Non-vanishing of Whittaker coefficients

Joint w/ Joakim Faergeman.
Setting: D-modules.

Thm (Arinkin, Braverman-Gaitsgory, Frenkel-Gaitsgory, Beilinson-Drinfeld):

For every irreducible $\check{G}$-local system σ on a smooth projective curve X (over k, $\mathrm{char}(k)=0$), $\exists$ a non-zero cuspidal Hecke eigensheaf

$\mathcal{F}_\sigma \in D(\mathrm{Bun}_G)$.

Thm (Faergeman-R.) $\mathcal{F}_\sigma$ can be taken to be perverse & irreducible on every connected component of Bun_G.

Thm, Beraldo

Thm: Cuspidal (more generally: tempered) D-modules on Bun_G have non-zero Whittaker coefficients.

Background on modular forms:

Setting: holomorphic modular forms of level 1 and fixed weight.

Reminder: These are fns $f: \mathbb{H} \to \mathbb{C}$
$\{\mathrm{im}(z) > 0\}$

that are holomorphic, satisfy $f(\tau+1) = f(\tau)$, some other stuff, $q := e^{2\pi i \tau}$

then $f(q)$, which is a holomorphic fn on $\{q \mid 0<|q|<1\}$, extends to a holomorphic fn on $\{|q|<1\}$.

This implies $f(q) = \sum_{n=0}^{\infty} a_n q^n$.

Language: a_0 is called the constant term of f, and a_n $n \geq 1$ are called the Whittaker coefficients, and f is cuspidal if its constant term vanishes $(a_0 = 0)$.

Trivial observation: Any non-constant modular form has non-vanishing Whittaker coefficients.

Generalizations:
- Automorphic forms for PGL_2.
- For GL_2
- For GL_n.

The same story holds: cuspidal aut. forms have non-zero Whittaker coefficients. Argument is a generalization of the above one ("mirabolic curse").

Emphatically false for other groups.

Geometric setting:

1) Whittaker coefficients

We'll start with an analogue of $\underbrace{f}_{\text{modular}} \longmapsto a_1(f)$.

Here: $\mathrm{Coeff}: D(\mathrm{Bun}_G) \longrightarrow \mathrm{Vect}$.

Input: Bun_N^{Ω} ← twisted form of Bun_N
$R\Gamma(X,\Omega)[1]$ for $G=PGL_2$

Normalized to have a canonical character $\psi: \mathrm{Bun}_N^{\Omega} \longrightarrow \mathbb{G}_a$

de Rham cohomology

Def: $\mathrm{coeff}(\mathcal{F}) := C_{dR}(\mathrm{Bun}_N^{\Omega}, \mathfrak{p}^! \mathcal{F} \overset{!}{\otimes} \psi^!(\exp))$
$[\dim \mathrm{Bun}_G - \dim \mathrm{Bun}_N^{\Omega}]$.

where $\mathfrak{p}: \mathrm{Bun}_N^{\Omega} \to \mathrm{Bun}_G$ is the structure map, $\exp \in D(\mathbb{G}_a)$ is the exponential / Artin-Schreier sheaf.

What about other Whittaker coefficients?

"generalized vanishing conjecture"

Thm (Gaitsgory): $\exists$ a canonical action of $\mathrm{QCoh}(LS_{\check{G}})$ on $D(\mathrm{Bun}_G)$ that refines the Hecke action.

<u>General construction</u>:

Given $\mathcal{A} \curvearrowright \mathcal{M}$ and $\lambda : \mathcal{M} \to \mathrm{Vect}$

($\mathcal{A}$ is dualizable, say)

Get $\lambda^{\mathrm{enh}} : \mathcal{M} \longrightarrow \mathcal{A}^\vee = \mathrm{Hom}(\mathcal{A}, \mathrm{Vect})$.

$\mathcal{F} \longmapsto (a \in \mathcal{A} \longmapsto \lambda(a * \mathcal{F}))$.

<u>NB</u>: $\mathrm{QCoh}(LS_{\check{G}})$ is self-dual b/c

$LS_{\check{G}}$ is a (qcpt) alg. stack.

$\Rightarrow$ $\exists$ a $\mathrm{QCoh}(LS_{\check{G}})$-linear functor

$\mathrm{coeff}^{\mathrm{enh}} : D(\mathrm{Bun}_G) \longrightarrow \mathrm{QCoh}(LS_{\check{G}})$

s.t. $\Gamma(LS_{\check{G}}, \mathrm{coeff}^{\mathrm{enh}}(\mathcal{F})) \simeq \mathrm{coeff}(\mathcal{F})$.

GL conj: $\mathrm{coeff}^{\mathrm{enh}}$ is "almost" an equivalence.

