

Deligne-Lusztig duality for automorphic sheaves, continuation

Last time:

Main thm: In $\text{End}(D(Bu_G))$

$$Ps\text{-Id}_* \circ Ps\text{-Id}_! [\text{shift}]$$

$$\simeq \text{CTAb} \left(\varinjlim_{p \neq G} (DL(p) \longrightarrow DL(G)) \right)$$

Only need to prove

Thm 2 The following diagram commutes

$$\begin{array}{ccc} \text{Par} & \xrightarrow{DL} & \text{End}(D(Bu_G)) \\ \downarrow (K & & \simeq \downarrow E \\ D(\overline{Bu}_G) & \xrightarrow{\overline{\Delta}_!} & D(Bu_G \times Bu_G) \end{array}$$

where: - $DL(p) = E_S^{\text{enh}} \circ C_T^{\text{enh}}_p$

$$\left(\text{II}(G, p) \begin{array}{c} \xrightarrow{E_S} \\ \xleftarrow{C_T} \end{array} D(Bu_G) \right)$$

- $E(f) = (f \circ \text{Id})(\Delta, k)$

$$- K(p) = i_{p,!} \circ i_p^* \circ j_{G,*} \circ r_!(k)$$

[corollary (p1)]

$$\cdot \bar{i}_p : \overline{B_{nG,p}} \longrightarrow \overline{B_{nG}}$$

$$\bar{j}_G = \bar{i}_G$$

$$- r : B_{nG} \longrightarrow B_{nG} \times_{|B|} \mathbb{Z}_G$$

is
 $\overline{B_{nG,G}}$

If we chose $r : A' \rightarrow \text{tad}^+ = A^r$

connecting G_p & G_G ,

$$\begin{array}{ccc} V \in B_{nG}^r & \longrightarrow & A' \\ \downarrow & & \downarrow r \\ V \in B_{nG} & \longrightarrow & A^r \end{array} \quad \text{then}$$

Thm 3

$$\begin{array}{ccc} \mathbb{F}_p(w) & = & \mathbb{F}_p^{\text{un}}(w) \subset D(V \in B_{nG} / G_p) \\ \downarrow & & \downarrow (-p\text{-th}) \\ K(p)[\text{shift}] & \in & D(\overline{B_{nG}}) \end{array}$$

$$\begin{array}{ccc}
 \text{Par} & \xrightarrow{\text{IDL}} & \text{End}(\text{D}(\text{Bn}_G)) \\
 \downarrow \text{IK} & & \downarrow \approx \text{E} \\
 \text{D}(\overline{\text{Bn}}_G) & \xrightarrow{\bar{\Delta}!} & \text{D}(\text{Bn}_G \times \text{Bn}_G)
 \end{array}$$

Proof of Thm 2:

Step 1: To construct

$$\bar{\Delta}_! \circ \text{IK} \longrightarrow \text{E} \circ \text{IDL}$$

$$\bar{\Delta}_! \circ \text{IK}(p) \longrightarrow (\text{E} \circ \text{IDL})(p)$$

Lemma 4:

$$\begin{array}{ccc}
 \overline{\text{Bn}}_{G,p} & \xrightarrow{\exists \text{ canonical}} & \text{Bn}_G^{p\text{-gen}} \times \text{Bn}_G \\
 \downarrow & & \downarrow \\
 \overline{\text{Bn}}_G & \longrightarrow & \text{Bn}_G \times \text{Bn}_G
 \end{array}$$

Lemma 5:

$\exists$ functorial (in p)

$$\overline{Bun_G, \mathbb{CP}} \longrightarrow Bun_G^{p\text{-gen}} \times Bun_G^{p^-\text{-gen}}$$

s.t. for $p = G$, it is Δ

Proof for SL_2 : Only need to prove
for $p = B$

$$VinBun_G, B = \{ \Sigma_1 \xleftarrow{\phi} \Sigma_2, \det \phi = 0 \}$$

$\phi \neq 0$
 $\text{Im } \phi$ is a general line bundle

$$\{ \text{in } \phi \hookrightarrow \Sigma_1 \} \in Bun_G^{B\text{-gen}}$$

$$\{ \Sigma_2 \rightarrow \text{in } \phi \} \in Bun_G^{B^-\text{-gen}} \quad \square$$

