Hitchin system gl_n + singular.

<u>Plan</u>:

① Classical picture for $K_{n+1,n}$ (more generally $K_{n+?,n}$)

② Conj. description for quantum version

Kernel: hol. - D-module with a section (cyclic vector)

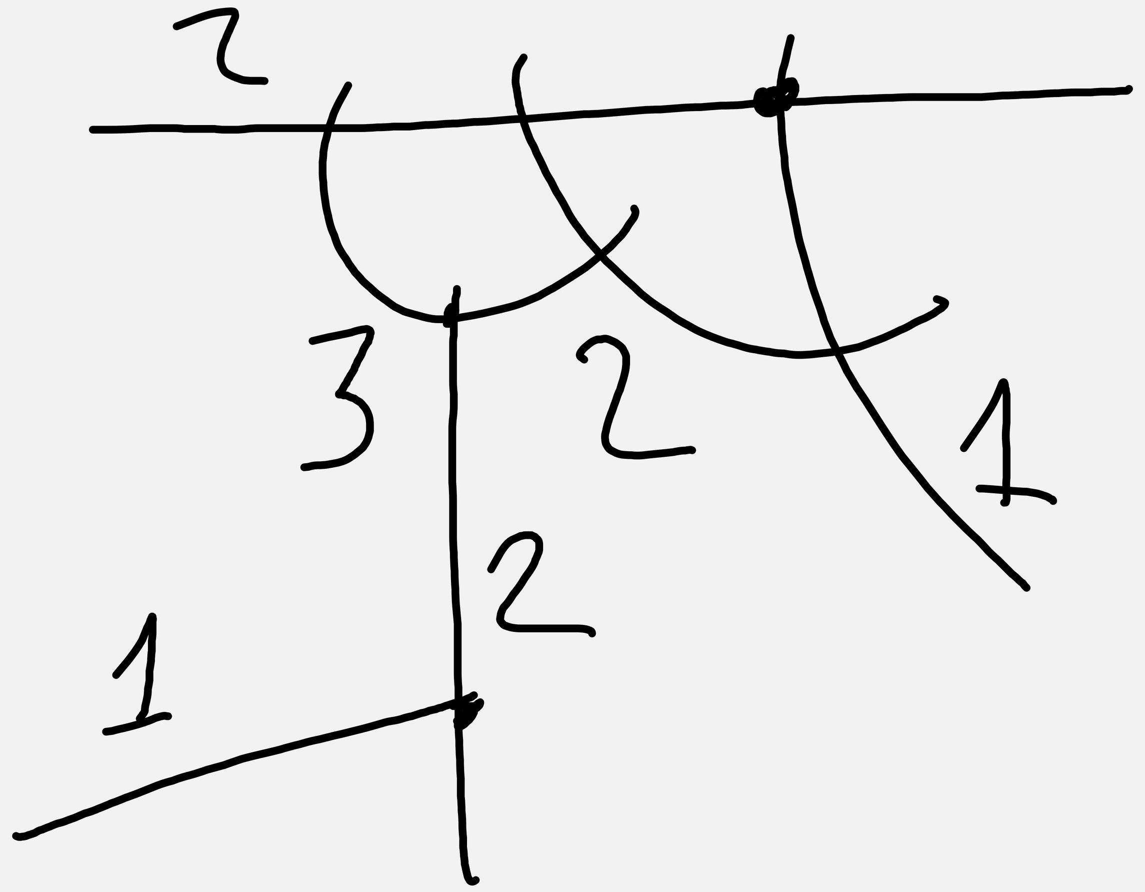
③ Speculations about generalizations to q-diff. eqns.

