The Spectral Hecke algebra

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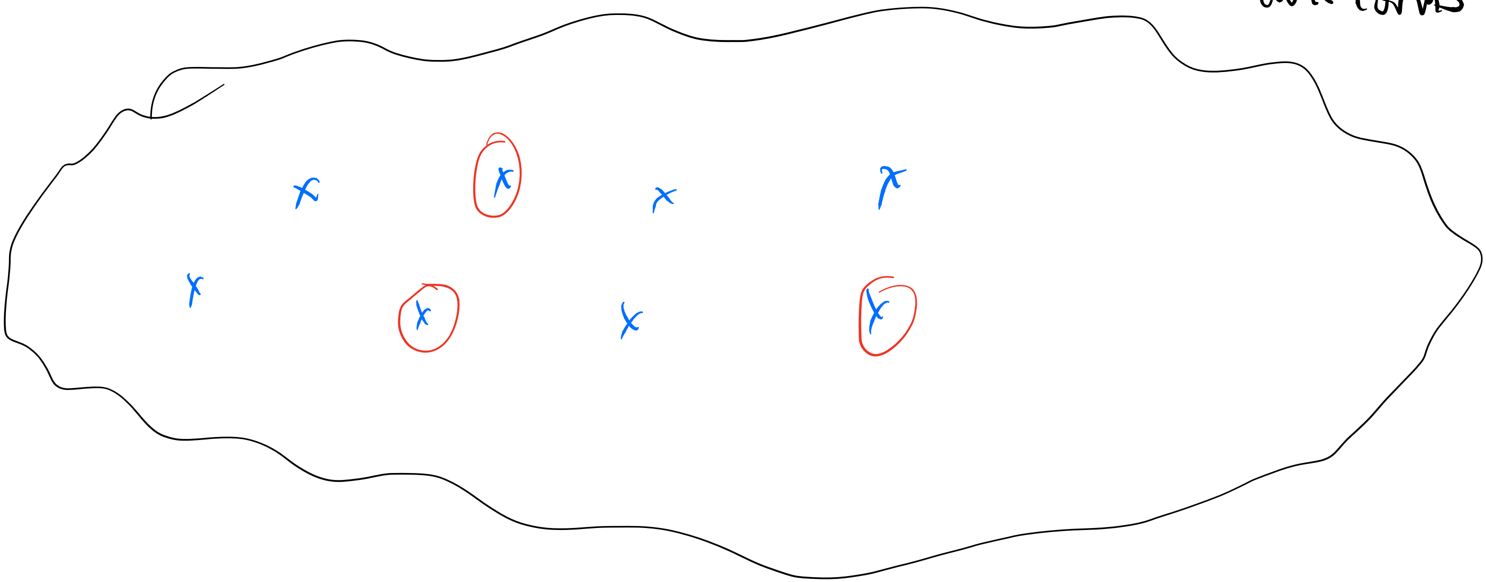
Various incarnations of Langlands duality

Number field	Function field	Geometric
$\mathbb{Q}$	$\mathbb{F}_q(t)$	X
Aut forms for G	Aut forms for G	D-modules on Bun_G
Galois reps to $\widehat{G}$	Galois reps to $\widehat{G}$	$\widehat{G}$-local systems

The landscape of automorphic forms/$\mathbb{Q}$



- alg. aut. forms $\longleftrightarrow$ Galois rep'ns

Maass

wt 1

- reg alg $\rightsquigarrow$ Hk eigensystems appear

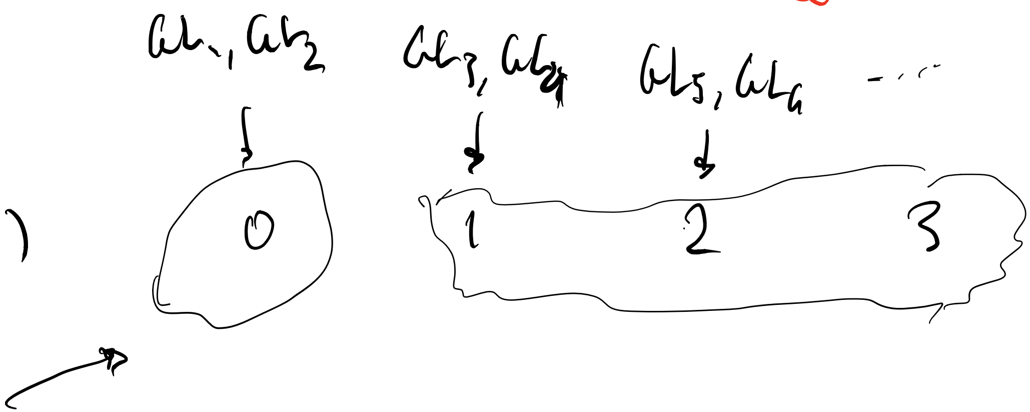
in H^*_{sing} (locally symm)

sing (space)