$$\begin{array}{ccc} \overline{B_{\infty, p}} & \xrightarrow{\exists \text{ canonical}} & B_{\infty}^{\rho\text{-gen}} \times B_{\infty} \\ \downarrow & & \downarrow \\ \overline{B_{\infty}} & \xrightarrow{\quad\quad\quad} & B_{\infty} \times B_{\infty} \end{array}$$

$$\begin{array}{ccc} \bigcirc & \xrightarrow{\text{!-push}} & \mathcal{F}_p \text{ (defined)} \\ \downarrow \text{!-push} & & \downarrow \\ k(p) & \xrightarrow{\quad\quad\quad} & \bar{\Delta}_! \circ k(p) \end{array}$$

Lemma 6 :

$$\mathcal{F}_p \in \mathcal{H}(G \times G, p \times G) \subset \mathcal{D}(B_{\infty}^{\rho} \times B_{\infty}^{\rho})$$

$$\begin{array}{ccc} \downarrow & & \downarrow \bar{\Delta}_!^{\text{enh}} \\ \bar{\Delta}_! \circ k(p) & \in & \mathcal{D}(B_{\infty} \times B_{\infty}) \end{array}$$

$$\bar{\Delta}_! \circ k(p) \in \mathcal{D}(B_{\infty} \times B_{\infty})$$

$$\bar{\Delta}_!^{\text{enh}}(\mathcal{F}_p) \simeq \bar{\Delta}_! \circ k(p)$$

By functoriality in P

$$\Rightarrow \overline{E}_S^{en}(\sigma_P) \longrightarrow \sigma_G^{\Delta!k}$$

$$\Rightarrow \theta_P : \mathcal{F}_P \longrightarrow \mathcal{C}_T^{en} \sigma_G$$

$$\Rightarrow \overline{E}_S^{en}(\theta_P) :$$

$$\simeq \overline{\Delta}_{!0}(K(p))$$

$$\overline{E}_S^{en} \Big|_{P \times G \rightarrow G \times G} (\sigma_P)$$

$\downarrow$

$$\overline{E}_{S \times P \times G \rightarrow G \times G}^{en} \circ \mathcal{C}_{G \times G \rightarrow P \times G}^{en} (\Delta!k)$$

is

$$(\overline{E}_{S \times P \times G}^{en} \circ \mathcal{C}_{G \times P}^{en} \otimes Id) (\Delta!k)$$

$$\simeq [E \circ Id]_L(p)$$

Claim $\Rightarrow$

$$\Delta_1(K \rightarrow E_0(1)L$$

Now goal:

$$\theta_p: \mathcal{F}_p \rightarrow \text{Cen}(\mathcal{F}_G)$$

is injective (in

$$\Pi(G \times G, p \times G)$$

Step 2:

$$\Pi(G \times G, p \times G) \xrightarrow{\text{Lin}_G} D(B_{\text{un}} \times B_{\text{un}})$$

$$D(B_{\text{un}}^{p\text{-ser}} \times B_{\text{un}}) \xrightarrow{\text{Lp} \times G} D(B_{\text{un}}^p \times B_{\text{un}})$$

$\uparrow q_{p \times G, *}$

Now goal: $\text{Lin}_G(\theta_p)$ is $\subseteq$

$$L_{M \times G} = D_{\text{mod}}^R \left(B_{M \times G} \xleftarrow{M \times G\text{-gen}} B_{P \times G} \xrightarrow{P \times G\text{-gen}} B_{G \times G} \right)$$

The (2nd algorithm $Y_{G \times G}$) $\left\{ \right\}$

$$L_{D_{\text{mod}}} \left(B_{M \times G} \xleftarrow{M \times G\text{-gen}} B_{P \times G} \xrightarrow{P \times G\text{-gen}} B_{G \times G} \right)$$

Rank: $C T_{P \times G, !}^{\text{gen}}$

- $\text{Thm}(1) - G$

$$\begin{aligned} \text{rep} &= P \times G & G \cap G \times G \\ & & \text{att} \approx P \times G \\ \text{log} &= M \times G & G \cap P \times G \\ & & \text{att} = P \times G \end{aligned}$$

$$D_{\text{mod}}^R (B_{M \times G} \xleftarrow{P} B_{P \times G} \xrightarrow{P} B_{G \times G})$$

