Lecture 2, 1 Sept 2020

Def: (a) A pcdh-topology is generated by $\{X_\alpha \to X\}$ s.t. $\coprod X_\alpha \to X$ is proper + has constructible section.

(b) A cdh'-topology is generated pcdh + Zariski coverings.

Def: An abstract blow-up is a

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \uparrow & & \uparrow i \\ Z' & \longrightarrow & Z \end{array} \quad (*)$$

s.t. f - proper, i - closed emb s.t. $f: (X' - Z')_{red} \to (X - Z)_{red}$

Lemma: (a) In the case $(*)$

$$X' \coprod_{Z'} Z \to X$$

is equiv in pcdh top.

(b) pcdh top is gen by maps of (a).

Not: Par = standard parabolics.

$\forall J \in \mathrm{Par} \mapsto Fl_J := [LG/P_J]$ - sheafify in ét(or Zar) top.

Thm: $\operatorname{colim}_{J \in \mathrm{Par}} Fl_J \xrightarrow[\text{pcdh}]{\sim} pt$ equivalence.

pf: Step 1: Enough to assume $G = G^{sc}$,

i.e., $\operatorname{colim}_{Par_G} Fl_J \xrightarrow{\sim} \operatorname{colim}_{Par_{G^{sc}}} Fl_J$

Idea: $Fl_G = \coprod_{w \in \Lambda = N_{LG}(I)/I} Fl_{G^{sc}} \cdot w$

$$W^{aff} \subseteq \widetilde{W}$$

Step 2: let $\mathcal{Y} = \left\{ \begin{array}{l} \text{finite } \underbrace{\text{subsets}}_{\text{closed}} \ Y \subseteq \widetilde{W} \ \Big| \\ \text{i.e. if } w \in Y,\ u \leq w \Rightarrow u \in Y \end{array} \right\}$

$$= \left\{ \begin{array}{l} \text{proj } I\text{-inv subschemes} \\ Fl^Y \subseteq Fl \end{array} \right\}$$

$\forall J \in Par$, $Y \in \mathcal{Y}$ let $Fl^Y_J \subseteq Fl_J$ be the image of $Fl^Y \subseteq Fl \twoheadrightarrow Fl_J$

Claim: $\exists$ functor $Par \times \mathcal{Y} \to Sch_k \subseteq PreSt_k$

$$(J, Y) \mapsto Fl^Y_J$$

Not: $\mathcal{F}l^Y := \operatorname{colim}_J Fl^Y_J \in PreSt_k$.

Thm 1': $\forall Y \in \mathcal{Y}$, $\mathcal{F}l^Y \xrightarrow{\sim} pt$ in pcdh.

lemma: Thm 1' $\Rightarrow$ Thm 1

Ex:
$$\begin{tikzcd} pt & \\ X \arrow[u] \arrow[r, "\sim"', "\uparrow pcdh"] & Y \end{tikzcd} \rightsquigarrow pt \coprod_X Y \cong pt$$

Ex1: $Y = \{1\}$, in this case,

$\underline{Fl}^Y = \operatorname{colim}_{par} pt \cong pt$, par has initial object.

Ex2: $G = SL_2$, $Y = \{1, s_0\}$, $Fl^Y = \mathbb{P}^1$,
$W = \langle s_0, s_1 \rangle$
$Fl^Y_{s_0} = pt$, $Fl^Y_{s_1} = \mathbb{P}^1$

$\underline{Fl}^Y = pt \coprod_{\mathbb{P}^1} \mathbb{P}^1 \simeq pt$

pf of lemma 1:

$$\operatorname{colim}_J Fl_J = \operatorname{colim}_J \left(\operatorname{colim}_Y Fl^J_Y\right) =$$
$$= \operatorname{colim}_Y \left(\operatorname{colim}_J Fl^J_Y\right) =$$
$$= \operatorname{colim}_Y \left(\underline{Fl}^Y\right) \underset{\text{Thm 1'}}{\cong} \operatorname{colim}_Y pt$$

$? \downarrow\wr$ pt ; $\operatorname{colim}_Y pt \downarrow\wr\ pt$. ☐

Pf of Thm 1': Induction on $|Y|$,

$|Y| = 1$, $Y = \{1\}$, it was example 1.

Assume $|Y| > 1$, pick maximal element $w \in Y$, $Y' := Y - w \subseteq Y$ closed.

<u>Want to show</u> $\begin{tikzcd} Fl^{Y'} \arrow[r, "\sim", "\text{pcdh}"'] & Fl^{Y} \end{tikzcd}$, $Y' = Y - w$

$$\begin{tikzcd} Fl^{Y'} \arrow[r, "\sim", "\text{pcdh}"'] \arrow[d, equal] & Fl^{Y} \arrow[d, equal] \\ \operatorname{colim} Fl^{Y'}_J & \operatorname{colim} Fl^{Y}_J \end{tikzcd}$$

$Fl_J^{Y'} \subseteq Fl_J^{Y}$ closed

$$Fl_J^{Y} - Fl_J^{Y'} = \begin{cases} 1)\ Fl_J^{w} = IwP_J/P_J, & w \text{ is minimal in } wW_J \\ 2)\ \emptyset, & \text{otherwise.} \end{cases}$$

$$Fl_J^{Y} = Fl_J^{YW_J} = Fl^{YW_J}/P_J, \quad YW_J - Y'W_J = \begin{cases} wW_J \\ \emptyset \end{cases}$$

In case 1),
$$\begin{tikzcd} Fl^{Y} \arrow[r] & Fl^{Y}_J \\ Fl^{Y'} \arrow[r] \arrow[u, hook] & Fl^{Y'}_J \arrow[u, hook] \end{tikzcd}$$
is abstract blow-up.

$$\left(Fl^{Y} - Fl^{Y'} = Fl^{w} \xrightarrow{\sim} Fl^{w}_J = Fl^{Y}_J - Fl^{Y'}_J \right)$$

2) $Fl^{Y'}_J \xrightarrow{\sim} Fl^{Y}_J$.

