\; LG^{YW_{P_J}} \times^{P_J} P_J"] & LG \\
\frac{I}{I} \arrow[r, "p_J"'] & \frac{P_J}{P_J} &
\end{tikzcd}$$

$p_{J!}\, \omega_{\frac{I}{I}} = S_{\frac{P_J}{P_J}} \circlearrowleft W_J$

Cor: $\exists$ functor $\mathcal{Y} \times Par \longrightarrow Shv^{\widetilde{W}}(LG)$

$$(Y, J) \longmapsto Ind^{\widetilde{W}}_{W_J}\left(S^{YW_J}_{W_J} \right) \in$$

Pf: – $\forall\, Y_1 \subseteq Y_2 \hookrightarrow \mathrm{Ind}_{W_J}^{\widetilde{W}}(S_J^{Y_1}) \to$ –"– ^{Y_2}

– $\forall\, J_1 \subseteq J_2$

$$\begin{tikzcd} \mathrm{ind}_{W_{J_1}}^{W_{J_2}}(S_{J_1}^{Y}) \arrow[r, "?"] & S_{J_2}^{Y} \end{tikzcd} \quad \Big| \quad \mathrm{ind}_{W_{J_2}}^{\widetilde{W}}(\ )$$

$$\begin{tikzcd} & \arrow[d, leftrightarrow, "\shortparallel"] \\ S_{J_1}^{Y} \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathrm{res}_{W_{J_2}}^{W_{J_1}} S_{J_2}^{Y} \arrow[d, phantom, "\shortparallel"] \\ S_{J_1}^{YW_{J_1}} \arrow[r] & S_{J_1}^{YW_{J_2}} \end{tikzcd} \quad (YW_{J_1} \subseteq YW_{J_2})$$

$\square$

Step 2: $\forall\, Y$ set

$$S_{\widetilde{W}}^{Y} := \mathrm{colim}_J\, \mathrm{ind}_{W_J}^{\widetilde{W}}(S_J^{Y})$$

Lemma: $\mathrm{colim}_Y\, S_{\widetilde{W}}^{Y} = S_{\widetilde{W}}$

Pf ① $S_{\widetilde{W}} = \mathrm{colim}_J\, \mathrm{ind}_{W_J}^{\widetilde{W}}(S_{W_J})$

$\shortparallel$

$\mathrm{colim}_J\, \mathrm{ind}_{W_J}^{\widetilde{W}}\, \mathrm{res}_{\widetilde{W}}^{W_J}(S_{\widetilde{W}})$

follows from

$$\mathrm{Shv}^{\widetilde{W}}(LG) \simeq \lim \mathrm{Shv}(LG)^{W_J}$$

$\mathcal{F} \xrightarrow{\sim} \mathrm{colim}\, \mathrm{ind}_J \mathrm{res}_J(\mathcal{F})$

$$S_{\widetilde{W}} = \mathrm{colim}_J \left(\mathrm{ind}_{W_J}^{\widetilde{W}} S_{W_J}\right) =$$

$$= \mathrm{colim}_J \left(\mathrm{colim}_Y\, \mathrm{ind}_{W_J}^{\widetilde{W}}\left(S_J^Y\right)\right)$$

$$= \mathrm{colim}_Y\, \mathrm{colim}_J \left(\qquad \right) =$$

$$= \mathrm{colim}_Y \left(S_{\widetilde{W}}^Y\right) \qquad \boxed{/}$$

$$S_{W_J} = \mathrm{colim}_Y\, S_{W_J}^{Y W_J}$$ (Thm!)

Cor: Enough to show $\forall a, n$

$\exists Y$ s.t. $\forall Y' \supseteq Y$

the map $S_{\widetilde{W}}^Y \to S_{\widetilde{W}}^{Y'}$ is

$\pi_{a,n}$-equivalence (isom after $\pi_{a,n}$

Step 3: $\forall w \in \widetilde{W}$ set $J_w = \{\alpha \in \Delta \mid w(\alpha) > 0\}$

Recall: $\forall J \subseteq J' \; \exists \; \mathrm{ind}_{W_J}^{W_{J'}}(S_J^{Y}) \to S_{J'}^{Y}$

Enough to show:

Thm'': $\forall a, n \; \exists N(a,n) \in \mathbb{N}$ s.t

$\forall w \in \widetilde{W} \; \forall \alpha \in J_w$ s.t $w(\alpha) > N(a,n)$

$\forall J \subseteq J_w - \alpha$ $\boxed{\mathrm{ind}_{W_J}^{W_{J \cup \alpha}} S_J^{w} \to S_{J\cup\alpha}^{w}}$

is $\pi_{a,n}$-equivalence

lemma: Thm'' $\Longrightarrow$ Thm':

Pf: let $Y_0 = \{w \in \widetilde{W} \mid \forall \alpha \in \Delta \; w(\alpha) \leq N(n,a)\}$

$\uparrow$ finite set. , Take $Y = \overline{Y_0}$

Claim: If satisfies Thm':

let $Y' \supsetneq Y$, $w \in Y' \setminus Y$ maximal

enough $S_{\widetilde{W}}^{Y'-w} \to S_{\widetilde{W}}^{Y'}$ is $\pi_{a,n}$-equiv

$$\iff \mathcal{J}_{\alpha,n}\ \mathrm{cof}\left(S^{y'-w}_{\widetilde{W}} \to S^{y'}_{\widetilde{W}}\right) = 0$$

$$S^{y'}_{\widetilde{W}} = \mathrm{colim}\ \mathrm{ind}^{\widetilde{W}}_{W_J}\left(S^{y'W_J}_{W_J}\right)$$

$$\mathrm{cof}\left(S^{(y'-w)W_J}_{W_J} \to S^{y'W_J}_{W_J}\right) = \begin{cases} 0, & J \not\subseteq J_w \\ S^{wW_J}, & J \subseteq J_w. \end{cases}$$

$$(y'-w)W_J = y'W_J \iff J \not\subseteq J_w$$

$$\boxed{\exists \alpha \in J;\ ws_\alpha < w \iff w(\alpha) < 0}$$

Step 4: Point case $J = \emptyset$

$$\forall \alpha, n\ \exists N\ \forall w\ \forall \alpha \in J_w\ \ w(\alpha) > N$$

$$\mathrm{ind}_1^{W_\alpha}(S^w) \to S^{wW_\alpha}_{W_\alpha} \text{ is } \mathcal{J}_{\alpha,n}\text{-equiv}$$

$$\Uparrow$$

$$\mathrm{Shv}^{W_\alpha}(LG)$$

Pf: Enough to show

(1) $S^w \to \left(S^{wW_\alpha}_{W_\alpha}\right)_{W_\alpha}$

(2) $S^w \to \left(S^{wW_\alpha}_{W_\alpha}\right)_{W_\alpha,\,sgn}$

$\mathcal{J}_{\alpha,n}$-equiv