Hecke

integral structure

$\delta = \delta(G)$



locally symm space is algebraic

all of function field Langlands, geometric

looks like res. alg $\delta = 0$

Theme of this talk: there are certain objects which come up naturally in Geometric Langlands, but have a "marginal" role in $\delta = 0$ situations (e.g. irrelevant to irreducible Galois representations).

However, they play <u>important</u> and <u>surprising</u> roles when $\delta > 0$.

- Derived Hecke alg
- Spectral Hecke alg

What is δ?

- $-\delta =$ "expected dimension" of the moduli space of Galois representations into $\widehat{G}$ at irreducible points.

- $\delta' =$ range of degrees in which *tempered* (reg. alg.) automorphic forms for G appear in singular cohomology.

- So, δ is a measure of "how derived" the Langlands correspondence is for G.

$\delta = \delta'$ "numerical coincidence"

The derived Hecke algebra

Notation:

$\mathbb{Z}/\ell^n$

- Λ = ℓ-adic coefficient ring (e.g. $\mathbb{F}_\ell, \mathbb{Z}_\ell, \mathbb{Q}_\ell$).
- $\mathbb{Q}_q$ = local field with residue field $\mathbb{F}_q$, characteristic $\neq \ell$.

$\mathbb{F}_q((t))$

Usual (spherical) Hecke algebra:

$$H_q(G, \Lambda) := \mathrm{Hom}_{G(\mathbb{Q}_q)}(\mathrm{c} - \mathrm{Ind}_{G(\mathbb{Z}_q)}^{G(\mathbb{Q}_q)} \Lambda, \mathrm{c} - \mathrm{Ind}_{G(\mathbb{Z}_q)}^{G(\mathbb{Q}_q)} \Lambda).$$

Derived Hecke algebra:

$$\mathcal{H}_q(G, \Lambda) := \mathrm{RHom}_{G(\mathbb{Q}_q)}(\mathrm{c} - \mathrm{Ind}_{G(\mathbb{Z}_q)}^{G(\mathbb{Q}_q)} \Lambda, \mathrm{c} - \mathrm{Ind}_{G(\mathbb{Z}_q)}^{G(\mathbb{Q}_q)} \Lambda).$$

Let's examine $\mathcal{H}_q^\bullet(G,\Lambda) = \mathrm{Ext}^\bullet_{G(\mathbb{Q}_q)}(\mathrm{c}-\mathrm{Ind}_{G(\mathbb{Z}_q)}^{G(\mathbb{Q}_q)}\Lambda, \mathrm{c}-\mathrm{Ind}_{G(\mathbb{Z}_q)}^{G(\mathbb{Q}_q)}\Lambda)$.

$$\left\{ G(\mathbb{Z}_q)\backslash G(\mathbb{Q}_q)/G(\mathbb{Z}_q) \ni x \longmapsto \underline{H^\bullet_{\text{cts}}(\mathrm{Stab}_x, \Lambda)} \right\}$$

compact

$\mathrm{Stab}_x \subset G(\mathbb{Z}_q)$ profinite, mostly pro-q

- $\Lambda = \mathbb{Q}_\ell$ $\quad \mathcal{H}_q(G,\Lambda) = H_q(G,\Lambda)$
- $\Lambda = \mathbb{Z}_\ell$ $\quad$ derived part "small"
- $\Lambda = \mathbb{F}_\ell$ or $\mathbb{Z}/\ell^n$

Analogy

$H_q(G) \longleftrightarrow \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \Lambda)$

$\mathcal{H}_q(G) \longleftrightarrow D_{L^+G}(\mathrm{Gr}_G, \Lambda)$

Tr (Frob)

Example: derived Hecke algebra of a split torus T.

$$\mathrm{Stab}_x = T(\mathbb{Z}_q)$$

only nontriv if $q \equiv 1 \bmod \ell$.

