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Proof of the trace conjecture

Reminder: Goal:

$$\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)) \simeq \mathrm{tr}(\mathrm{Frob}_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)).$$

Recap: last time, gave a lot of formal background.

1) $\mathrm{Rep}\check{G}_{\mathrm{Ran}}$ ← universal source of Hecke functors.

$\forall I$ finite set, $\exists$ canonical functor

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(X^I) \longrightarrow \mathrm{Rep}\check{G}_{\mathrm{Ran}}.$$

$\mathrm{Rep}\check{G}_{\mathrm{Ran}}$ is symmetric monoidal, unit object called triv

call the binary product by $*$.

$\operatorname{Rep} \check{G}_{Ran} \circlearrowright \operatorname{Shv}(\operatorname{Bun}_G)$.

It acts in a super nice way.

$\forall\, V \in \operatorname{Rep} \check{G}_{Ran}$, corresponding functor

$$\operatorname{Shv}(\operatorname{Bun}_G) \xrightarrow{V * -} \operatorname{Shv}(\operatorname{Bun}_G)$$

is defined by kernels in <u>both</u>

upper-! lower-* & upper-* lower-!

senses.

$V \in \operatorname{Rep} \check{G}_{Ran} \longmapsto$

$K_V \in \operatorname{Shv}(\operatorname{Bun}_G \times \operatorname{Bun}_G)$

mean corr. upper-* lower-! kernel.

Example: $K_{triv} = \Delta_! \overline{\mathbb{Q}}_{\ell, Bun_G}$.

We also have a canonical functor

$\mathrm{Loc}: \mathrm{Rep}\,\check{G}_{Ran} \longrightarrow \mathrm{QCoh}(LS^{restr}_{\check{G}})$.

$x \in X$, $V \in \mathrm{Rep}\,\check{G}$ $\rightsquigarrow$ $V_x \in \mathrm{Rep}\,\check{G}_{Ran}$

by construction, maps to corr. vector bundle on $LS^{restr}_{\check{G}}$.

Remind: $\mathrm{Rep}\,\check{G}_{Ran}$ is canonically self-dual as a DG category.

Define a favorite object $\mathcal{R} \in \mathrm{Rep}\,\check{G}_{Ran}$, called Beilinson spectral projector,

$$\begin{array}{ccc}
\text{unit} & \in & \operatorname{Rep}\check{G}_{Ran} \otimes \operatorname{Rep}\check{G}_{Ran} \\
\Big\downarrow & & \Big\downarrow {\scriptstyle Loc \otimes id} \\
 & & \operatorname{QCoh}(LS^{restr}_{\check{G}}) \otimes \operatorname{Rep}\check{G}_{Ran} \\
 & & \Big\downarrow {\scriptstyle \Gamma_! \otimes id} \\
\mathcal{R} & \in & \operatorname{Rep}\check{G}_{Ran}.
\end{array}$$

What to know about $\mathcal{R}$:

Prop: $Loc(\mathcal{R}) = \mathcal{O}_{LS^{restr}_{\check{G}}}$ $(= Loc(triv))$.

Thm: $\mathcal{R} * - : Shv(Bun_G) \longrightarrow Shv(Bun_G)$ is the projector onto $Shv_{Nilp}(Bun_G)$.

Last thing: $\mathcal{R}$ is related to the self-duality for Shv_{Nilp} that Dennis spoke about two weeks ago.

Thm: $\exists$ self-duality for $Shv_{Nilp}(Bun_G)$

s.t. the evaluation is:

$$\begin{tikzcd} Shv_{Nilp} \otimes Shv_{Nilp} \arrow[r, "\Delta^*(- \boxtimes -)"] & Shv(Bun_G) \arrow[d, "H^*_c(Bun_G, -)"] \\ & Vect \end{tikzcd}$$

and the coevaluation is:

$$\begin{tikzcd} Vect \arrow[r, dashed] \arrow[dr] & Shv_{Nilp} \otimes Shv_{Nilp} \arrow[d, phantom, "\cap"] \\ & Shv(Bun_G \times Bun_G) \end{tikzcd}$$

$$\overline{\mathbb{Q}}_\ell \longmapsto K_{\mathcal{R}}.$$

We'll use these tools to prove the trace conjecture.

First: I want to start by computing

$\mathrm{tr}(\mathrm{Frob}_* \curvearrowright \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$.