$\left\{ \right\}$

$$L_{D_{\text{mod}}} (B_{M \times G} \xleftarrow{P} B_{P \times G} \xrightarrow{P} B_{G \times G})$$

- $\text{Thm}(\text{Braden})$

$$\text{For } G \cap \mathbb{Z}$$

$$D_{\text{nd}}^R (z^{\text{fix}} \xleftarrow{\quad} z^{\text{att}} \xrightarrow{\quad} z)$$

$$D_{\text{nd}}^L (z^{\text{fix}} \xleftarrow{\quad} z^{\text{rep}} \xrightarrow{\quad} z)$$

for G_m -monodromic objects
on Z .

— General framework by Drinfeld:

If b is a nice enough assignment

$$D_{\text{nd}}^R (b(z^{\text{fix}}) \xleftarrow{\quad} b(z^{\text{att}}) \xrightarrow{\quad} b(z))$$

$$D_{\text{nd}}^L (b(z^{\text{fix}}) \xleftarrow{\quad} b(z^{\text{rep}}) \xrightarrow{\quad} b(z))$$

$$\left(\begin{array}{l} G_n \cap G \text{ s.t. } G^{\text{att}} = P \\ b = B_n() \quad G^{\text{rep}} = P^- \\ G^{\text{fix}} = M \end{array} \right)$$

Drinfeld $\Rightarrow$ D-G

New goal:

$$CT_{p \times G, !}^{gr}(\mathcal{F}_p) \xrightarrow{\text{G} \swarrow \text{Nat}} CT_{!}^{gr} \circ CT^{enh}(\mathcal{F}_G)$$

is

$$CT_{p \times G, !}(\mathcal{F}_G)$$

$$\text{Wet: } \theta_p: \mathcal{F}_p \rightarrow CT_{G \hookrightarrow p \times G}^{\text{enh}}(\mathcal{F}_G) \rightarrow \simeq$$

$$L_{M \times G}^{\text{enh}}(\theta_p)$$

 $\Uparrow$
 $\rightarrow \simeq$
 $\Uparrow \text{ (2nd adj.)}$

$$CT_{p \times G, !}^{\text{sen}}(\theta_p) \rightarrow \simeq$$

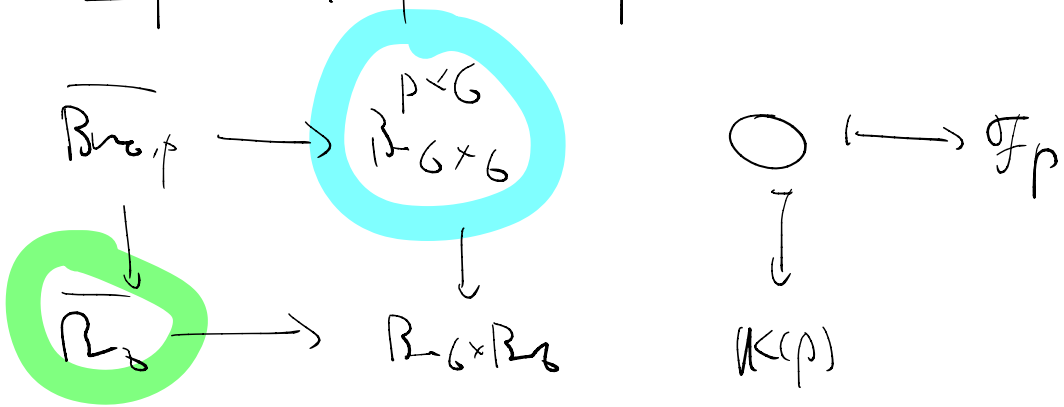
$$CT_{p \times G, !}^{\text{sen}} : B_{M \times G} \xleftarrow{!} B_{p \times G}^{M \times G} \xrightarrow{*} B_{G \times G}^{p \times G}$$

$$\underline{CT_{p \times G, !}^{\text{sen}}(\mathcal{F}_p)} \xrightarrow{\gamma_p} \underline{CT_{p \times G, !}^{\text{sen}}(\mathcal{F}_G)}$$

$$\left(CT_{p \times G, *} \simeq L_{M \times G}^{\text{enh}} \circ CT_{G \hookrightarrow p \times G}^{\text{enh}} \right)$$