&

$$H^*(\mathrm{Stab}_x) = H^*(T(\mathbb{F}_q)_{(\ell)}, \Lambda)$$

$$\mathcal{H}_q(T, \Lambda) = H_q(T, \Lambda) \otimes H^*(T(\mathbb{F}_q)_{(\ell)}, \Lambda)$$

The basic structure of derived Hecke algebras remains mysterious.

Q Is $\mathcal{H}_q^{\bullet}(G, \Lambda)$ commutative?

Thm (Venkatesh) $q = 1 \in \Lambda$, $\ell > |W|$

$$\mathcal{H}_q^{\bullet}(G, \Lambda) \xrightarrow{\sim} \mathcal{H}_q^{\bullet}(T, \Lambda)^W$$

given by restriction.

Sketch Smith theory w.r.t. $\mathbb{Z}/\ell \hookrightarrow \mathbb{F}_q^\times$

$$\xrightarrow{2\ell} G(\mathbb{F}_q)$$

Cohomology of locally symmetric spaces

"The derived Hecke algebra acts on the derived $G(\mathbb{Z}_q)$-invariants of any $G(\mathbb{Q}_q)$-representation".

$\mathcal{H}_q(G,\Lambda) := \mathrm{RHom}_{G(\mathbb{Q}_q)}(\mathrm{c}-\mathrm{Ind}_{G(\mathbb{Z}_q)}^{G(\mathbb{Q}_q)}\Lambda, \mathrm{c}-\mathrm{Ind}_{G(\mathbb{Z}_q)}^{G(\mathbb{Q}_q)}\Lambda)$ acts on

$\mathrm{RHom}_{G(\mathbb{Q}_q)}(\mathrm{c}-\mathrm{Ind}_{G(\mathbb{Z}_q)}^{G(\mathbb{Q}_q)}\Lambda, ???) = \mathrm{RHom}_{G(\mathbb{Z}_q)}(\Lambda, \sim)$

away from q

$$Y_G = G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/K^q\, G(\mathbb{Z}_q) \sim \mathrm{Bun}_G$$

$$Y_G \uparrow Y_{G,\infty} \qquad G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/K^q\, G(\mathbb{Z}_q) \xleftarrow{G(\mathbb{Z}_q)} G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/K^q$$

$$Y_{G,\infty} = G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/K^q \circlearrowleft G(\mathbb{Q}_q) \sim \mathrm{Bun}_G^{\boxtimes q}$$

$$\Rightarrow \quad C^*(Y_a) = C^*(Y_{a,\infty})^{hG(\mathbb{Z}_q)}$$

$$\circlearrowright$$

$$\mathcal{H}_q(a, s)$$

adding unipotent

add Iwahori

radical of Iwah.

Example.

level

$$\begin{tikzcd} & Y_0(q) \arrow[dl] \arrow[dr] & \\ Y & & Y \end{tikzcd}
\qquad
\begin{tikzcd} Y_1(q) \arrow[d, "T(\mathbb{F}_q)"'] & \\ Y_0(q) \arrow[r] & BT(\mathbb{F}_q) \end{tikzcd}$$

$\alpha \in H^1(BT(\mathbb{F}_q))$

$$\begin{tikzcd} & H^*(Y_0(q)) \arrow[r, "\cdot\alpha"] & H^{*+1}(Y_0(q)) \arrow[dr] & \\ H^*(Y) \arrow[ur] \arrow[rrr] & & & H^{*+1}(Y) \end{tikzcd}$$

example of derived Hecke operator

$G =$ semisimple, $\quad \delta(G) = \mathrm{rk}\, G(\mathbb{R}) - \mathrm{rk}\, K_\infty$

e.g. $\quad \delta(SL_n) = \lfloor \frac{n-1}{2} \rfloor$ $\quad$ e.g. $\delta = 0$ $\quad Y_G$ "Shimura"

$$H^*(Y_G)_{\mathfrak{m}}$$ localized at tempered eigensystem

Thm (Borel-Wallach) $H^*(Y_G;\mathbb{Q})_{\mathfrak{m}}$ supp $* \in [q_0, q_0+\delta]$

$$\mathrm{rk}\; H^{q_0+j}(Y_G;\mathbb{Q})_{\mathfrak{m}} = (\mathrm{mult}) \binom{\delta}{j}$$

Dream: derived Hecke operators commute with each other, and generate all of $\mathfrak{m}$-isotypic cohomology from the bottom degree.