Convenient to rewrite this in other terms.

Y_0 alg stack $/\mathbb{F}_q \rightsquigarrow Y/\overline{\mathbb{F}}_q$.

Then $\mathrm{Frob}_! = \mathrm{Frob}_*$ as functors

$\mathrm{Shv}(Y)_\circ \longrightarrow \mathrm{Shv}(Y)$. (Also, these are isos.)

So I can replace Frob_* w/ $\mathrm{Frob}_!$.

Next: Fact: $(\mathrm{Frob}_*)^\vee = (\mathrm{Frob}_!)^\vee = \mathrm{Frob}^*$

$(= \mathrm{Frob}^!)$. With respect to self-duality

on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$!

~~Note~~ All this uses is evaluation & coevaluation are Frobenius equivariant.

$\Rightarrow tr(Frob_*) = tr(Frob^* \circlearrowright Shv_{Nilp}(Bun_G))$.

~~Examples~~ ~~Explain comp~~

By definition: $tr(Frob^* \circlearrowright Shv_{Nilp})$

is:

$H^*_c\ \Delta^* (Frob \times id)^* (K_R)$.

$= H^*_c\ Graph^*_{Frob} K_R$.

In general: $T : \mathcal{C} \to \mathcal{C}$

$tr(T) = ev\ (T \otimes id)(unit)$.

$$Vect \xrightarrow{unit} \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{T \otimes id} \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{ev} Vect$$

Convenient: generalize this construction.

<u>Define</u>: $\mathrm{Sht} : \mathrm{Rep}\, \check{G}_{\mathrm{Ran}} \longrightarrow \mathrm{Vect}$

as $\quad \mathcal{V} \longmapsto$

$$H_c^* \, \mathrm{Graph}_{\mathrm{Frob}}^* K_{\mathcal{V}} .$$

<u>Just saw</u>: $\mathrm{Sht}(\mathcal{R}) = \mathrm{tr}(\mathrm{Frob}_*, \mathrm{QShv}_{\mathrm{Nilp}})$.

<u>OTOH</u>: $\mathrm{Sht}(\mathrm{triv}) = \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$

b/c: $\mathrm{Sht}(\mathrm{triv}) = H_c^* \, \mathrm{Graph}_{\mathrm{Frob}}^* K_{\mathrm{triv}}$

$= H_c^* \, \mathrm{Graph}_{\mathrm{Frob}}^* \ddot{\Delta}_! \overline{\mathbb{Q}_\ell} =$

$H_c^* \left(\mathrm{Bun}_G^{\mathrm{Frob}=\mathrm{id}}, \overline{\mathbb{Q}_\ell} \right)$

$= H_c^* \left(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}_\ell} \right).$

<u>Thm</u>: $\exists$ a unique factorization of

Sht as:

$$\mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \xrightarrow{\mathrm{Loc}} \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \xrightarrow{\mathrm{Sht}^{\mathrm{enh}}} \mathrm{Vect}.$$

We'll deduce from work of Xue.

Assuming this, we then have

$$\mathrm{Sht}(R) = \mathrm{Sht}^{\mathrm{enh}}\,\mathrm{Loc}(R) = \mathrm{Sht}^{\mathrm{enh}}(\mathcal{O}_{\mathrm{LS}^{\mathrm{restr}}_{\check{G}}})$$

$$\mathrm{Sht}(\mathrm{triv}) = \mathrm{Sht}^{\mathrm{enh}}\,\mathrm{Loc}(\mathrm{triv}) = \mathrm{Sht}^{\mathrm{enh}}(\mathcal{O}_{\mathrm{LS}^{\mathrm{restr}}_{\check{G}}})$$

$\rightsquigarrow$ our isomorphism.

~~Cor.~~ $\mathrm{Shit}^{\mathrm{enh}} : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}) \longrightarrow \mathrm{Vect}$

$\exists$ dual object in $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})^{\vee} = \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$

called Drinf.

Basic properties:

1) $\Gamma_c(\mathrm{Drinf}) = \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q))$.

2) $\Gamma_c(\mathcal{V}_x \otimes \mathrm{Drinf}) =$ Shtuka cohomology

for $x \in X$ & $V \in \mathrm{Rep}\,\check{G}$.

Reduced to showing $\mathcal{S}_{\mathrm{Lit}}$ factors through Loc.