This proves everything. (Exercise)

<u>Cor</u>: let $LG \circlearrowright X$ - prestack, then

$$\operatorname*{colim}_{J \in \mathrm{Par}} X/P_J \longrightarrow X/LG \text{ is cdh-equiv.}$$

<u>Pf</u>: $X/P_J = (X \times LG/P_J)/LG$

$\operatorname{colim}(LG/P_J) \xrightarrow{\sim} pt$ (Thm 1)

$\Downarrow$

$\operatorname{colim}(X \times LG/P) \xrightarrow{\sim} X$

$\Downarrow$

$\operatorname{colim}_{LG} \backslash (X \times LG/P) \xrightarrow{\sim}_{G} \backslash X$

$LG \backslash Z := \operatorname{colim}(LG^n \times Z)$

Cor 2: $Shv(X/LG) \simeq \lim_J Shv(X/P_J)$

pf: Cor 1 + ($X \to Shv(X)$ is a "sheaf" in cdh-topology. $\blacksquare$

Shv = ℓ-adic, D-modules

recall "categorical distribution"

- $Shv(LG) = Shv^!(LG) = \operatorname{Ind}(Shv^!(LG)_c)$
- $Shv^*(LG)_c$

① $\forall \mathcal{F}, \mathcal{G} \in Shv(LG)$, $\mathcal{F} \overset{!}{\otimes} \mathcal{G} \in Shv(LG)$

② $\forall_c \mathcal{A} \in Shv(LG)$, $\mathcal{B} \in Shv^*(LG)_c$

$\mathcal{A} \overset{*\text{-conv}}{*} \mathcal{B} \in Shv(LG)$

classical analog: $f \in C^\infty(G(F))$, $\mu \in \mathcal{M}_c^\infty(G(F))$

$f * \mu \in C^\infty(G(F))$

categorical analog:

$\mathcal{A} \in Pro(Shv^*(LG)_c)$, $\mathcal{B} \in Shv^!(LG)_c$

$\rightsquigarrow \mathcal{A} \overset{!\text{-ren}}{*} \mathcal{B} \in Pro(Shv^*(LG)_c) \underset{\text{Verdier}}{\rightsquigarrow}$

$m: L^+G \times L^+G \to L^+G$

$m_*(\mathcal{A} \boxtimes \mathcal{B})$

$\cap$

$Shv(L^+G \times L^+G)$

$L^+G \times L^+G$

$\downarrow$

$(K_n \backslash L^+G) \times L^+G \xrightarrow{m_*} K_n \backslash L^+G$

Def: $Shv_{dist}(LG) = \{ \mathcal{F} \in Shv(LG) \mid$

$(\mathcal{F} \overset{!}{\otimes} \omega_{LG^{\le 4}}) * 1_{I_n} \simeq (\mathcal{F} * 1_{I_n}) \overset{!}{\otimes} \omega_{LG^{\le 4}}$

$\in Shv(LG)_c$

$\iff \{ \mathcal{F} \mid (\mathcal{F} \overset{!}{\otimes} \mathcal{A}) * \mathcal{B} \in Shv(LG)_c$

$\forall \mathcal{A} \in Shv(LG)_c,\ \mathcal{B} \in Shv^*(LG)_c \}$

Let $\tilde{W}$ - gp.

$Shv^{\tilde{W}}_{dist}(LG) = \{ \mathcal{F} \in Shv^{\tilde{W}}(LG) : \forall \mathcal{A}, \mathcal{B}$

$(\mathcal{F} \overset{!}{\otimes} \mathcal{A}) * \mathcal{B} \in Shv^{\tilde{W}}(LG)_c$

<u>Recall</u> $p : \widetilde{LG} \to LG$ Affine Springer ind-proper fibration

$S := p_! \, \omega_{\widetilde{LG}} \in Shv(LG)$

<u>FACT</u>: If $f : X \to Y$ is ind-fp-proper then $\exists f_! \dashv f^!$ ■

<u>Lemma</u>: p ind-fp-proper

<u>pf</u>: $\widetilde{LG} \hookrightarrow \{(g,\gamma) \in Fl \times LG \mid g^{-1}\gamma g \in I\}$

$$\begin{array}{ccccc}
\widetilde{LG} & \hookrightarrow & Fl \times LG & \xrightarrow[\text{ind-proper}]{\pi} & LG \\
\downarrow & \square & \downarrow & & \\
\frac{I}{I} & \underset{\text{fp-closed}}{\hookrightarrow} & \frac{LG}{I} & &
\end{array}$$

■

<u>Recall</u>: S has a natural $\widetilde{W}$ upgrade to $S_{\widetilde{W}} \in Shv^{\widetilde{W}}(LG)$

<u>Thm 4</u>: $S_{\widetilde{W}} \in Shv^{\widetilde{W}}_{dist}(LG)$, i.e.,

$\forall n, 4$; $\left(S_{\widetilde{W}} \otimes^! \omega_{LG^{\le 4}}\right) * \delta_{I_n} \in Shv^{\widetilde{W}}(LG)^{c}$