Geometric Langlands seminar

Motivation: Let F - local n.a field

$[F:\mathbb{Q}_p]<\infty$ or $F=\mathbb{F}_q((t))$

Local Langlands conj studies

$\mathrm{Dist}(G(F))^{G(F)}$

Our approach "construct" them

(Bezrukavnikov, Kazhdan, ...)

using geometry. For simplicity

$F=\mathbb{F}_q((t))$, $k=\overline{\mathbb{F}_q}$

Want to study $\mathrm{Shv}^{\mathcal{L}G}(\mathcal{L}G)=\mathrm{Shv}\left(\frac{\mathcal{L}G}{\mathcal{L}G}\right)$

$\mathcal{L}G$ - loop gp of G $(A\mapsto G(k((t)))$

problem: $\mathcal{L}G$ is "huge", e.i,

ind-scheme = ind-pro-alg var.

Actual goal of my talks discuss

"finiteness" = "stabilization" issues.

(I) Let $\mathcal{L}G$ acts on prestack X

Main example: $\mathcal{L}G \curvearrowright \mathcal{L}G$ by adj: action.

Q: can one describe $Shv(\mathcal{L}G \backslash X) = Shv^{\mathcal{L}G}(X)$ in more explicit terms?

Cor1: $Shv^{\mathcal{L}G}(X) \simeq \lim_P Shv^P(X)$

P runs over standard parahorics $P \supseteq I$ $\{P\} \overset{1-1}{\longleftrightarrow} \{J \underset{\neq}{\subseteq} \tilde{\Delta}$ - simple affine roots of $G\}$

Ex: $Shv^{\mathcal{L}G}(\mathcal{L}G) = \underleftarrow{\lim}_P Shv(\mathcal{L}G/P)$

ind scheme / ~~alg sp~~ scheme

part case: $X = \mathcal{L}G$, $\mathcal{L}G$ acts by right multiplication, $\mathcal{L}G \backslash X = pt$

Cor2: $Shv(pt) \simeq \lim_P Shv(\mathcal{L}G/P)$

$\Uparrow$

Thm: The natural map

$\operatorname{colim}_P (\mathcal{L}G/P) \to pt$ is isom.

in ~~PreSt~~ St_k in cdh-topology.

Rem: If $G = G^{sc}$, then $P \in Par$ - partially

$\operatorname{colim}_{p \in par}$

$par = \{$partially ordered sets of $I \subsetneq P\}$

In general

$par = \begin{cases} Ob = \{J \subsetneq \tilde{\Delta}\} \\ Mor_{par}(J_1, J_2) = \{w \in \Omega \mid w(J_1) \subseteq J_2\} \end{cases}$

sim

$\tilde{W} = X_*(T) \rtimes W$ extended affine Weyl gp; $\Omega = \operatorname{Stab}_{\tilde{W}}(\tilde{\Delta})$

as a set

II — discrete version of I — needed for the rest.

<u>Thm 2</u>: $\operatorname{colim}_{p \in par} \tilde{W}/W_p \xrightarrow{\sim} pt$ (spaces)

<u>Cor 1</u>: $\operatorname{colim}_p B(W_p) \xrightarrow{\sim} B(\tilde{W})$

Pf: divide Thm 1 by $\tilde{W}$.

Def: $\forall$ ∞-category $\mathcal{C}$ discrete gp A,

$\mathcal{C}^A := \mathrm{Fun}(BA, \mathcal{C})$ - category of A-equiv objects in $\mathcal{C}$.