Obviously false!

Enemy: no interesting degree shifting operators.

conditional on TW assumptions

*

Venkatesh proves: subspace of $\mathrm{End}(H^*(Y_K; \mathbb{Z}_\ell)_\mathfrak{m})$ which are approximated to arbitrary ℓ-adic precision by derived Hecke operators, generates over the bottom degree.

$\mathbb{Z}_\ell \to \mathbb{Z}/\ell^n$

$q \equiv 1 \in \mathbb{Z}/\ell^n$

Derived Galois deformation rings

Philosophy: (In the most favorable situations) cohomology of locally symmetric spaces should be "free of rank 1" over the correct deformation ring.

Hence, when $\delta > 0$, the correct deformation ring needs to be derived.

aut forms

- Loc sys

(char 0)

Geometric LocSys classical + smooth of

irred. local systems!

dim = 0

Function Fields Def^{ρ} classical smooth of

irred local systems

dim = 0

Number fields

Def^{ρ} derived LCI of

irred local systems

expected dim = $-\delta$ }

dim = 0

The "extra obstructedness" for Galois representations over number fields could be thought of as coming from Hodge-theoretic restrictions on ~~families~~ of motives, e.g. "p-adic Hodge theory".

ρ = rep'n of $\pi_1[R]$ over $\mathbb{F}_\ell$

$$\mathrm{Def}_R^{\rho} = \{\text{deformations to Artin}/\mathbb{F}_\ell\} \quad /\mathbb{Z}_\ell$$

$S \supseteq \{\ell, \text{ramified primes}\}$

$$\begin{tikzcd}
\mathrm{Def}^{\mathrm{cryst}}_{\mathbb{Z}[1/S]} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Def}^{\rho}_{\mathbb{Z}[1/S]} \arrow[d] \\
\mathrm{Def}^{\mathrm{crys}}_{\mathbb{Q}_\ell} \arrow[r, hook] & \mathrm{Def}^{\rho}_{\mathbb{Q}_\ell}
\end{tikzcd}$$

Derived Gal def'n space

(Derived) Galois deformation rings are understood via the Taylor-Wiles method.

Idea: geometry improved by "passing to infinite level", then descend.

- At "infinite level", global singularities are smoothed out.
- At "infinite level", derived rings are discrete.

<u>Prop</u> [Galatius–Venkatesh] Pick $q \notin S$

$$\mathrm{Def}^{crys}_{\mathbb{Z}[1/S]} \xrightarrow{\sim} \mathrm{Def}^{crys}_{\mathbb{Z}[1/SQ_n]} \times^{h} \mathrm{Def}_{\mathbb{Z}_{Q_n}} \quad \mathrm{Def}_{\mathbb{Q}_q}$$

$$Q_n = \{q_1, \dots, q_r\} \qquad q_i \equiv 1 \mod \ell^n$$

" $n \to \infty$ "

Tw

$\mathrm{Def}_{\mathbb{Q}_n} = \mathrm{Def}_{\mathbb{Q}_{\ell_1}} \times \mathrm{Def}_{\mathbb{Q}_{\ell_2}} \times \cdots$

$$\mathrm{Def}^{\mathrm{crys}}_{\mathbb{Z}[1/S]} \simeq_S \text{“}\mathrm{Def}^{\mathrm{crys}}_{\mathbb{Z}[1/S\,\mathbb{Q}_\infty]}\text{”} \times^{h} \text{“}\mathrm{Def}_{\mathbb{Z}\mathbb{Q}_\infty}\text{”}$$

$$\text{“}\mathrm{Def}_{\mathbb{Q}_{\mathbb{Q}_\infty}}\text{”}$$

$$\mathbb{Z}_\ell[[x_1, \dots, x_{57-\delta}]] \qquad \mathbb{Z}_\ell[[x_1, \dots, x_{57}]] \qquad \mathbb{Z}_\ell$$

