Joint w/ Arinkin, Gaitsgory, Kazhdan, Rozenblyum, Varshavsky.

<u>Problem</u>: Compute $tr(\mathrm{Frob} \circlearrowright Shv_{N.lp}(Bun_G))$.

<u>Motivation</u>: A weakness of usual Langlands correspondence is that it concerns aut. rep'ns, not automorphic forms. By contrast, usual (D-module) geometric Langlands describes D-mod(Bun_G) $\leftarrow$ analogue of space unramified automorphic forms.

<u>Program</u>: Give a version of categorical geometric Langlands for ℓ-adic sheaves, deduce a conjecture about unramified aut. forms via trace of Frobenius.

Setting: $X_0/\mathbb{F}_q$ smooth, geom. connected projective curve, $X/k = \overline{\mathbb{F}_q}$ its base-change.

Bun_G k-stack of G-bundles on X.

$\mathrm{Shv}(-) :=$ ind-constructible ℓ-adic ($\overline{\mathbb{Q}_\ell}$) sheaves.

$\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \subseteq \mathrm{Shv}(\mathrm{Bun}_G)$ via Beilinson's definition of singular support.

$\mathrm{Frob} : \mathrm{Bun}_G \longrightarrow \mathrm{Bun}_G$.

Thm: $\mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)) \xrightarrow{\simeq}$
$$\mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q))$$
isomorphism is given by trace of Frobenius construction.

<u>Today</u>: Construction of the iso (ignore compatibility w/ usual Shv → fns dictionary).

<u>Preview</u>: Dima will give a conjecture describing $Shv_{Nilp}(Bun_G)$ in spectral terms.

<u>Rem'k</u>: Last time, we discussed some properties of Shv_{Nilp} are also satisfied by lisse sheaves on smooth proper varieties.

Trace of Frobenius property is not satisfied in that generality:

a) A_0 ab. variety $/\mathbb{F}_q$

$$\mathrm{tr}(\mathrm{Frob}, \mathrm{Lisse}(A)) \xrightarrow{\simeq} \mathrm{Fun}(A(\mathbb{F}_q))$$

b) $\mathbb{P}^1$

$$\mathrm{tr}(\mathrm{Frob}, \mathrm{lisse}(\mathbb{P}^1)) \underbrace{\longrightarrow}_{\text{not an iso}} \mathrm{Fun}(\mathbb{P}^1(\mathbb{F}_q)).$$

What do we need to prove?

What is trace?

Reminder: $\mathcal{C}$ dualizable DG cat

$$T : \mathcal{C} \to \mathcal{C} \iff T \in \mathcal{C} \otimes \mathcal{C}^\vee \ni u$$

$$T = (T \otimes \mathrm{id})(u)$$

$$\mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{ev} \mathrm{Vect}$$

$$ev(T) = \mathrm{tr}(T).$$

Last time: Dennis explained

$$\mathcal{C} = \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G), \quad \exists\ \mathcal{C}^\vee = \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$$

w/ properties:

1)
$$\begin{tikzcd}
\mathcal{C}\otimes\mathcal{C}^\vee \arrow[r] \arrow[d, equal] & \mathrm{Vect} \\
\mathrm{Shv}_{Nilp}\otimes\mathrm{Shv}_{Nilp} \arrow[ur, "\Gamma_c(\mathrm{Bun}_G,\, \Delta^*(- \boxtimes -))."'] &
\end{tikzcd}$$

2)
$$\begin{tikzcd}
\mathrm{Vect} \arrow[r, "\mathrm{coev}"] & \mathcal{C}\otimes\mathcal{C}^\vee \arrow[d, equal] \\
& \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)\otimes\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \arrow[d, "\simeq"', "\mathrm{Thm}"] \\
& \mathrm{Shv}_{Nilp\times Nilp}(\mathrm{Bun}_G\times\mathrm{Bun}_G) \arrow[d, shift right, "\mathrm{emb}"'] \\
& \mathrm{Shv}(\mathrm{Bun}_G\times\mathrm{Bun}_G). \arrow[u, shift right, "\mathrm{emb}^R"']
\end{tikzcd}$$

$$\mathrm{coev}(\overline{\mathbb{Q}}_\ell) \leftarrow\!\shortmid\ u = \mathrm{emb}^R \Delta_! \overline{\mathbb{Q}}_\ell.$$

<u>Note</u>: $\exists$ canonical map $u = \mathrm{emb} \circ \mathrm{emb}^R \Delta_! \overline{\mathbb{Q}_\ell}$

$$\begin{array}{c} u = \mathrm{emb} \circ \mathrm{emb}^R \Delta_! \overline{\mathbb{Q}_\ell} \\ \downarrow \\ \Delta_! \overline{\mathbb{Q}_\ell}. \end{array}$$

~~Prom~~ ~~Nat~~: Promise: easier to compute
$\mathrm{Tr}(\mathrm{Frob}^* \curvearrowright \mathrm{Shv}_{N_i lp}(\mathrm{Bun}_G))$,
will identify w/ $\mathrm{Tr}(\mathrm{Frob}_*)$ later if time permits.