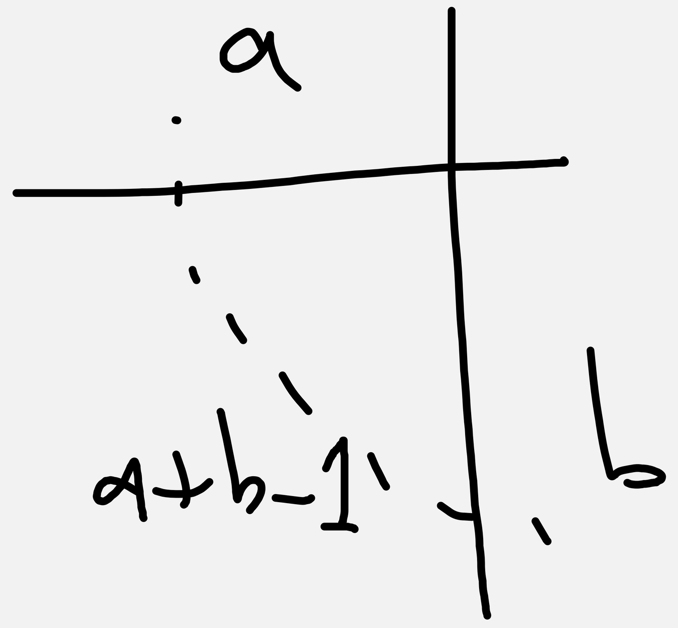
C smooth compact curve $/\mathbb{C}$

$\to S = T^*C$ surface with sympl. form ω

$\overline{S} := S \cup C_\infty = \mathbb{P}(T_C^* \oplus \mathcal{O}_C)$

Poisson surface



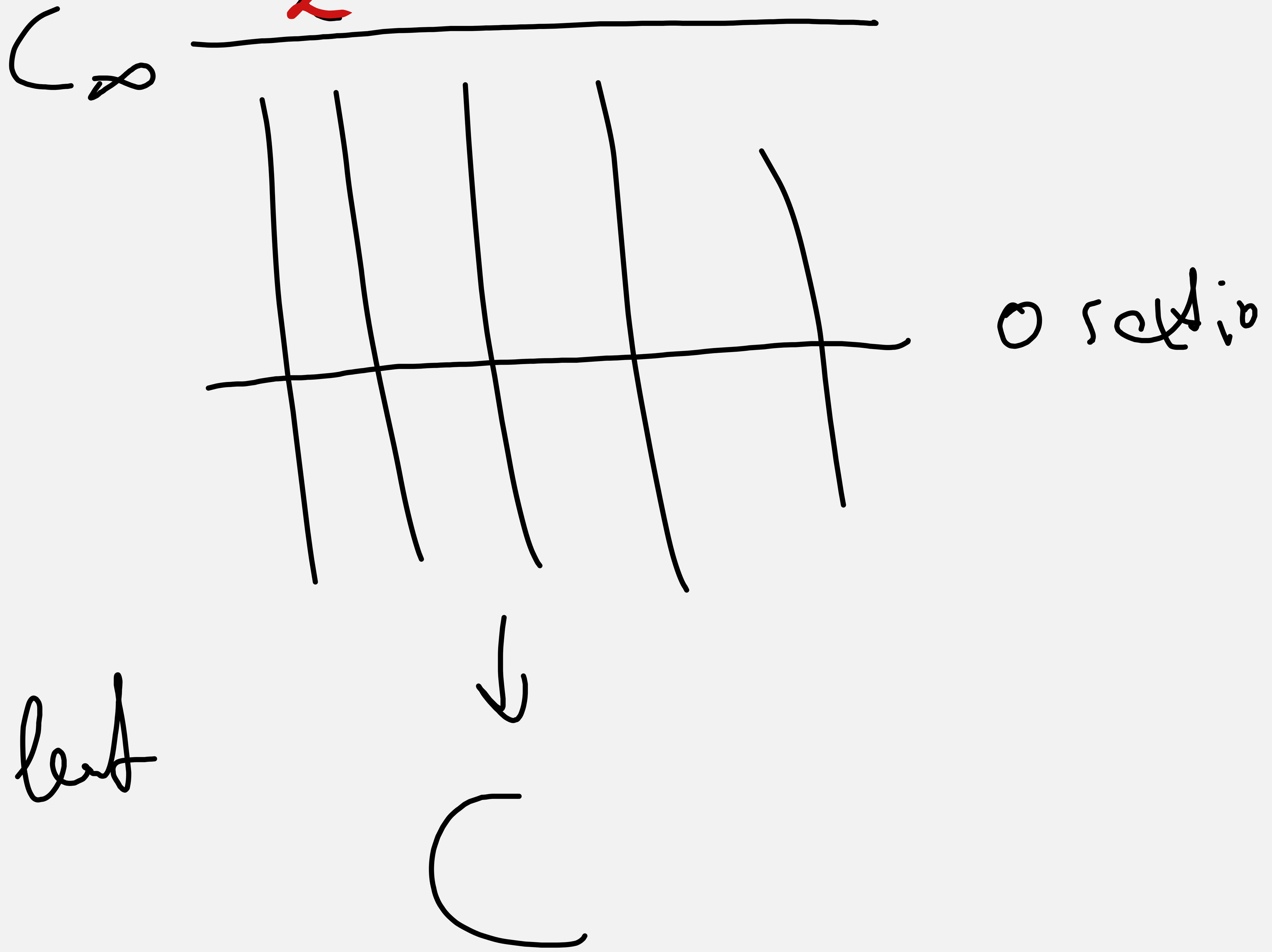


new Poisson surface $P > S$
symple leaf

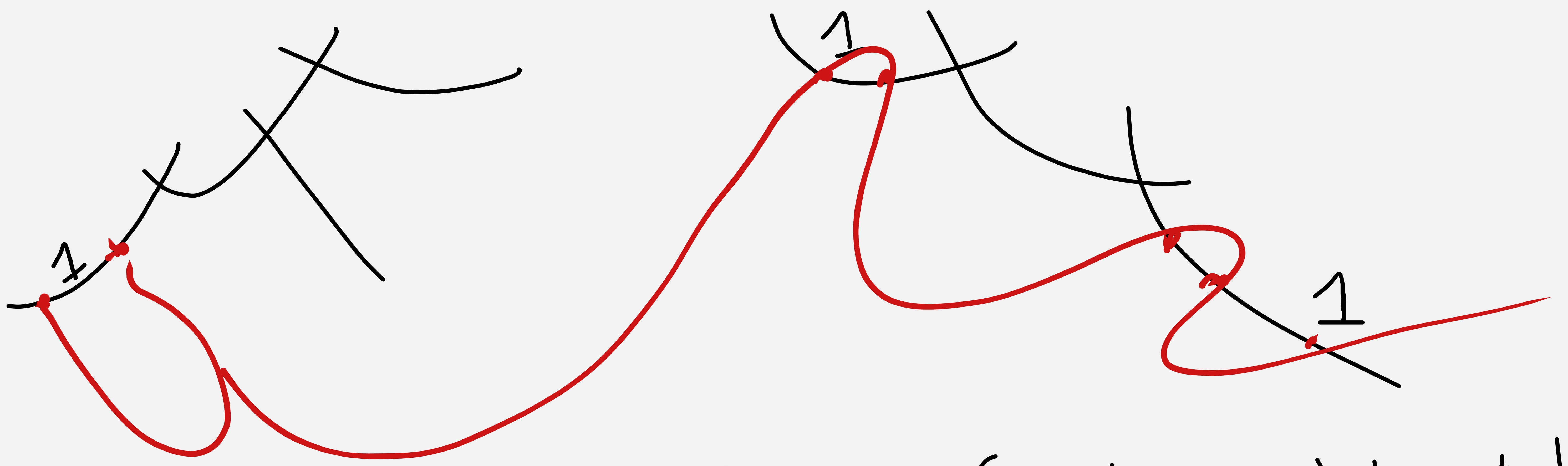
$S = T^*C$
$\downarrow$ fibers $\simeq \mathbb{A}^1$
C

$\mathbb{A}^1 \subset \mathbb{P}^1$

ω pole of order 2



P



Lemma: all divisors where ω has pole of order 1 are disjoint, each $\underset{\text{canon.}}{\simeq} \mathbb{A}^1$

Fix a coll. of pts $\{p_\alpha\} \subset \coprod \mathbb{A}^1$

1st order pole

Assume gen. member is smooth

$$\mathcal{B} = \Big\{ \underset{\text{spectral curves}}{\Sigma \subset P} \;\Big|\; \Sigma \underset{\text{trans.}}{\pitchfork} (P-S), \quad \cap \;=\; \{p_\alpha\} \Big\} \simeq \underset{\text{affine space}}{\mathbb{A}^n}$$

$$n = g(\Sigma)_{\text{generic}} = g$$

$$\begin{array}{c} \mathrm{Bundle}_d \\ \downarrow \\ \mathcal{B} \end{array} \quad \text{fiber}\big|_\Sigma : \ \mathrm{Pic}_d(\Sigma) \qquad d \in \mathbb{Z}.$$

$$\Big\downarrow \mathrm{Pic}_d(\Sigma)$$

Base

$d = g$

$$\underset{\substack{\text{canon}\\ \text{bir. symplect.}}}{\simeq} \quad \mathrm{Sym}^g(T^*C) \underset{\text{bir.}}{\simeq} T^*(\mathrm{Sym}^g C)$$