δ = Euler char of $T(\mathrm{Def}^{\mathrm{crys}}_{\mathbb{Z}[1/S]})$

$$= H^*(\mathbb{Z}[1/S], \mathrm{Ad}\,\rho)$$

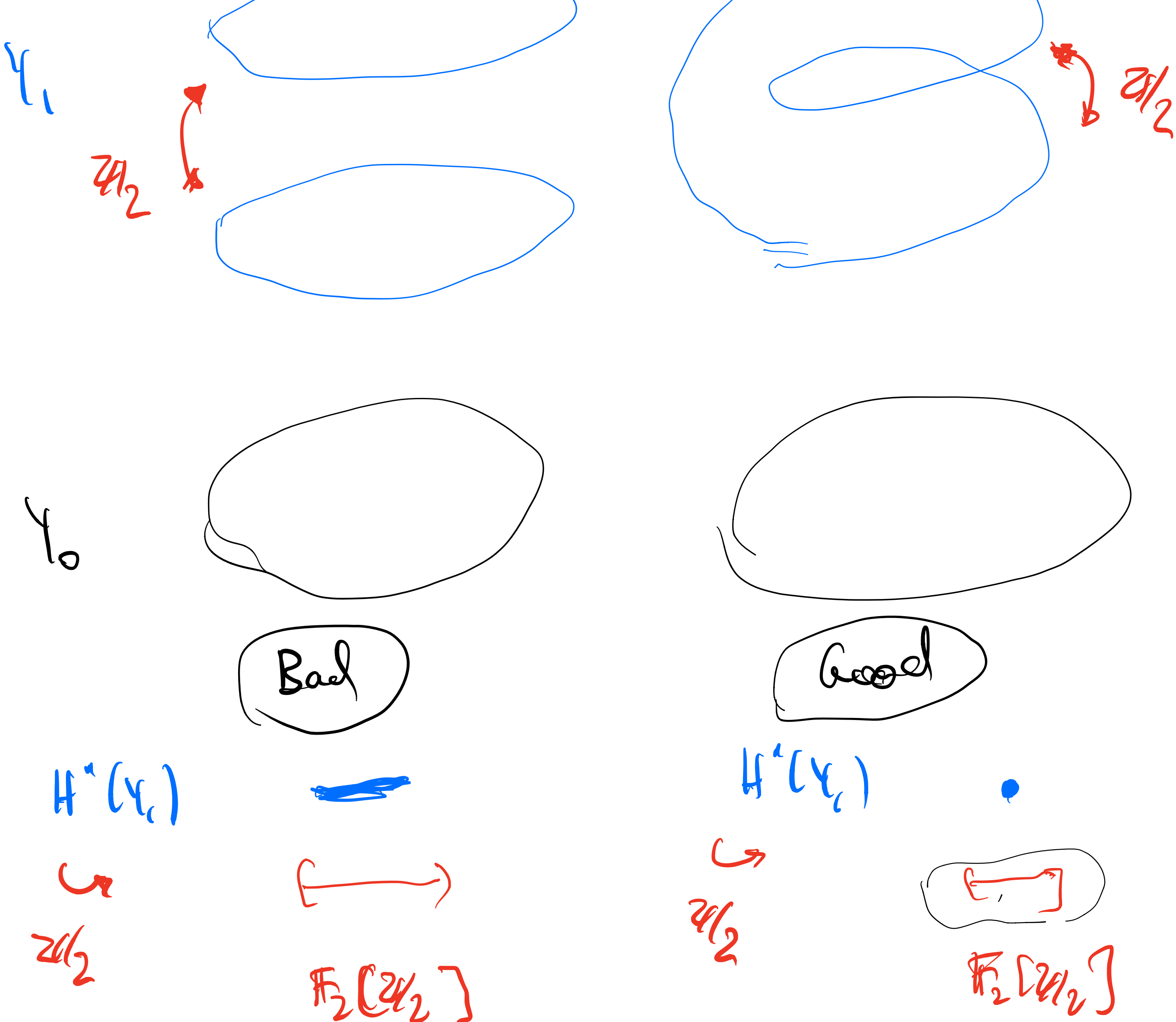
$\Rightarrow$ $\boxed{\text{derived Hecke action is "big"}}$

Toy model

$$H^*(Y)_m \qquad \begin{matrix} 1 \\ 3 \\ 3 \\ \underline{1} \end{matrix}$$

$\parallel$

Y_b



Y_1
$\mathbb{Z}/2$
$\mathbb{Z}/2$
Y_0
Bad
Good
$H^*(Y_1)$
$H^*(Y_1)$
$\mathbb{Z}/2$
$\mathbb{Z}/2$
$\mathbb{F}_2[\mathbb{Z}/2]$
$\mathbb{F}_2[\mathbb{Z}/2]$

$$\left.\begin{array}{cc} H^*(Y_0) & H^*(Y_0) \\ \hookrightarrow & \hookrightarrow \\ H^*(B\mathbb{Z}/2) & H^*(B\mathbb{Z}/2) \\ \cong & \\ \mathbb{F}_2[x_1] & \end{array}\right\} \text{DGA}$$