So $\mathrm{tr}(\mathrm{Frob}^* \curvearrowright \mathrm{Shv}_{N_i lp}(\mathrm{Bun}_G))$ is

$$\mathrm{ev}\,(\mathrm{Frob} \times \mathrm{id})^* u$$
$$\shortparallel$$
$$\Gamma_c(\mathrm{Bun}_G, \Delta^*(\mathrm{Frob} \times \mathrm{id})^* u) \longrightarrow$$
$$\Gamma_c(\mathrm{Bun}_G, \cancel{\Delta^*} \mathrm{Graph}^*_{\mathrm{Frob}} \Delta_! \overline{\mathbb{Q}_\ell})$$
$$= \Gamma_c(\mathrm{Bun}_G \underset{\mathrm{Bun}_G \times \mathrm{Bun}_G}{\times} \mathrm{Bun}_G, \overline{\mathbb{Q}_\ell}) = \Gamma_c(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}_\ell})$$
$$= \text{cptly supp'd functions.}$$

Content: the map here is an isomorphism

Need to understand u a little better.

Claim: $\mathrm{emb}^{R}(u)$ can be expressed using Beilinson's "spectral projector."

Need to introduce $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}} / \overline{\mathbb{Q}}_\ell$.

Will be discussed more by Dima, but I'm going use some properties as a black box.

Def'n: $S/\overline{\mathbb{Q}}_\ell$ test scheme, a map

$$S \longrightarrow LS^{\mathrm{restr}}_{\check{G}} \quad \Leftrightarrow$$

$$\mathrm{Rep}\,\check{G} \longrightarrow \mathrm{QCoh}(S) \otimes \mathrm{Lisse}(X).$$

$\otimes$-functor.

Turns out to give fairly reasonable geometry.

Roughly, $LS^{restr}_{\check{G}}$ looks like a disjoint union of the formal completions of Artin stacks along closed substacks.

~~Claim~~. One property that follows:

$\exists$ canonical self-duality for ~~...~~ $QCoh(LS^{restr}_{\check{G}})$. In particular, there is a canonical functor

$$\Gamma_c : QCoh(LS^{restr}_{\check{G}}) \longrightarrow Vect.$$

<u>Claim</u>: $QCoh(LS^{restr}_{\check{G}})$ are equivalent to ~~...~~ data of natural maps

$$Rep(\check{G}^I) \longrightarrow Lisse(X^I)$$

$I \in Set_{<\infty}$.

<u>Construction</u>: $\mathcal{F} \in \mathrm{QCoh}(LS^{restr}_{\check{G}})$

$$\mathrm{can}: \mathrm{Rep}\,\check{G} \longrightarrow \mathrm{QCoh}(LS^{restr}_{\check{G}}) \otimes \mathrm{Lisse}(X)$$

symmetric monoidal

$$\rightsquigarrow \mathrm{Rep}\,\check{G}^{\otimes I} \longrightarrow \mathrm{QCoh}(LS^{restr}_{\check{G}})^{\otimes I} \otimes \mathrm{Lisse}(X)^{\otimes I}$$

$$\begin{tikzcd} \mathrm{QCoh}(LS^{restr}_{\check{G}})^{\otimes I} \otimes \mathrm{Lisse}(X)^{\otimes I} \arrow[d, "\otimes \text{ on first factor}"] \\ \mathrm{QCoh}(LS^{restr}_{\check{G}}) \otimes \mathrm{Lisse}(X)^{\otimes I} \arrow[d, "\Gamma_c(\mathcal{F} \otimes -) \otimes \mathrm{id}"] \\ \mathrm{Lisse}(X)^{\otimes I}. \end{tikzcd}$$

<u>Claim</u>: This is an equivalence.