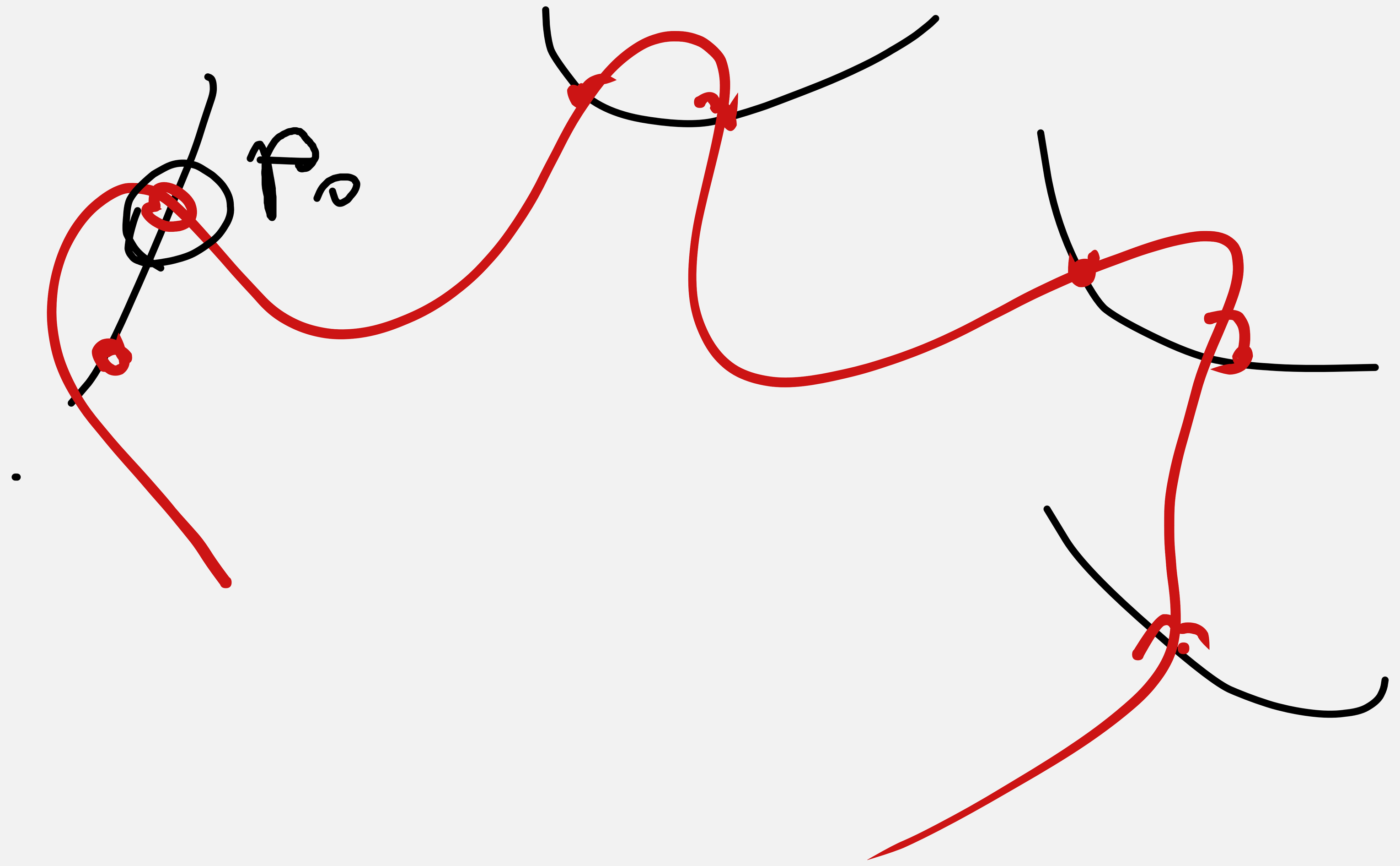
$\mathrm{Pic}_g(\Sigma) = \mathrm{Sym}^g \Sigma$

