# Whittaker coefficients II

$G/k$ reductive group.

$\mathfrak{g}$ = Lie algebra.

$\{e, h, f\} \subseteq \mathfrak{g}$ principal $\mathfrak{sl}_2$.

## Classical picture:

$\mathcal{N}$ := nilpotent cone $\subseteq \mathfrak{g}$

stack quotient

$$f + \mathfrak{b}/N = \text{Kostant slice} \longrightarrow \mathfrak{g}/G$$

then: $$f+\mathfrak{b}/N \underset{\mathfrak{g}/G}{\times} \mathcal{N}/G = \{f\}.$$

Globalize this picture:

<u>Reminder</u>:

$\text{Higgs}_G$ ← moduli of $P_G$ a $G$-bundle on $X$ (our sm. proj. connected curve)

& + $\varphi \in \Gamma(X, \mathfrak{g}_{P_G} \otimes \Omega^1_X)$.

<u>Example</u>: $G = GL_n$ : $\mathcal{E}$ rk $n$ v.b. on $X$ + $\varphi : \mathcal{E} \to \mathcal{E} \otimes \Omega^1$.

Std argument w/ Serre duality & Kodaira-Spencer-style deformation theory: $\text{Higgs}_G = T^* \text{Bun}_G$

(uses $\mathfrak{g} \simeq \mathfrak{g}^*$ $G$-eq.).

<u>Analogy</u>: $\text{Higgs}_G \approx \mathfrak{g}/G$.

<u>Properly</u>: $\text{Higgs}_G = \text{Maps}(X, \mathfrak{g}/(G \times \mathbb{G}_m)) \times \{\Omega^1_X\}$.

$\text{Bun}_{\mathbb{G}_m}$

Let:

$\mathcal{N}ilp \subseteq Higgs_G = T^*Bun_G$ be

the space of nilpotent Higgs bundles.

(Replace $\check{\mathfrak{g}}/G$ in our fmla by $\mathcal{N}/G$).

Studied a lot recently in my joint work

w/ Arinkin - Gaitsgory - Kazhdan - Rozenblyum - Varshavsky

(AGKRRV).

Thm (Faltings): $\mathcal{N}ilp$ is a "(singular) Lagrangian"

(at least, $\dim \mathcal{N}ilp = \dim Bun_G$) in $T^*Bun_G$.

We let $\mathcal{N}ilp^{reg} \subseteq \mathcal{N}ilp$ be the open

consisting of nilpotent Higgs bundles

$(P_G, \varphi)$ s.t. $\varphi$ is regular nilpotent

at the gen'c pt of $X$.

For a $k$-pt $(P_G, \varphi) \in \mathrm{Nilp}^{reg}$, there are two natural numerical invariants:

<u>First</u>: $c_1(P_G) = \deg(P_G) \in \pi_0(\mathrm{Bun}_G) = \pi_1^{\text{true}}(G)$.

<u>Second</u>: $\exists!\, P_B$ red'n of $P_G$ to $B$

s.t. $\varphi \in \Gamma(n_{P_B} \otimes \Omega^1_X)$

$$\longrightarrow \bigoplus_{i \in I_G} \Gamma(\Omega^1_X \otimes \mathcal{L}_{P_B}^{\alpha_i}).$$

$\rightsquigarrow$ effective divisors on $X$ indexed by the simple roots

$$\begin{pmatrix} 0 & \circledast & * \\ 0 & 0 & \circledast \\ 0 & 0 & 0 \end{pmatrix}$$

$\Leftrightarrow$ a $\check{\Lambda}^+_{G^{ad}}$-valued divisor on $X$.

its degree we call $\mathrm{disc}(P_G, \varphi)$. $\leftarrow$ <u>discrepancy</u>.

<u>Prop</u> (Beilinson–Drinfeld): irreducible components of $\mathcal{N}ilp^{reg}$ are classified by degree & discrepancy.

<u>Also</u>: There's an analogue of the Kostant slice in $Higgs_G$.

<u>Special case</u>: $G = GL_2$:

$$Kost^{glob} \longrightarrow Higgs_G$$

$$\| $$

$$\left\{ 0 \to \Omega^{1/2} \to \mathcal{E} \to \Omega^{-1/2} \to 0 \ + \ \varphi \text{ Higgs bundle on } \mathcal{E} \right\}.$$

s.t. the induced map

$$\underbrace{\Omega^{1/2} \hookrightarrow \mathcal{E} \xrightarrow{\varphi} \mathcal{E} \otimes \Omega^1 \twoheadrightarrow \Omega^{1/2}}_{\text{identity}}$$

<u>Rem</u>: $\mathrm{Kost}^{glob}$ maps isomorphically onto the Hitchin base.

<u>Also</u>: $\mathrm{Kost}^{glob}$ is Lagrangian. $(g>1)$.