Abstract nonsense 2:

$$\mathrm{Rep}\,\check{G}_{\mathrm{Ran}} := \underset{\substack{\mathrm{TwArr} \\ I \to J}}{\mathrm{colim}}\ \mathrm{Rep}\,\check{G}^{\otimes I} \otimes \mathrm{Shv}(X)^{\otimes J}$$

Sample objects: $x_1, \dots, x_n \in X$ $\quad V_1, \dots, V_n$

$$\rightsquigarrow V_{1,x_1} \otimes \dots \otimes V_{n,x_n} \in \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$$

$\mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \curvearrowright \mathrm{Shv}(\mathrm{Bun}_G)$ via Hecke functors.

$$\mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \xrightarrow{\mathrm{Loc}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$$

$x \in X$

$V \in \mathrm{Rep}\,\check{G}$

$\mathrm{Loc}(V_x) = ?$

$$\begin{array}{ccc} \operatorname{Rep}\check{G} & \xrightarrow{\text{Can}} & \mathrm{QCoh}(LS_{\check{G}}^{\text{restr}}) \otimes \mathrm{Lisse}(X) \\ & & \downarrow{\scriptstyle \mathrm{id} \otimes \text{fiber at } x} \\ \end{array}$$

$$\operatorname{Rep}\check{G} \ni V \longmapsto \mathrm{Loc}(V_x) \in \mathrm{QCoh}(LS_{\check{G}}^{\text{restr}}),$$

Loc is symmetric monoidal in the natural sense.

Important object: $\mathcal{R} \in \operatorname{Rep}\check{G}_{\mathrm{Ran}}$.

idea of construction: A comm. alg in

$\operatorname{Rep}\check{G} \rightsquigarrow A_{\mathrm{Ran}} \in \operatorname{Rep}\check{G}_{\mathrm{Ran}}$.

Apply to regular rep'n of $\check{G}$

$\rightsquigarrow \mathcal{R}_0 \in \operatorname{Rep}(\check{G} \times \check{G})_{\mathrm{Ran}} = \operatorname{Rep}\check{G}_{\mathrm{Ran}} \otimes \operatorname{Rep}\check{G}_{\mathrm{Ran}}$

Apply

$$\begin{array}{c} \operatorname{Rep}\check{G}_{Ran} \otimes \operatorname{Rep}(\check{G})_{Ran} \\ \big\downarrow (\Gamma_c \circ Loc) \otimes id \\ \operatorname{Rep}\check{G}_{Ran} . \end{array}$$

Get my object: $\mathcal{R} \in \operatorname{Rep}\check{G}_{Ran}$.

Important thm:

$$\begin{tikzcd} H_{\mathcal{R}} : Shv(Bun_G) \arrow[rr] \arrow[dr, twoheadrightarrow, "emb^R"'] & & Shv(Bun_G) . \\ & Shv_{Nilp}(Bun_G) \arrow[ur, hook] & \end{tikzcd}$$

I.e.: $H_{\mathcal{R}}$ is the projector onto $Shv_{Nilp}(Bun_G)$.

$V \in \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \rightsquigarrow K_V \in \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$.

corresponding kernel
in upper-$*$ lower-$!$
sense.

in other words:

$$H_V(\mathcal{F}) = p_{1!}\left(p_2^*(\mathcal{F}) \overset{*}{\otimes} K_V\right).$$

$\mathcal{F} \in \mathrm{Shv}_\phi(\mathrm{Bun}_G)$

E.g.: $\mathrm{triv} \in \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \rightsquigarrow K_{\mathrm{triv}} = \Delta_! \overline{\mathbb{Q}}_\ell$.

Last time: Dennis asserted

$$\mathcal{R} \in \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \longmapsto K_{\mathcal{R}} = \underset{\substack{\shortparallel \\ \mathcal{U}}}{\mathrm{emb}^R} \Delta_! \overline{\mathbb{Q}}_\ell$$

By construction $\exists$ canonical map
$\mathcal{R} \longrightarrow \mathrm{triv} \in \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ compatible w/ above.

<u>Problem</u>: $\mathcal{R} \in \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$

$\rightsquigarrow K_{\mathcal{R}} \in \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$.

$\underset{\mathcal{U}}{=}$

Need to compute

$\Gamma_c(\mathrm{Bun}_G, \mathrm{Graph}_{\mathrm{Frob}}^{*}(K_{\mathcal{R}}))$.

Want to see: $\mathcal{R} \longrightarrow \mathrm{triv}$ induces an iso on applying the above.

<u>Next time</u>: Explain how to see this using Cong Xue's theorem on lisseness of Shtuka cohomology.