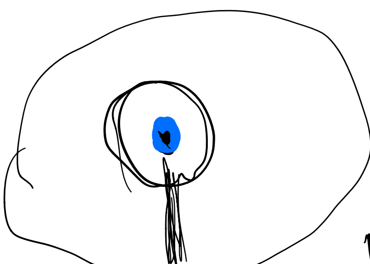
"attach Gal rep to art." $\longrightarrow$ $\mathrm{Def}^{crys}_{\mathbb{Z}[1/S]}$

structure of derived deformation ring

$H^*(Y_\infty)$ over $\mathbb{F}_2[\varprojlim \mathbb{Z}/2^n] = \mathbb{F}_2[[t]]$

$$\delta' \le \operatorname{codim} H^*(Y_\infty) \le \operatorname{codim} \mathrm{Def}^{crys}_{\mathbb{Z}[1/S\mathbb{Q}_\infty]}$$

$$\le \text{expected codim} \le \delta$$



$\mathrm{Def}\ \mathbb{Q}_{\infty}$

The spectral Hecke algebra

We want "reciprocity laws" for derived Hecke operators.

Reciprocity for classical Hecke operators: Hecke eigenvalues $\sim$ Frobenius eigenvalues.

What is the *spectral counterpart* to derived Hecke operators?

The analogous question and answer exist in Geometric Langlands.

Motivation from geometric Langlands

$$D(Bun_G) \simeq Coh(LocSys_{\hat{G}})$$

$$\circlearrowright^{*} \qquad\qquad\qquad \circlearrowright^{*}$$

$$D(Hk^{aut}) \xrightarrow{\sim} \boxed{Coh(Hk^{spec})}$$

$$\wr \qquad\qquad \text{Bez-Finkel}$$

$$D_{L^+G}(Gr_G)$$

The automorphic Hecke stack classifies:

> *"Two G-bundles on a disk, plus an isomorphism of their restrictions to the punctured disk".*

The spectral Hecke stack classifies:

> *"Two $\widehat{G}$-local systems on a disk, plus an isomorphism of their restrictions to the punctured disk".*

Arithmetic $\bar{\rho}$ mod ℓ.

"Two $\pi_1(\mathbb{Z}_q)$-rep'ns, + ~~isom~~ "

as $\pi_1(\mathbb{Q}_q)$-rep'ns

repl'd by Spectral Hecke alg

$$\begin{tikzcd} \mathrm{Hk}_q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Def}_{\mathbb{Z}_q} \arrow[d] \\ \mathrm{Def}_{\mathbb{Z}_q} \arrow[r, hook] & \mathrm{Def}_{\mathbb{Q}_q} \end{tikzcd}$$

Structure of the spectral Hecke algebra

$$\mathrm{Def}^{\bar\rho}_{\mathbb{Z}_q} \hookrightarrow \mathrm{Def}^{\bar\rho}_{\mathbb{Q}_q}$$

What does this look like?

- all deformations tame

  Inertia $\to \ker\left(\hat{G}(\mathbb{Z}_\ell) \to \hat{G}(\mathbb{F}_\ell)\right)$

- $\bar\rho(\mathrm{Frob}_q)$ (strongly) regular $\Rightarrow$ all deformations abelian

[$q \equiv 1 \bmod \ell$, $\bar\rho(\mathrm{Frob}_q)$ dist e.g.]

$$\mathrm{Def}^{\bar{T}} \longrightarrow \mathrm{Def}^{\bar{G}}$$

$$\mathrm{Hom}\left(\underbrace{(\text{tame Gal})^{ab}}_{\mathbb{F}_q^\times \times \hat{\mathbb{Z}}} , \bar{T} \right)$$