New goal: γ_p is $\simeq$

Step 3: Pickout γ_p



By base-change,

$$LHs(\gamma_p) = CT_{p \times G, 1}^{gen}(\sigma_F)$$

$$= D_{mod}^L(Br_{M \times G} \leftarrow Br_{p-}^{M-gen} \times_{Br_G^{P-gen}} \overline{Br}_{G,p} \rightarrow \overline{Br}_G) \quad (H^1(K(p)))$$

$$RHs(\gamma_p) =$$

$$1) D_{mod}^L(Br_{M \times G} \leftarrow Br_{p-} \times_{Br_G} \overline{Br}_{G,G} \rightarrow \overline{Br}_G) \quad (H^1(G))$$

$\nwarrow \quad \nearrow$
 $Pr_{13} \quad Br_{M \times Br \times Br_G}$

Lemma 7: $\exists$ canonical open subscheme

$$(B_{\mathcal{O}_P} \times_{B_{\mathcal{O}_G}} \overline{B_{\mathcal{O}_G, \geq p}})^{\text{gen}}$$

$$B_{\mathcal{O}_P} \times_{B_{\mathcal{O}_G}} \overline{B_{\mathcal{O}_G, \geq p}} \quad \text{s.t.}$$

$$\beta \quad B_{\mathcal{O}_{M \times G}} \leftarrow (\quad)^{\text{gen}} \rightarrow \overline{B_{\mathcal{O}_G}}$$

$$\downarrow$$

$$T_{\text{ad}, \geq p} / T$$

the $\mathcal{P}$ -strata of the above corr. is
the corr. in $\text{Chs}(\mathcal{O}_P)$

- - - $\mathcal{G}$ -strata of - - -
- - - $\text{Rhs}(\mathcal{O}_P)$

Proof for SL_2 :

$$B_{m_B^-} \times_{B_G} V_{in} B_{m_C} = \{ L_1 \leftarrow \Sigma_1 \leftarrow \Sigma_2 \mid \phi \neq 0 \}$$

$(\quad)^{gen}$ is s.t. $\Sigma_L \rightarrow L_1$
is nonzero

Claim: $Corr^L(\beta) \circ \llcorner_K$ sends

$$p \rightarrow G \quad \text{to} \quad \delta p$$

New goal: $Corr^L(p)$ sends

$$\llcorner_K(p) \rightarrow \llcorner_K(G) \quad \text{to} \quad \simeq$$

$$\beta := \left(B_{m_{m_C}} \leftarrow (B_{m_B^-} \times_{B_G} \overline{B_{m_C} \rightarrow p})^{gen} \rightarrow \overline{B_{\delta}} \right)$$

Step 4: restoring the symmetry

Lemma 8:

$$B_{M \times G} \leftarrow ()^{\text{gen}} \quad \nwarrow \quad \nearrow \quad \textcircled{\exists}$$

$$B_{M \times G} \times B_G^{p^{-\text{gen}}}$$

$$\left| \text{Pref SL: } \{ L_1 \leftarrow \varepsilon_1 \leftarrow \varepsilon_2, \text{ comp. } \neq 0 \} \right.$$

$$\left. \sim \right) L_1 \leftarrow \varepsilon_2 \in B_G^{p^{-\text{gen}}}$$

New goal: replace β by β' :

$$B_{M \times G}^{M \times p^{-\text{gen}}} \leftarrow ()^{\text{gen}} \longrightarrow \overline{B_G}$$

i.e. $\text{Dnd}^L(\beta') \text{ ok and } p \nmid G \text{ to } \sim$

Claim: The image is contained in

$$\mathcal{I}(M \times G, M \times p^-) \subset \mathcal{D}(B_{M \times G}^{M \times p^{-\text{gen}}})$$

□

As before:

