Non-vanishing IV

Goal for today: Wrap everything up.

First: Want to finish discussion from last time.

Thm: $\mathcal{F} \in Shv_{N:\psi\text{-irreg}}(Bun_G) \Rightarrow \mathcal{F}$ is anti-tempered.

Reduced to:

Thm: $G \curvearrowright \underbrace{\mathcal{Y}}_{\text{alg. stack}}$, $\mathcal{F} \in \underbrace{Shv_{G\text{-irreg}}}_{\text{means: } \mu(SS(\mathcal{F})) \subseteq \mathfrak{g}^*_{\text{irreg}}}(\mathcal{Y})^B$

$\Rightarrow Av_!^{N,\psi}(\mathcal{F}) \in D(\mathcal{Y})^{N,\psi}$ is zero.

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4) <u>Goal</u>: Reduce to case where $Y = G$ ($Y_0 = pt$).

<u>Thm</u> (Kashiwara - Schapira):

Given smooth schemes$/\mathbb{C}$ Y & Z, $z \in Z$

$\Lambda \subseteq T^*Y$, $\mathcal{F} \in Shv_{\Lambda \times T^*Z}(Y \times Z)^c$,
closed & conical

the sheaf $(id \times i_z)^!(\mathcal{F}) \in Shv(Y)$

has $SS \subseteq \Lambda$.

Application of this thm:

$y \in Y_0$ a k-pt. $\Lambda = \mu^{-1}(\mathfrak{g}_{irreg})$.

$$\begin{tikzcd}
Shv(G \times_B Y_0) \arrow[r] \arrow[d, "(id \times i_y)^!"] & D(G \times Y_0)^{N^-, \psi} \arrow[d, "id \times i_y^!"] \\
Shv(G/B) \arrow[r] & D(G)^{N^-, \psi}
\end{tikzcd}$$

preserves G-irregularity

$\Rightarrow$ reduction.

5) $\mathcal{Y} = G$

$\mathcal{F} \in \mathrm{Shv}(G/B)$ w/ irreg. SS

$\mathcal{F}$ perverse & simple.

Then I have:

$$\begin{tikzcd}
\mathrm{Shv}(G/B) \arrow[d, hook, "\Gamma(G/B,\, RH^{-1}(-))"', "\text{fully-faithful, exact by Beilinson-Bernstein}"] \arrow[rd, "\mathrm{Av}_!^{\psi}"] & \\
\mathfrak{g}\text{-mod}_0 \arrow[r, "\mathrm{Loc}^{\psi}"] & D(G)^{N^-,\psi}
\end{tikzcd}$$

Std argument (Borho-Brylinski):

$\mathcal{F} \in D(G/B)^c$

$$\begin{array}{ccc}
SS(\mathcal{F}) & \& & SS(\Gamma(\mathcal{F})) \subseteq \mathrm{Spec}(\mathrm{gr}(U(\mathfrak{g})_0)) \\
\cap & & = \mathcal{N} \\
T^*G/B & & \\
\| & & \\
\tilde{\mathcal{N}} & & \text{are related by the}
\end{array}$$

formula:

$$SS(\Gamma(\mathcal{F})) = \pi(SS(\mathcal{F}))$$

$$\pi : \tilde{\mathcal{N}} \longrightarrow \mathcal{N}.$$

$$\Rightarrow \underbrace{\Gamma(\mathcal{F})}_{\mathcal{M}} \in \mathfrak{g}\text{-mod}_0^{\mathcal{S}?}$$

simple module w/ irreg. SS.

What we did last time:

Show $Loc^{\psi}(\mathcal{M}) = 0$ in such a situation. //

Pf of Thm 2 $\Rightarrow$ Thm 1.

Back to main thms:

<u>Goal</u>: $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ w/ $\mathrm{coeff}^{\mathrm{enh}}(\mathcal{F}) \in$

$\mathrm{QCoh}(LS_{\check{G}})$

being zero,

then WTS: $\mathcal{F}$ is anti-tempered.

By Thm 1:

suffices to show $\mathcal{F}$ has irregular singular support.

Suppose otherwise.

Then $\exists$ some cpt of $\mathcal{N}ilp^{reg}$

lying in $SS(\mathcal{F})$.

For simplicity, suppose G is simply-connected.

Then Talk 2: $\mathcal{N}ilp^{reg} = \coprod_{\check{\lambda} \in \check{\Lambda}^+} \mathcal{N}ilp^{reg, \check{\lambda}}$.

$\check{\lambda} \leftarrow$ discrepancy of a generically regular nilpotent elt.

$\check{\lambda} = 0$ cpt is $\mathcal{N}ilp^{Kos}$.

Take $\check{\lambda}$ so $\mathrm{Nilp}^{\mathrm{reg},\check{\lambda}} \subseteq SS(\mathcal{F})$.

<u>Show</u>; $x \in X$,

$$SS\left(H_x^{\check{\lambda}}(\mathcal{F})\right) \supseteq \mathrm{Nilp}^{\mathrm{Kos}}.$$

~~coeff~~ Can recover $\mathrm{coeff}_{\check{\lambda}\cdot x}(\mathcal{F})$ from

$\mathrm{coeff}^{\mathrm{enh}}(\mathcal{F})$. If $\mathrm{coeff}^{\mathrm{enh}}(\mathcal{F}) = 0$

$\Rightarrow \mathrm{coeff}_{\check{\lambda}\cdot x}(\mathcal{F}) = 0$.

$\Rightarrow \mathrm{coeff}\left(H_x^{\check{\lambda}}(\mathcal{F})\right) = 0$.

However, we showed:

$$\mathrm{coeff} : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \longrightarrow \mathrm{Vect}$$

is t-exact (up to shift), and

for $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\heartsuit,c}$ (maybe replace Bun_G by $U \subseteq \mathrm{Bun}_G$ qcpt open)

$$\chi(\mathrm{coeff}(\mathcal{F})) = \mathrm{ord}_{\mathrm{Nilp}^{\mathrm{Kos}}}\, CC(\mathcal{F}).$$

$\Rightarrow$

$\mathrm{Ker}(\mathrm{coeff}: \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \longrightarrow \mathrm{Vect})$

is exactly $\mathrm{Shv}_{\underbrace{\mathrm{Nilp}^{\neq \mathrm{Kos}}}}(\mathrm{Bun}_G)$.

union of all cpts besides $\mathrm{Nilp}^{\mathrm{Kos}}$.

Case of gen'l D-modules:

Follows from $\mathrm{Shv}_{\mathrm{Nilp}}$ case & :

<u>Thm</u> (AGKRRV): For any non-zero $\mathcal{F} \in D(\mathrm{Bun}_G)$, $\exists\, k'/k$ s.t. ~~the~~ the ~~object~~ image of $\mathcal{F}$ under

$$D(\mathrm{Bun}_G) \longrightarrow D(\mathrm{Bun}_{G,k'}) \xrightarrow[\text{r.adj. to embedding}]{} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G,k'})$$

is non-zero.

~~[illegible]~~ Suitably understood, this r.adj. is computed by a Hecke functor.