"derived part" is $\Lambda \overset{\mathbb{L}}{\otimes}_{\Lambda[T(\mathbb{F}_q)]} \Lambda = H_*(T(\mathbb{F}_q), \Lambda)$

matches "derived Hecke alg" $= \left((\quad) \otimes H^*(T(\mathbb{F}_q)) \right)^W$

completed at max'l ideal

(Co)-action on the derived Galois deformation ring

$Hk_q^{\bar{\rho}}$ groupoid / $\mathrm{Def}_{\mathbb{Z}_q}^{\bar{\rho}}$

$$\mathrm{Def}_{\mathbb{Z}_q}^{\bar{\rho}} \simeq \mathrm{Def}_{\mathbb{Z}_q}^{\bar{\rho}} \times^h_{\mathrm{Def}_{\mathbb{Q}_q}^{\bar{\rho}}} [-] \quad \text{''}$$

$Hk_q^{\bar{\rho}}$ acts on

$$\circlearrowleft$$

$$\mathrm{Def}^{crys}_{\mathbb{Z}[1/S]} = \mathrm{Def}^{crys}_{\mathbb{Z}[1/S_q]} \times^h_{\mathrm{Def}_{\mathbb{Q}_q}} \mathrm{Def}_{\mathbb{Z}_q}$$

Venkatesh's reciprocity law

Conj: $\mathrm{Ad}^* \rho_{\mathfrak{m}}$ exists as a motive

$[H^1(\mathbb{Q}, \mathrm{Ad}^* \rho(1))^\vee]$

Venkatesh conjectures that

HLTT, Scholze

$$H^*(Y_G; \mathbb{Q})_{\mathfrak{m}} \textit{ is free over } \wedge^\bullet H^1_{\mathcal{M}}(\mathbb{Q}, \mathrm{Ad}^* \rho_{\mathfrak{m}}(1))^\vee.$$

$\mathfrak{m} \rightsquigarrow \rho_{\mathfrak{m}}$

ℓ

$\mathbb{Z}_\ell$ $\mathbb{F}_\ell$

$H^*_{ct}(U(1,n),)$

Classical reciprocity law: Hecke operators T_q, a priori indexed by q, are actually parametrized by *global* data (e.g. image of Frob_q in $\widehat{G}(\mathbb{F}_\ell)$).

$q \equiv 1 \mod \ell^n$

Derived reciprocity law: derived Hecke operators, a priori indexed by $H^1(T(\mathbb{F}_q); \mathbb{F}_\ell)$, are actually parametrized by a *global* Galois cohomology group $H^1(\mathbb{Q}, \mathrm{Ad}^* \rho_{\mathfrak{m}}(1))^\vee$.

$(H_T \otimes H^*(T/\mathbb{F}_\ell))^W$

$H^1(T(\mathbb{F}_p))$ $H^1(T(\mathbb{F}_\ell))$

~~We will sketch a fictional proof of the reciprocity law.~~

DHA introduced

Thm action is big.

"aut. forms"

SHA $\circlearrowright$ $H^*(Y_a)$

Derived Galois deformation

Thm Understand structure of derived Gal. deform.

"R = T"

SHA at "good" q.

DHA

$$
\begin{tikzcd}
H^1(Y)_m \arrow[d, phantom, "\wr\wr"] & \\
H^1(Y_0(q))_{\tilde m} \arrow[r] \arrow[dd] & H^1(Y_1(q)) \arrow[dd] \\
\color{red}{\mathrm{Def}_{\mathbb{Z}[1/S]}} & \color{red}{\mathrm{Def}_{\mathbb{Z}[1/S_q]}} \\
\mathrm{Def}_{\mathbb{Z}_q} \arrow[r] & \mathrm{Def}_{\mathbb{Q}_q}
\end{tikzcd}
$$

SHA

$k \otimes k \qquad k \otimes k \subset H^1(Y_0(q))$

SHA $= k[\Delta] \longrightarrow$ Diamond operators

A

$H^*(Y_{\mathfrak{p}}(\mathfrak{q}))$

Conj (2hw)

$\mathcal{H}_q(G, \Delta) \simeq R\mathrm{Hom}\left(\mathcal{O}_{Loc^{un}}, \mathcal{O}_{Loc^{un}}\right)^{op}$

$Loc_q^{\hat{G}}$

Tr(Frob)

Tr(Frob)

$D_{L\text{-}tak}(G_A, \Delta)^{ren} \simeq (\quad\quad)$

Arinkin–Bez

Derived Satake isomorphisms

(Joint work in progress with Dennis Gaitsgory).

Categorical trace of Frobenius

Application to character varieties