General setting: $\mathcal{S} : \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \longrightarrow \mathrm{Vect}$.

Q: When does $\mathcal{S}$ factor through Loc?

There is an easy criterion. I finite set.

$$\mathrm{Rep}\,\check{G}^{\otimes I} \otimes \mathrm{Shv}(X^I) \longrightarrow \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \xrightarrow{\mathcal{S}} \mathrm{Vect}$$

Verdier duality

$$\Longleftrightarrow \mathrm{Rep}\,\check{G}^{\otimes I} \xrightarrow{\mathcal{S}_I} \mathrm{Shv}(X^I).$$

Criterion: $\forall I$, the functor $\mathcal{S}_I$ takes values in $\mathrm{Lisse}(X^I)$.

Lem: Criterion $\Leftrightarrow$ existence of S^{enh} with $S = S^{enh} \circ \mathrm{Loc}$. Any such S^{enh} is unique.

Idea of pf: pass to dual ~~cats~~ categories

$$S^\vee \in (\mathrm{Rep}\,\check{G}_{\mathrm{Ran}})^\vee = \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} = \mathrm{Hom}_{\circ\text{-}\mathrm{EfSet}}(\mathrm{Rep}\,\check{G}^{\circ}, \mathrm{Shv}(X^{\circ}))\,.$$

OTOH: last time we said

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) = \mathrm{Hom}_{\circ\text{-}\mathrm{EfSet}}(\mathrm{Rep}\,\check{G}^{\circ}, \mathrm{Lisse}(X^{\circ}))\,.$$

Clearly $\Rightarrow$ $\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \overset{\mathrm{Loc}^\vee}{\subseteq} \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$.

For $S = \mathrm{Sht}$, get functors

$$\mathrm{Sht}_I : \mathrm{Rep}\, \check{G}^{\otimes I} \longrightarrow \mathrm{Shv}(X^I).$$

Exercise in unwinding def'ns:

$\mathrm{Sht}_I(V) \in \mathrm{Shv}(X^I)$ is

"shtuka cohomology" as

defined by Drinfeld, Varshavsky, V. Lafforgue.

Thm (Xue): Sht_I takes values in $\mathrm{Lisse}(X^I)$.

Proved using excursion operators.

$\Rightarrow$ criterion is satisfied for us

$\Rightarrow \exists\, \mathrm{Sht}^{enh}$. So we're done.

<u>Topics</u>:

1) $(LS^{restr}_{\check{G}}) \circlearrowleft$ Frob automorphism.

$$LS^{arthm}_{\check{G}} := (LS^{restr}_{\check{G}})^{Frob = id}.$$

(Considered by Scholze & Zhu also, defined non-Tannakianly.)

<u>Thm</u>: $LS^{arthm}_{\check{G}}$ is an (honest) algebraic stack (non-formal, unlike $LS^{restr}_{\check{G}}$.)

Quasi-coherent sheaves on $LS^{arthm}_{\check{G}}$ are described like for $LS^{restr}_{\check{G}}$, but with partial Frobenius maps.

So upshot: our object Drinf localizes further onto $LS_{\check{G}}^{\text{arthm}}$.

I.e. $\exists$ $\text{Drinf}^{\text{arthm}} \in QCoh(LS_{\check{G}}^{\text{arthm}})$ whose pushfwd to $LS_{\check{G}}^{\text{restr}}$ is Drinf.

$$1') \quad QCoh(LS_{\check{G}}^{\text{restr}}) \overset{\text{Thm}}{\curvearrowright} Shv_{Nilp}(Bun_G).$$

compatibly w/ Frobenius on both sides.

Formally implies that $tr(\text{Frob}_* \curvearrowright Shv_{Nilp}(Bun_G))$ upgrades to an object of $QCoh(LS_{\check{G}}^{\text{arthm}})$.

The two constructions 1) & 1') are equivalent.

2) Construction of Drinf works in the ramified setting as well.

But the realization as trace(... of what?) is not presently clear.

3) Conjecture Dima will explain:

$$\mathrm{Drinf}^{\mathrm{arithm}} \in \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{arithm}})$$

$$\parallel$$

$$\omega_{\mathrm{LS}_{\check{G}}^{\mathrm{arithm}}}.$$

Follows from a form of geometric Langlands conjecture in this setting.