$\mathrm{Ob}\,\mathcal{C}^A = \{x \in \mathrm{Ob}\,\mathcal{C} + A \to \mathrm{Aut}_{\mathcal{C}}\, x\}$

Cor2: $\forall \mathcal{C}$ $\quad \mathcal{C}^{\widetilde{W}} \xrightarrow{\sim} \lim_p \mathcal{C}^{W_p}$

pf: LHS $\mathrm{Fun}(B\widetilde{W}, \mathcal{C})$

RHS $= \lim_p \mathrm{Fun}(BW_p, \mathcal{C}) =$

$= \mathrm{Fun}(\mathrm{colim}_p BW_p, \mathcal{C})$

Cor1 $\quad\square$

Cor3: $\exists$ equivalence

$\widetilde{W}\text{-mod} = \{\mathcal{C} \in \mathrm{Cat}_\infty + \text{action of } \widetilde{W}\}$

$\wr\!\wr$

$\lim_p \{W_p\text{-mod}\}$

Pf: Apply Cor 2 for Cat_∞

Cor 4: $\mathrm{Colim}^{\text{Mon Cat}}\, W_p \xrightarrow{\sim} \widetilde{W}$

pf: Want $\forall$ mon cat $\mathcal{M}$

$$\mathrm{Hom}_{\text{Mon Cat}}(\widetilde{W}, \mathcal{M}) \cong \lim \mathrm{Hom}_{\text{Mon Cat.}}(W_p, \mathcal{M})$$

Cor 3 $\Rightarrow$ the same for $\mathcal{M} = \mathrm{End}(\mathcal{C})$

☐

Disgression:

It might happen Thm 1 $\Rightarrow$

$$\begin{array}{c} \mathrm{Sh}(\mathbb{Z}G)\text{-mod} \\ \downarrow\wr \\ \lim_p \{\mathrm{Sh}(P)\text{-mod}\} \end{array}$$

III Grothendieck-Springer affine sheaf.

consider GS resolution

$$\mathcal{I}G \xleftarrow{p} \widetilde{\mathcal{I}G} = \left\{ (g,x) \in Fl \times \mathcal{I}G \;\middle|\; g^{-1}xg \in I \right\}$$

($Fl = \mathcal{I}G/I$)

let $\omega_{\widetilde{\mathcal{I}G}} \in Sh(\widetilde{\mathcal{I}G})$ - dualizing sheaf.

set $S = p_!(\omega_{\widetilde{\mathcal{I}G}}) \in Sh(\mathcal{I}G)$

Grothendieck-Springer sheaf.

Thm 3: S is equipped with action of $\widetilde{W}$.

"pf": By Cor 2 of Thm 2, it is enough to construct comp system of W_P-actions.

- W_p actions are induced from W-action on f.d GS-sheaf
- compatibility follow from GS-sheaves are perverse $\Rightarrow \mathrm{Ext}^{-n}(-,-)=0$, $n>0$.

$$\begin{tikzcd}
\underline{\Phi}/\underline{\Phi} \arrow[r] \arrow[d] & \underline{P}/P \arrow[dl] \\
\underline{LG}/LG &
\end{tikzcd}$$

<u>Cor 1</u>: $\forall\, \tau \in \mathrm{Rep}_{fd}(\widetilde{W})$, the

$$S_\tau := S \otimes_{\widetilde{W}} \tau \in \mathrm{Shv}(LG)$$

is "categorical distribution",

<u>e.g.</u>

e.g ~~fty~~ sheaf function corr
$S_{\mathcal{I}} \rightsquigarrow Tr(Fr_q, S_{\mathcal{I}}) \in \mathcal{D}ist(G(F))^{G(F)}$

what are sheaves?

$Shv = Shv^! : Prest_k \to DGCat$

- $\forall X$ (affine) scheme f.t/k
$Shv(X) = Ind\, D^b_c(X, \overline{\mathbb{Q}}_\ell)$
- $\forall X$ affine scheme write
$X = \varprojlim X_\alpha$, X_α - sch f.t/k
and set $Shv^!(X) = colim^{!-pb}\, Shv(X_\alpha)$
$\left(Shv^*(X) = colim^{*-pb}\, Shv(X_\alpha) \right)$

- $\forall \mathcal{X} \in Prest_k = Fun(Aff_k^{op}, Spaces)$
$$Shv(\mathcal{X}) = \lim_{\text{affine} \to X \xrightarrow{\pi} \mathcal{X}}{}^{!-pb}\, Shv(X)$$

Ex: Assume X = ind-scheme

$X = \mathrm{colim}\, X_i$, $\underbrace{X_1 \hookrightarrow X_2 \hookrightarrow \cdots}_{\text{fp-closed embeddings.}}$