$\forall$ smooth Σ

$[\Sigma] \in B \qquad \overline{b_1, \dots, b_g \in \Sigma \,/\, \mathrm{perm}} \rightsquigarrow \overline{b_1 \dots b_g \in T^*C \,/\, \mathrm{perm}}$

Conversely: $b_1 \dots b_g \in T^*C \qquad \exists!\, \Sigma \ni b_1, \dots, b_g$

Pick one of points $p_0 \in \{p_\alpha\}$



Using p_0 we can identify all

Bundles$_d$ for $d \in \mathbb{Z}$

opposite sympl. str.

$\rightsquigarrow$ Lagr. subvariety $L_{g+1,g} \subset \mathrm{Sym}^{g+1}\overline{T^*C} \times \mathrm{Sym}^g T^*C$

or $L_{g+k,g}$

$\forall k \geq 1$

$$\left\{(a_1, \dots, a_{g+1}; b_1, \dots, b_g) \;\middle|\; \begin{array}{c} \exists \Sigma \ni a_1, \dots, a_{g+1}, b_1, \dots, b_g \\ a_1 + \dots + a_{g+1} = b_1 + \dots + b_g + p_0 \\ \text{in } \mathrm{Pic}(\Sigma) \end{array} \right\}$$

Lesson: need $p_0 \in \{p_n\}$

Meaning of Divisors $<$ P-S

$\cong \mathbb{A}^1$, 1st order pole

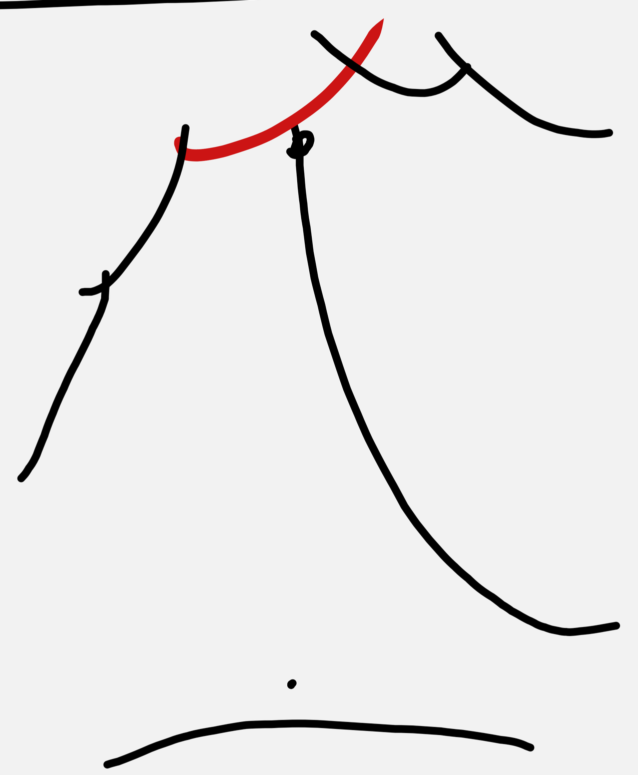
$\xleftrightarrow{1:1}$ point of $\mathbb{C}$

+ irreg. term at this point:

$e^{F(x)}$

F: Puiseux series

graph dF



$\mathbb{C}$ $\quad x$ local coordinate.

$x=0$

Bundles on $\mathbb{C}((x))$ with connection.

Ab. category. HLT theorem

$= \bigoplus$ blocks

SS

∞ Local syst on S^1

$\sum_{a/b \in \mathbb{Q}_{<0}} c_{a/b}\, x^{-a/b}$ $\quad \not\exists$ sum rep. sing.

