Separation of variables
(E. Sklyanin)

$f_{\lambda_1 \dots \lambda_n}(x_1, \dots, x_n)$

$\rightsquigarrow$ Math kernel function in $3n$ variables

$\Psi_{\lambda_1 \dots \lambda_n}(x)$ $\uparrow$ 1 variable

$x \in C$ curve

$f_{\lambda_1 \dots \lambda_n}(\underbrace{x_1, \dots x_n}_{\text{Symm.}}) := \Psi_{\lambda_1 \dots \lambda_n}(x_1) \cdots \cdots \Psi_{\lambda_1 \dots \lambda_n}(x_n)$

$\mathrm{Sym}^n C$

$T^* \mathrm{Sym}^n C$

$\approx T^*(\mathrm{Bun}_{GL_r})$

Birat. symplect

$n = \dim \mathrm{Bun}_{GL_r} = g(\text{spectral curve})$

$\Psi_{\lambda_1 \dots \lambda_n}(x) = e^{\lambda_1 x + \lambda_2 x^2 + \dots + \lambda_n x^n}$

$x \in \mathbb{A}^1$

$f_{\lambda_1 \dots \lambda_n}(x_1 \dots, x_n) = e^{\lambda_1 (\underbrace{x_1 + \dots + x_n}_{})} e^{\lambda_2 (\underbrace{x_1^2 + \dots + x_n^2}_{})} \cdots e^{\lambda_n (x_1^n + \dots + x_n^n)}$

coordinates on $\mathrm{Sym}^n \mathbb{A}^1$

$$\rightsquigarrow \exists!\, K_{n+1,n}\big(\overbrace{x_1 \dots x_{n+1}}^{\Sigma_{n+1}}, \overbrace{y_1 \dots y_n}^{\Sigma_n}\big)\, dy_1 \dots dy_n$$

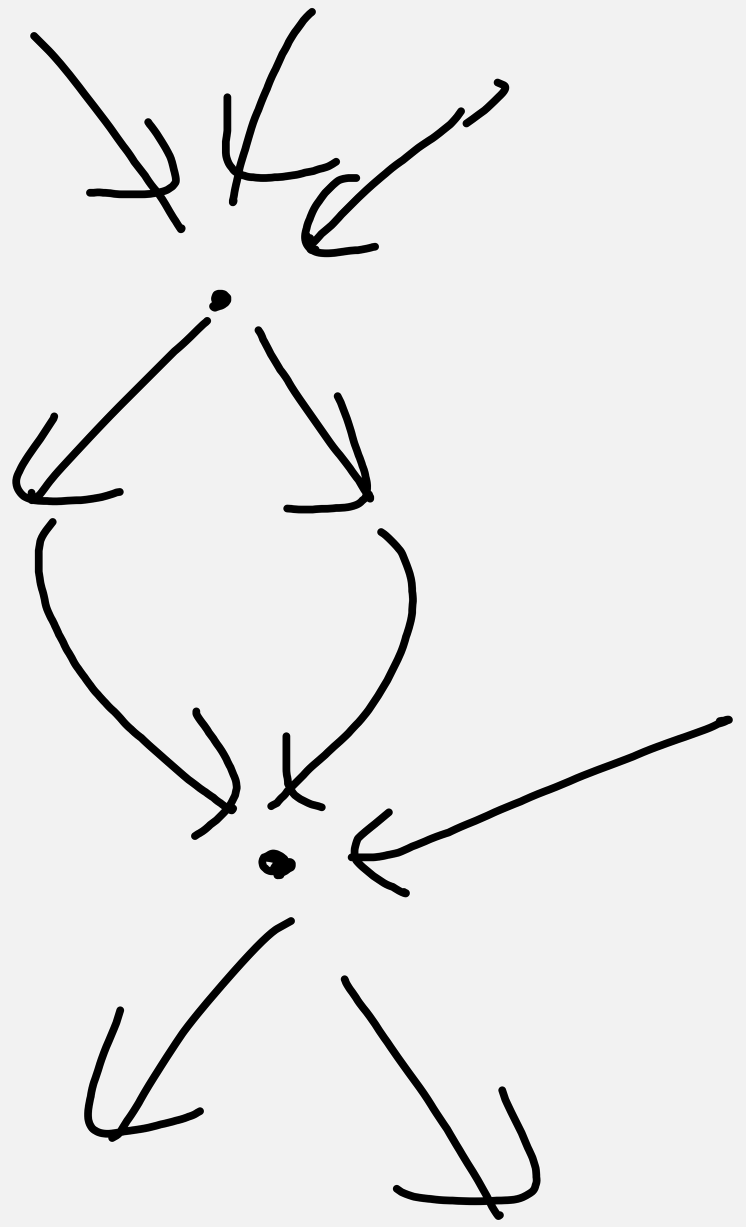
$$\psi_{\lambda_1\dots\lambda_n}(x_1)\cdots\psi_{\lambda_1\dots\lambda_n}(x_{n+1}) = \int K_{n+1,n}(x_1 \dots x_{n+1}, y_1 \dots y_n)\,\psi_{\lambda_1\dots\lambda_n}(y_1)\dots\psi_{\lambda_1\dots\lambda_n}(y_n)\, dy_1 \dots dy_n$$

