Whittaker coefficients III

Last time: "Big" singular support $\Rightarrow$ non-vanishing of Whittaker coefficients.

Today: Temperedness $\Rightarrow$ big SS.

Thm 1: If $\mathcal{F} \in Shv_{Nilp_{irreg}}(Bun_G) \Rightarrow$

$\mathcal{F}$ is $\underbrace{\text{anti-tempered}}_{\text{Projection to tempered quot. is zero.}}$.

Here: $Nilp_{irreg} \subseteq Nilp \subseteq T^*Bun_G$

$Nilp \setminus Nilp_{irreg} = Nilp^{reg}$.

$Shv \leftarrow$ (ind-) regular holonomic D-modules.

Thm (AGKRRV): $D\text{-mod}_{Nilp} \cong Shv_{Nilp}$.

How to think about temperedness:

Reminder from first talk:

$G(K) \curvearrowright \mathcal{C}$ (local around a pt $x \in X$).

We defined $\mathcal{C}^{G(O)} \to \mathcal{C}^{G(O),\text{temp}}$.

There's something called the Whittaker category of $\mathcal{C}$,

$$\mathrm{Whit}(\mathcal{C}) = \mathcal{C}^{N(K),\psi}.$$

There are adjoint functors

$$\mathcal{C}^{G(O)} \underset{\mathrm{Av}_*^{G(O)}}{\overset{\mathrm{Av}_!^{\psi}}{\rightleftarrows}} \mathcal{C}^{N(K),\psi} = \mathrm{Whit}(\mathcal{C}).$$

<u>Lem</u>: a) The functor $\mathcal{C}^{G(O)} \xrightarrow{Av_!^{+}} \mathrm{Whit}(\mathcal{C})$ factors through $\mathcal{C}^{G(O),\mathrm{temp}}$.

b) The induced functor

$\mathcal{C}^{G(O),\mathrm{temp}} \longrightarrow \mathrm{Whit}(\mathcal{C})$ is conservative.

<u>I.e.</u>:

<u>Cor</u>: $\mathrm{Ker}(Av_!^{+}) = \mathcal{C}^{G(O),\text{anti-temp}}$.

Follows from a compatibility of derived Satake.

<u>Upshot</u>: Need to prove $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}_{\mathrm{irreg}}}$ has vanishing $Av_!^{+}$ at a point $x \in X$.

<u>Goal</u>: reduce this to a finite-dimensional statement.

<u>Setting</u>: $G \curvearrowright Y$ ← smooth algebraic stack.

$\rightsquigarrow \mu: T^*Y \longrightarrow \mathfrak{g}^* \simeq \mathfrak{g}$.

I'll write $Shv_{G\text{-}irreg}(Y) \subseteq Shv(Y)$

for the subcat. of objects w/

$SS \subseteq \mu^{-1}(\mathfrak{g}_{irreg})$.

<u>Thm 2</u>: The functor

$$Shv_{G\text{-}irreg}(Y)^{B^-} \xrightarrow{Av_!^{N,\psi}} D(Y)^{N,\psi}$$

is zero.

<u>Claim</u>: Thm 2 $\Rightarrow$ Thm 1.

Gen'l <u>statement</u>: $G(K) \curvearrowright \mathcal{C}$.

Subgps of $G(K)$:

$$\begin{array}{c} I^- \subseteq G(\mathcal{O}) \\ \cup \\ I \\ \cup \\ \mathring{I} \end{array}$$

opposite Iwahori

$\leftarrow$ radical of Iwahori

$I^- \cap \mathring{I} = K^1_{\mathcal{O}}$ $\leftarrow$ first cong. subgp.

$H \subseteq G(K)$, I'll write H_1 for

$Ad_{-\check{\rho}(t)}(H)$.