$e^{\left(\overline{\mathbb{C}((x))}/\text{ring of integers}\right)} / \hat{\mathbb{Z}} = \mathrm{Gal}\left(\overline{\mathbb{C}((x))}/\mathbb{C}((x))\right)$

"Same" data P

$\{p_\alpha\} \subset \coprod \mathbb{A}^1$

gives a class of Bundles with irr singul. $/\mathbb{C}$ — pts (given irr. term)

$$e^{\sum_{a/b<0} c_{a/b,\alpha} x^{-a/b}} \cdot x^{\lambda_\alpha}$$

$p_\alpha = \lambda_\alpha \in \mathbb{A}^1$

Get a $2g$-dim. sympl. variety

$\cup$

g-dim Lag. subvariety of opers,

$g = \deg\left(\begin{array}{c}\Sigma \\ \downarrow \\ C\end{array}\right)$

$\boxed{\text{Distinguished pt } P_0}$

pf p_0: Assume corr. irrg. term $e^{\sum c_{a/b,0} x^{-a/b}} \cdot x^{\mu_0}$ $\boxed{a/b \in \mathbb{Z}_{\geq 1}}$

$\lambda \in$ Opers

$(\mathbb{A}^1)^g$ dep. on g parameters

$\lambda_1, \dots, \lambda_g$

$\deg \lambda_i = i$

$\deg P_k = k$

$\exists!$ solution of DPER

expansion at my point

$\mathbb{C}((x))$

$$e^{\sum_{a=1,2,\dots} c_{a,0} x^{-a}} x^{\mu_0} \left(1 + \dots \underbrace{x}_{\mathbb{C}[\lambda_\cdot][[x]]}\right)$$

$$P_0(\lambda) + P_1(\lambda)x + P_2(\lambda)x^2 + \dots$$

$P_0(\lambda) = 1$

$$\left[\frac{\partial}{\partial x} x(x-1)(x-1)\frac{\partial}{\partial x} + x - \lambda\right]\psi(x) = 0$$

$$\exists!\ \psi = 1 + \lambda x + \left(\frac{\lambda^2}{2} + \dots\right)x^2 + \dots$$

$P_0 = 1,\ P_1,\ P_2 \dots$

Claim: $\{P_{i_1} P_{i_2} \dots P_{i_g}\}_{i_1 \le \dots \le i_g}$ is a linear basis of $\mathbb{C}[\lambda_1, \dots, \lambda_g]$

$= \mathcal{O}(\mathrm{Opers})$

$\forall\, i_1 \dots i_{g+1},\ j_1 \dots j_g$

$$C^{j_1 \dots j_g}_{i_1 \dots i_{g+1}} := \text{Coeff of } P_{i_1} \dots P_{i_{g+1}} \text{ for basis element } P_{j_1} \dots P_{j_g}$$

$$K_{g+1,g}(x_1, \dots, x_{g+1}; y_1 \dots y_g) := \sum C^{j_1 \dots j_g}_{i_1 \dots i_{g+1}} x_1^{i_1} \dots x_{g+1}^{i_{g+1}} \cdot y_1^{-j_1} \dots y_g^{-j_g} \cdot \prod_1^g \frac{dy_i}{y_i}$$

"$g=1$" $\quad e^{\lambda x} = 1 + \lambda x + \frac{\lambda^2}{2!}x^2 + \frac{\lambda^3}{3!}x^3 + \dots$

$$\lambda = \lambda_1 \quad P_i(\lambda) = \frac{\lambda^i}{i!} \qquad C^j_{i_1 i_2} = \delta_{j, i_1+i_2} \frac{j!}{i_1! \, i_2!}$$

$$\rightsquigarrow K(x_1, x_2, y) = \frac{dy}{y - x_1 - x_2}$$

Invariant meaning: $\left(\mathbb{C}[[y_1 \dots y_g]][y_1^{-1}, \dots, y_g^{-1}] \cdot\right) \mathbb{C}[[x_1 \dots x_{g+1}]]$ / part where for at least one y_i all exp ≥ 0

is module over diff. operators on $\mathbb{C}^{2g+1}$ near pt. $\mathrm{proj}(p_0)^{2g+1}$
dy.

Conjecture: $K_{g+1,g}$ (and $K_{g+k,g}$) generates a holonomic D-module (exp. + mult. character + motivic)

Claim: holds for other connections! "Loc Sys"

$\mathcal{E}_{\lambda}$: oper corr. to λ

Oper $\mathfrak{g} \subset \mathfrak{z}_{\mathfrak{g}}$

$$\int K_{g+1,g}(\underbrace{\mathcal{E}_{\lambda} \otimes \cdots \otimes \mathcal{E}_{\lambda}}_{g}) = \underbrace{\mathcal{E}_{\lambda} \otimes \cdots \otimes \mathcal{E}_{\lambda}}_{g+1 \text{ copy}}$$

(Π_1)

rational

$$(S, \omega) \longrightarrow$$

symp. surface

trigon.