In $\otimes$-categories: V vector space

$$Sym^{n+1} V \to Sym^n V$$

(usual comm. ass. $n=1$)

Axiom: compose with itself is $\Sigma_{n+2} \times \Sigma_n$-invariant.



$\Rightarrow Sym^n V$ is a comm. assoc. algebra

V = functions in 1 variable x

Input: $t_1 \dots t_{n+2} \in \mathbb{C}$ $(t_{n+3} = \infty)$ n+3 pts

$n \geq 1$

$k_1 \dots k_{n+2} \in \mathbb{C}$. $s \in \mathbb{C}$

(k_{n+3})

$$\rightsquigarrow \quad P(x) = \prod_{i=1}^{n+2} (x - t_i) \quad \exists ! \, Q \in \mathbb{C}[x] \text{ s.t. } Q(t_i) = k_i^2 \, P'(t_i)^2$$

$P \in \mathbb{C}[x]$ $\deg Q \leq n+1$

$$D_x = \frac{\partial}{\partial x} \cdot P \cdot \frac{\partial}{\partial x} - s(s+n+1)\, x^n - \frac{Q}{P} + (\lambda_1 x^{n-1} + \lambda_2 x^{n-2} + \dots + \lambda_n)$$

dep. on $\lambda_1 \dots \lambda_n$

Sol. of eqn $D_{x, \lambda_1 \dots \lambda_n} \psi_{\lambda_1 \dots \lambda_n} = 0$

if $x \to t_i$ $\psi_{\lambda_1 \dots \lambda_n} \sim (x - t_i)^{\pm k_i}$

if $x \to \infty$ $\psi_{\lambda_1 \dots \lambda_n} \sim x^{s}$ or x^{-s-n-1}

$$K_{n+1,n}(x_1, \dots, x_{2n+1}) =$$

Σ_{2n+1} inv.

$$= \underbrace{\int \cdots \int}_{(n+2)\text{ dim}} \prod_{i=1}^{n+2} \left(w_{i,+}^{+k_i} \, w_{i,-}^{-k_i} \right) \cdot \left(\sum_i \left(w_{i,+} + w_{i,-} \right) \right)^{+2s} \bigwedge \frac{d w_i^{+}}{w_j^{+}}$$

in fact $(n+1)$-dim

constraint

$$\left(\sum_i \left(w_{i,+} + w_{i,-} \right) \right)^{+2s} \rightsquigarrow \times \text{const} \left(\sum_i w_{i,+} \right)^{+s-k_1-\cdots-k_{n+2}} \left(\sum_i w_{i,-} \right)^{+s+k_1+\cdots+k_{n+2}}$$

$$\forall i \quad w_{i,+} \cdot w_{i,-} = \frac{\prod_{\alpha=1}^{2n+1} (x_\alpha - t_i)}{p'(t_i)^2}$$

$= 1 \sim n+2$

$n=1$ 4 pt

2-dim $\int$.

$$\int_{\triangle} q_1^{*} \, q_2^{*} \, (1-q_1-q_2)^{*} \left(q_1 q_2 + A q_1 + B q_2 \right)^{*} dq_1 \, dq_2$$

1

0 1

C : compact curve$/\mathbb{C}$