Then:

$$
\begin{tikzcd}
\mathcal{C}^{G(O)} \arrow[rr, "Av_!^{\psi}"] \arrow[dr, "oblv"] \arrow[dd, "Av_*"'] & & \mathcal{C}^{N(K),\psi} \\
& \mathcal{C}^{I} \arrow[dl, "1/2"] & \\
\mathcal{C}^{I_1^-} \arrow[rr, "Av_!^{\psi}"] \arrow[d, equal] & & \mathcal{C}^{\mathring{I}_1,\psi} \arrow[uu, "Av_!"'] \arrow[d, equal] \\
(\mathcal{C}^{K'_1})^{B^-} & & (\mathcal{C}^{K'_1})^{N,\psi}
\end{tikzcd}
$$

conservative

where $G \curvearrowright \mathcal{C}^{K'_1}$

~~Applications~~

For us:

Say G is adjoint for simplicity.

$$\begin{tikzcd}
\mathrm{Shv}_{\mathrm{Nilp\,irreg}}(\mathrm{Bun}_G) \arrow[r] \arrow[ddr, bend right, "T"'] & \mathrm{Shv}(\mathrm{Bun}_G^{I,x}) \arrow[d, "\wr"] & \\
& \mathrm{Shv}(\mathrm{Bun}_G^{I^-}) \arrow[r] \arrow[d, "\mathrm{Oblv}"] & D(\mathrm{Bun}_G^{K_1,x})^{N_x} \\
& \mathrm{Shv}(\mathrm{Bun}_G^{x}) &
\end{tikzcd}$$

$\leftarrow$ affine Hecke elt

Enough to see: For $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp\,irreg}})$

$T(\mathcal{F})$ lies in $\mathrm{Shv}_{G\text{-irreg}}(\mathrm{Bun}_G^{x})$.

(For VD: $\mathcal{Y} = \mathrm{Bun}_G^{x}$).

<u>Idea</u>: Nadler–Yun idea $\Rightarrow$

map into $\mathrm{Shv}_{\mathrm{Nilp\,irreg}}(\mathrm{Bun}_G^{I^-})$.

<u>Observation</u>: moment map for Bun_G^{x} is taking a residue; irreg nilp on $X \setminus x$ $\Rightarrow$ this res. is irregular.

Now; indicate pf of Thm 2.

Argument goes by red'n to:

Thm (Losev): Let $\mathcal{M} \in \mathfrak{g}\text{-mod}_0^{\heartsuit}$ ← ab. cat.

0 ← fixed central char = that of triv.

be a simple, faithful module of $\mathcal{U}(\mathfrak{g})_0$ ← central quot.

Then: $SS(\mathcal{M}) \cap \mathcal{N}^{reg}$ is non-empty.

$$\begin{array}{c} SS(\mathcal{M}) \\ \cap \\ \mathcal{N} \end{array} \leftarrow \text{nilpotent cone of } \mathfrak{g} = gr^{PBW}(\mathcal{U}(\mathfrak{g})_0)$$

Let's build out from this.

Define: $Loc^\psi : \mathfrak{g}\text{-mod}_0 \longrightarrow D(G)^{N,\psi}$

as the G-eq. functor corresponding

to $\mathrm{Hom}_G(\mathfrak{g}\text{-mod}_0, D(G)^{N,\psi})$:

$$\mathfrak{g}\text{-mod}_0^{N,\psi} = \mathrm{Vect} \ni k.$$

Thm 3: For $M \in \mathfrak{g}\text{-mod}_0^\heartsuit$ w/ simple

$SS(M) \subseteq \mathcal{N}_{irreg}$, $Loc^\psi(M) = 0$.

Pf: Let $A = U(\mathfrak{g})_0 / Ann(M)$. By Losev,

the map $U(\mathfrak{g})_0 \twoheadrightarrow A$ has non-trivial

kernel. Note: $M \in A\text{-mod}^\heartsuit \subseteq \mathfrak{g}\text{-mod}_0^\heartsuit$.

We'll show: the composition

$$A\text{-mod} \xrightarrow{\mathrm{Oblv}} \mathfrak{g}\text{-mod}_0 \xrightarrow{\mathrm{loc}^\psi} D(G)^{N,\psi}$$

is zero.

Note: G acts on A-mod compatibly w/ Oblv by usual HC formalism.

$\Rightarrow$ Enough to show $A\text{-mod}^{N,\psi} = 0$.

This category has a t-str; enough to show $A\text{-mod}^{N,\psi,\heartsuit} = 0$.

Then $A\text{-mod}^{N,\psi,\heartsuit} \subseteq \mathfrak{g}\text{-mod}_0^{N,\psi,\heartsuit} = \mathrm{Vect}^\heartsuit$.

Kostant: any object of $\mathfrak{g}\text{-mod}_0^{N,\psi,\heartsuit}$ is a faithful $U(\mathfrak{g})_0$-module.

Losev kicks in: not in $A\text{-mod}^{N,\psi,\heartsuit}$. //

More next time!