New goal $\xleftarrow{\text{conservatively}}$

$$\text{Dnd}^R (B_{m \times p} \leftarrow B_{m \times p} \rightarrow B_{m \times p - g}) \circ \text{Dnd}^L(\beta')$$

$o|K$ sends $p \rightarrow G$ to $\subseteq$

2nd option

$\delta \Leftarrow$

$$\text{Dnd}^L (B_{m \times m} \xleftarrow{m \times m - \text{gen}} B_{m \times p} \xrightarrow{m \times p - g} B_{m \times G}) \circ \text{Dnd}^L(\beta')$$

$o|K$ - - - - -

New goal: $\text{Dnd}^L(\delta, \beta') \circ |K$ sends

$p \rightarrow G$ to $\subseteq$

$\delta \circ \beta'$:

$$B_{m \times m} \leftarrow Y_{K|}^P / Z_m \rightarrow B_G$$

$$\begin{aligned}
 & \overline{Ex}(SL_2) \\
 & Y_{|h|}^B = \left\{ \begin{array}{l} L_1 \xrightarrow{f} L_1 \xrightarrow{\phi} L_1 \xrightarrow{g} L_2 \\ f \in B \sim B^-, g \in B \sim B \\ \phi \in VA_{B_0} \\ \text{Group} \neq 0 \end{array} \right\} \\
 & \cong \left(B \sim_{B_0} B^- \times VA_{B_0} \times B \sim_{B_0} B \right)^{\text{group}}
 \end{aligned}$$

Step 5:

$$\begin{aligned}
 \text{Fact: } B_{M \times M} & \longleftarrow Y_{|h|}^P / \mathbb{Z}_m \\
 & \uparrow \swarrow \text{canon.} \\
 & H_{m, G\text{-ps}} / \mathbb{Z}_m
 \end{aligned}$$

$$\begin{aligned}
 & \overline{Ex}(SL_2) \\
 & H_{m, G\text{-ps}} = \{ L_1 \leftarrow L_2 \text{ no zero } \},
 \end{aligned}$$

$$\varphi_{p \rightarrow p}: H_{m, G, p}/ZM \leftarrow Y_{G, p}/ZM \rightarrow \overline{B_{r, G, p}}$$

$$\text{New goal: } \text{Dom}^L(\varphi_{p \rightarrow p}) \circ j_{p \rightarrow p}^* \text{ ok}$$

$$\text{Sees } D \rightarrow G \text{ to } \simeq$$

Claim: $\text{Dom}^L(\varphi_{p, 2p})$ (2nd adjoints type)

$$\subseteq \text{Dom}^R \left(\left(H_{m, G, p}/ZM \leftarrow \right. \right. \right. \\ \left. \left. \left. \text{def str } \overline{B_{r, G, p}} \xrightarrow{\quad} \overline{B_{r, G, 2p}} \right) \right)$$

Fact: $\overline{B_{r, G, p}}$ has a defect str.

$$\text{def str } \overline{B_{r, G, p}} := \bigsqcup_{\text{def str}} \text{strata}$$

$$\text{Ex. sl}_2: \forall \lambda \in \mathfrak{h}^*, \lambda = \{ \tau_1 \in \tau_2, \text{def str} \}$$

$$\tau_1 \hookrightarrow \mu_1 \hookrightarrow \mu_2 \hookrightarrow \tau_2, \deg(\mu_2) - \deg(\mu_1)$$

New goal:

$$\text{Dnd}(j) \circ j_p^* \circ \text{ll} \text{ sets } p \rightarrow G$$

$$\Leftarrow \text{ip} \circ j_p^* \circ \text{ll} \text{ sets } p \rightarrow G$$

to =

$$\text{ip} : \overline{B_{G,p}} \longrightarrow \overline{B_G}$$

$$\text{ll}(p) := \text{ip}_! \circ \text{ip}^* \circ j_{G,*} \circ r_!(h) [\text{-code}(p)]$$

New goal:

$$\text{ip}^* \circ j_{G,*} [\text{-code}(p)] \longrightarrow \text{ip}_! \circ j_{G,!}$$

sets $r_! h$ to $\cap$

This is correct for $\text{rank}(p) = 1$

$\square SL_2$ for much then