$\mathbb{C}^\times \times \mathbb{C}^\times$

$$\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$$

A skew

elliptic

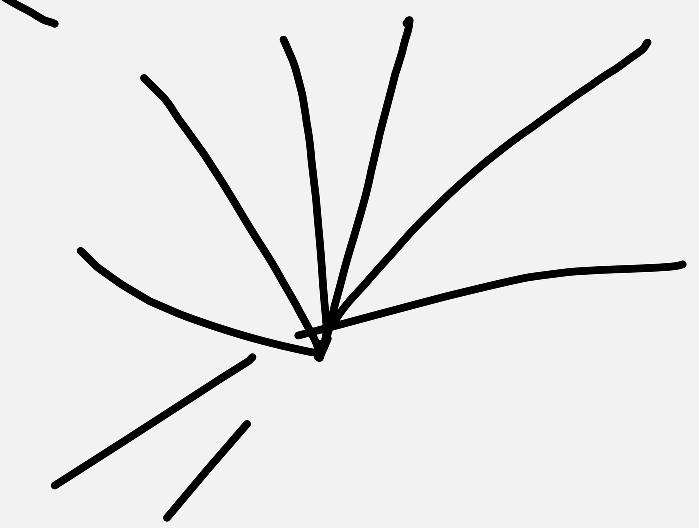
$\mathbb{CP}^2 -$ cubic curve

$$\frac{dx\,dy}{P_3(x,y)}$$

$[\omega]=0 \in H^2(S)$

div. pole of order $=1$

$= \mathbb{Q}$ rays in $\mathbb{Z}^2$



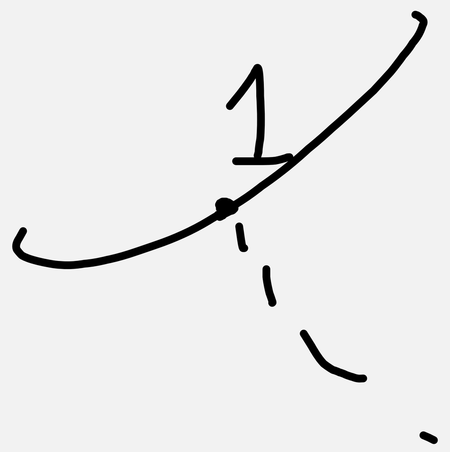
$E \times \mathbb{C}^\times_z$

ell. curve

1. form ~~A~~ $\frac{dz}{z}$

? Motivic

$T^*\mathbb{C} \cup C_\infty$



$K_{g+1,g} \in A^{g+1}\text{-mod-}A^g$

For D-modules: X alg. variety $/\mathbb{Q}$

Exponential-Motivic ?

$\mathcal{E} \in$ hol. D-module

$\rightsquigarrow$ Model $\forall p \neq 2,3,5$; $p \gg 2$

$\mathcal{E}$ mod p module over Diff op. mod p

Center

$(\mathbb{C}^\times)^{2n}$

q-diff. eqns.

A: $XY = qYX$

$X^{\pm 1}\ Y^{\pm 1}$

$(1+X+Y)\Psi = 0$

3d CS quantum

H^{b} ; $\mathbb{C}^\times$ ↓ conical

Total space 3 CY noncompact variety

$$\underset{\text{spectral curve}}{\Sigma} \subset \underset{\text{symp}}{S} \subset \underset{\text{Poiss}}{P}$$

$$S = T^*C$$

$$\begin{array}{ccc} & & 3\,CY \\ & \swarrow & \downarrow \text{ fiber ALE} \\ T^*C & & \\ & \searrow & \\ & & C \end{array}$$

ALE sm: $\mathbb{C}^2/\mathbb{Z}_n$, $\mathbb{C}^2/ADE$

$$uv = P(x,y) \subset \mathbb{C}^4$$

$$P \in \mathbb{C}[x,y]$$

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