Then $\mathrm{Shv}(X) = \mathrm{Ind}\big(\mathrm{Shv}(X)_{\mathrm{comp.}}\big)$

$\mathrm{Shv}(X)_{\mathrm{comp}} = \mathrm{colim}\big(\mathrm{Shv}(X_i)_{\mathrm{comp.}}\big)$

$X_i = \varprojlim_{\alpha} X_{i,\alpha}$

$\mathrm{Shv}(X_i)_{\mathrm{comp}} = \mathrm{colim}\, \mathrm{Shv}(X_{i,\alpha})_{\mathrm{comp.}}$

Ex: $X = LG$; $LG/I = Fl = \mathrm{colim}_{Y}\, Fl^{Y}$

$Fl^{Y} \subseteq Fl$ is I-inv proj. variety.

$\rightsquigarrow LG = \mathrm{colim}\, \underbrace{LG^{Y}}_{\text{schemes.}}$

$LG(\mathbb{F}_q) = \bigcup \underbrace{LG^{Y}(\mathbb{F}_q)}_{\text{compact subset.}}$

"Bern center"

consider "all measures" $\rightarrow$

$$\begin{array}{ccc} Shv(LG) & \supset & ② \\ \cap & & \cup \\ ① & \supset & Shv(LG)_{comp} \end{array}$$

② "= smooth measures with comp supp"

"smooth measures"

① $\mathcal{F} \in Shv(LG) \mid \forall\, \imath_{\mathcal{Y}} : \mathcal{I}G^{\mathcal{Y}} \hookrightarrow LG$

$\imath_{\mathcal{Y}!}\, \imath_{\mathcal{Y}}^{*}\, \mathcal{F} \overset{!}{\in} Shv(LG)_{comp}$

$\cong$

$\mathcal{F} \overset{!}{\otimes} \omega_{\mathcal{I}G^{\mathcal{Y}}}$

② $\forall n$ let $I_n \subseteq I$ n-th congruence subgp.

$\mathcal{F} \in Shv(LG)_{comp}$ $\quad \mathcal{F} * \omega_{I_n} \in Shv(LG)_{comp}$

③ "=" ① $\cup$ ② $\mathcal{F}$ $\forall\, \mathcal{I}G^{\mathcal{Y}}$ $\forall n$

$$\left(\mathcal{F} \overset{!}{\otimes} \omega_{\mathcal{I}G^{\mathcal{Y}}}\right) * \omega_{I_n} = \left(\mathcal{F} * \omega_{I_n}\right) \overset{!}{\otimes} \omega_{\mathcal{I}G^{\mathcal{Y}}} \in Shv(\ )_{comp}$$

categorical distr.

Rem: Recall

$$\mathrm{Shv}^!(LG) \xrightarrow[\sim]{\text{ID}} \mathrm{Fun}_{\mathrm{cont}}(\mathrm{Shv}^*(LG), \mathrm{Vect})$$

$$\cup \qquad\qquad \| $$

$$\mathrm{Fun}_{ex}(\mathrm{Shv}^*(LG)_{comp}, \mathrm{Vect})$$

$$\cup$$

categorical distributions $\to$ ③ $\overset{}{\underset{\text{Ex.}}{\hookrightarrow}}$ $\mathrm{Fun}_{ex}(\mathrm{Shv}^*(LG)_{comp}, \mathrm{Vect}_c)$

Thm 4: S is "$\widetilde{W}$-distribution", i.e.,

$\forall\, LG^y \subseteq LG$, $\forall n$

$$(S * \omega_{I_n}) \overset{!}{\otimes} \omega_{LG^y} \in \mathrm{Shv}^{\widetilde{W}}(LG)_{comp}$$

$$\uparrow$$

$$\mathrm{Shv}(LG^y/I_n, \mathbb{Q}_\ell[\widetilde{W}])_{comp}$$

$LG^y = \varprojlim LG^y/I_n \qquad \pi_n : LG^y \to LG^y/I_n$

$S * \omega_{I_n} = \pi_n^! \; \pi_{n*}^{ren}(S)$