Precisely (after Arinkin-Gaitsgory):

$\mathrm{coeff}^{\mathrm{enh}}$ is fully-faithful on compact objects of $D(\mathrm{Bun}_G)$ and induces an equivalence

$$\begin{array}{ccc} & & \mathrm{Perf} \\ & & \cap \\ D(\mathrm{Bun}_G)^{\mathrm{cpt\ objs}} & \xrightarrow{\simeq} & \mathrm{Coh}_{\mathrm{Nilp}}(LS_{\check{G}}). \\ & & \cap \\ & & \mathrm{Coh}. \end{array}$$

Rem: Suppose D is a $\check{\Lambda}^+$-valued divisor on X. Say: $D = \sum_{i=1}^{n} \check{\lambda}_i \cdot x_i$.

$\rightsquigarrow \mathcal{E}_D$ on $LS_{\check{G}}$ a vector bundle

$\sigma \in LS_{\check{G}}$:

Explicitly: $(\mathcal{E}_D)|_\sigma = \bigotimes_{i=1}^{n} (\sigma_{V^{\lambda_i}}) \Big|_{x_i}$.

Up to me getting signs right,

$$\mathrm{coeff}_D(\mathcal{F}) := \Gamma(LS_{\check{G}}, \mathcal{E}_D \otimes \mathrm{coeff}^{\mathrm{enh}}(\mathcal{F})) \in \mathrm{Vect}.$$

is a reasonable definition of an analogue of a_n for $n \geq 1$

$\underset{\Rightarrow}{\text{by}}$ the geometric Casselman-Shalika formula of Frenkel-Gaitsgory-Vilonen.

2) Tempered D-modules

Goal: Define $D(\mathrm{Bun}_G)^{\mathrm{temp}}$ (after Arinkin-Gaitsgory).

Something funny: definition depends on a point $x \in X(k)$ and is purely local.

Abstract setup: $O :=$ Taylor series (at x)

$K :=$ Laurent series

$G(K) =$ loop gp, $\underbrace{G(O)}_{\text{arc gp.}} \subseteq G(K)$

$Gr_G := G(K)/G(O)$

Suppose: $G(K) \curvearrowright \mathcal{C}$. (Means: $\mathcal{C} \in DGCat_{cont}$, I know how to convolve w/ D-modules on $G(K)$).

$\mathcal{H}_{sph} := D({}_{G(O)}\backslash Gr_G) = D(Gr_G)^{G(O)}$.

$\uparrow$ monoidal DG category

$\mathcal{H}_{sph} \curvearrowright \mathcal{C}^{G(O)}$.

"derived Satake"

Thm (Bezrukavnikov - Finkelberg):

$$\mathcal{H}_{sph} \simeq \mathrm{IndCoh}_{Nilp}\left(pt/\check{G} \underset{\check{\mathfrak{g}}/\check{G}}{\times} pt/\check{G}\right).$$

as monoidal DG categories.

In particular: I ~~can~~ obtain a monoidal functor

$$\mathcal{H}_{sph} \xrightarrow{P} \underbrace{\mathrm{QCoh}(\mathbb{B}\check{G} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \mathbb{B}\check{G})}_{\mathcal{H}^{temp}_{sph}}.$$

By derived Satake, P admits a fully faithful left adjoint ($\mathcal{H}_{sph}$-linear).

<u>Def</u>: $\mathcal{C}^{G(O),\text{temp}} := \mathcal{C}^{G(O)} \otimes \mathcal{H}_{\text{sph}}^{\text{temp}}$.

$\mathcal{H}_{\text{sph}}$

By functoriality: I obtain

$$\mathcal{C}^{G(O)} \underset{p}{\overset{p^L}{\rightleftarrows}} \mathcal{C}^{G(O),\text{temp}}.$$

In particular, p is a quotient.

~~Definition~~

<u>Example</u>: $\mathcal{C} = D(\text{Bun}_G^{\text{level},x}) \circlearrowleft G(K)$

$$\mathcal{C}^{G(O)} = D(\text{Bun}_G)$$

$$\rightsquigarrow D(\text{Bun}_G)^{\text{temp},x} = D(\text{Bun}_G)^{\text{temp}}.$$

Prehistory:

<u>Thm</u> (Faergeman, R., '21): $D(Bun_G)^{temp,x}$ is independent of $x \in X$.

<u>Thm</u> (Beraldo, '21): Cuspidal D-modules are tempered:

<u>i.e.</u>:

$$D_{cusp}(Bun_G) \subseteq D(Bun_G)^{temp} \overset{P^L}{\subseteq} D(Bun_G).$$

<u>Alternatively</u>: $D_{cusp}(Bun_G) \to D(Bun_G) \xrightarrow{P} D(Bun_G)^{temp}$

is fully-faithful.

Proper statement of our main thm for these talks:

$$D(Bun_G)^{temp} \xrightarrow{coeff^{enh}} QCoh(LS_{\check{G}})$$

is conservative.