From Page 1: $\mathrm{Kost}^{glob} \cap \mathrm{Nilp} = \{f^{glob}\}$

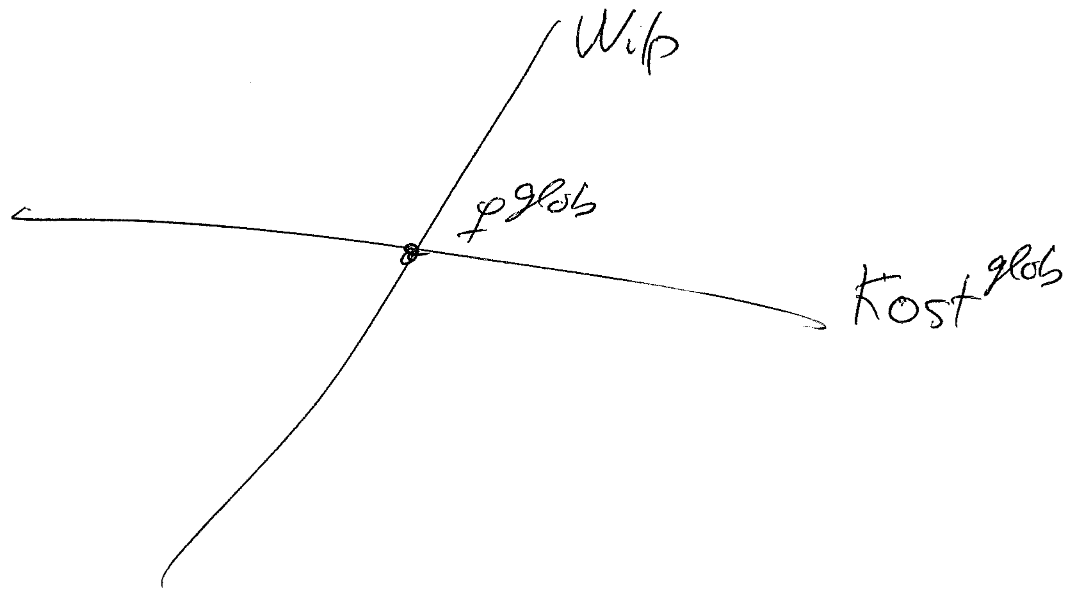
<u>Actual definition</u>:

acts by

$$\mathrm{Kost}^{glob} = \mathrm{Maps}\left(X, (f+\mathfrak{b}/N)/\mathbb{G}_m\right) \times \{\Omega_X^{1/2}\}.$$

$\mathrm{Bun}_{\mathbb{G}_m}$

<u>Picture</u>:



Let $\mathcal{N}ilp^{Kos} \subseteq \mathcal{N}ilp^{reg}$ be the component containing $f^{glob}$. ($f^{glob}$ is a smooth pt).

*Example*: $G = \mathbb{G}_m$: $\mathcal{N}ilp = \mathrm{Bun}_{\mathbb{G}_m} \subseteq T^*\mathrm{Bun}_{\mathbb{G}_m}$

$\mathrm{Kos}^{glob}$ = conormal to the trivial bundle.

$$f^{glob} = (\mathcal{O}_X, 0).$$
($0 \leftarrow \varphi$.)

*Heuristic*: $\mathrm{Kos}^{glob}$ is related to the coefficient functor.

Precise statement:

last time:

$$\begin{array}{ccc} & \mathrm{Bun}_N^{\Omega} & \\ {}^{\psi}\swarrow & & \searrow^{\phi} \\ \mathbb{A}^1 & & \mathrm{Bun}_G \end{array}$$

$$\mathbb{A}^1 \overset{pt}{\underset{df}{\rightrightarrows}} T^*\mathbb{A}^1$$

$$\begin{array}{ccccc} & T^*\mathbb{A}^1 \times_{\mathbb{A}^1} \mathrm{Bun}_N^{\Omega} & & T^*\mathrm{Bun}_G \times_{\mathrm{Bun}_G} \mathrm{Bun}_N^{\Omega} & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \\ T^*\mathbb{A}^1 & & T^*\mathrm{Bun}_N^{\Omega} & & T^*\mathrm{Bun}_G \end{array}$$

the fiber product is $\mathrm{Kost}^{glob}$

(on the nose).

$Shv_{Nilp}(Bun_G) \subseteq D(Bun_G)$ be the subcategory ~~[illegible]~~ of objects with nilpotent singular support.

<u>Thm</u> (Faergeman-R.):

1) The functor coeff : $Shv_{Nilp}(Bun_G) \longrightarrow Vect$ is t-exact and commutes w/ Verdier duality.

2) If $\mathcal{F} \in Perv_{Nilp}(Bun_G)$, then

$$\dim \operatorname{coeff}(\mathcal{F}) = \operatorname{ord}_{Nilp^{reg}}(CC(\mathcal{F})).$$

(ord ← multiplicity)

<u>Rem</u>: Our argument is automorphic. Uses a forthcoming result of Kevin Lin.

Nadler–Taylor have forthcoming work w/ a topological argument.

Example: $G = \mathbb{G}_m$:

$$\mathcal{D}(\mathrm{Bun}_{\mathbb{G}_m}) \xrightarrow{\text{coeff} = i^!_{\text{triv}}[\dim \mathrm{Bun}_{\mathbb{G}_m}]} \mathrm{Vect}$$

$\subseteq$

$\mathrm{Shv}_0(\mathrm{Bun}_{\mathbb{G}_m})$

$\parallel$

$\mathrm{Lisse}(\mathrm{Bun}_{\mathbb{G}_m})$

Gen'l heuristic:

Kashiwara–Schapira "say":

$Y/\mathbb{C}$ smooth

$\Lambda \subseteq T^*Y$ Lagrangian

$\mathcal{F} \in \mathrm{Shv}_\Lambda(Y)$

$\rightsquigarrow$ local system $\mu(\mathcal{F})$ on $\underbrace{\Lambda^{sm}}_{\text{smooth locus}} \subseteq \Lambda$

whose fibers have dim'n = orders of characteristic cycles.

fibers: are called "microstalks" of $\mathcal{F}$.

Heuristic (Nadler-Taylor make it precise):

coeff$(\mathcal{F})$ = microstalk of $\mathcal{F}$ at $f^{glob} \in Nilp$

for $\mathcal{F} \in Shv_{Nilp}(Bun_G)$.