"Multiplication kernels"

(joint work with A. Odessky, in preparation, based on discussions with V. Golyshev, V. Roubtsov, D. van Straaten).

$$e^{\lambda x}\cdot e^{\lambda y} = e^{\lambda(x+y)}$$

$$x^{\lambda}\cdot y^{\lambda} = (xy)^{\lambda}$$

$e^{\lambda x}$: solution $\frac{d}{dx}-\lambda$ normalized $f|_{x=0}=1$

x^{λ} : solution of $\frac{x\,d}{dx}-\lambda$ $f|_{x=1}=1$

$$J_0(\lambda x)\cdot J_0(\lambda y) \stackrel{?}{=} \int_{|y-x|}^{y+x} \frac{J_0(\lambda z)}{\mathrm{Area}\left(\triangle_{x,y,z}\right)}\,\frac{z\,dz}{2\pi}$$

Sonine-Gegenbauer formula

$$J_0(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}=\frac{1}{2\pi i}\oint e^{x\left(\frac{u-u^{-1}}{2}\right)}\frac{du}{u}$$

$J_0(\lambda x)$: $\left(\frac{d}{dx}\right)^2+x^{-1}\frac{d}{dx}+\lambda^2$

$|_{x=0}=1$

$\forall\lambda$

$$f_\lambda(x)\cdot f_\lambda(y) = \int K(x,y,z)\, f_\lambda(z)\, dz$$

$f_\lambda(x)$ solves

$$\frac{d}{dx} x(x-1)(x-t)\frac{d}{dx} + x - \lambda$$

$$K(x,y,z) = \frac{1}{\sqrt{(xy+yz+zx-t)^2 + 4xyz\,(1+t-(x+y+z))}}$$

General considerations in symm. mon. categories.

V vector space$/k$ $\dim V < +\infty$

Family of operators $f : \mathcal{U} \to End(V) \iff \mathcal{U} \otimes V \to V$

constraint. comm. family $\mathcal{U} \otimes \mathcal{U} \otimes V \to V$

is Σ_2-invariant.

$Sym(\mathcal{U}) \longrightarrow End\, V$

comm. alg.

$V = \bigoplus \mathcal{L}_\alpha$ $\dim \mathcal{L}_\alpha = 1$ $\forall \alpha$

Assume: spectrum is simple

basis $\{e_\alpha\}$ up to perm. & rescaling

$e_\alpha \in \mathcal{L}_\alpha - 0$

$V \otimes V \to V \otimes V$ $e_\alpha \otimes e_\alpha \longmapsto e_\alpha \otimes e_\alpha$

$e_\alpha \otimes e_\beta \longmapsto 0$ $\alpha \neq \beta$

$\Sigma_2 \times \Sigma_2$-inv

$\approx$ V is a free one-dim. module / comm. algebra

$V^{\otimes 3} \to V^{\otimes 3}$ is $\Sigma_3 \times \Sigma_3$ inv.

Assume add. that we choose a cyclic vector

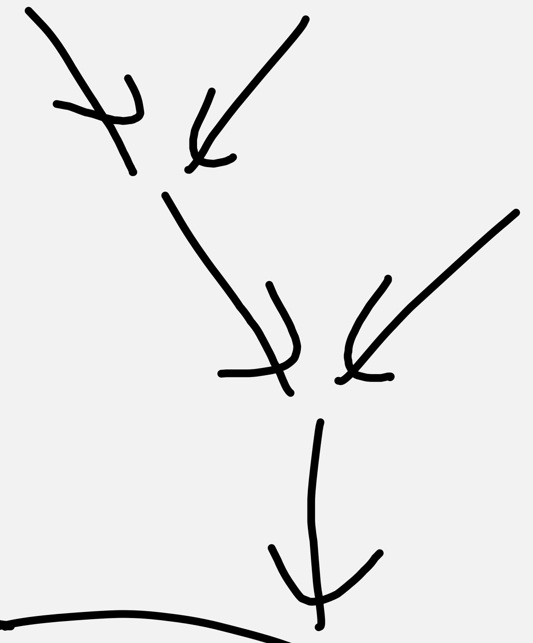
$v = \sum e_\alpha$ $\quad$ ($\Leftrightarrow$ choose $e_\alpha \in L_\alpha - 0$)

V is comm. unital assoc. alg.

$\mathbb{1} \to V$ $\qquad 1 \mapsto v = \sum e_\alpha$

$V \otimes V \to V$ $\qquad e_\alpha \otimes e_\alpha \mapsto e_\alpha$

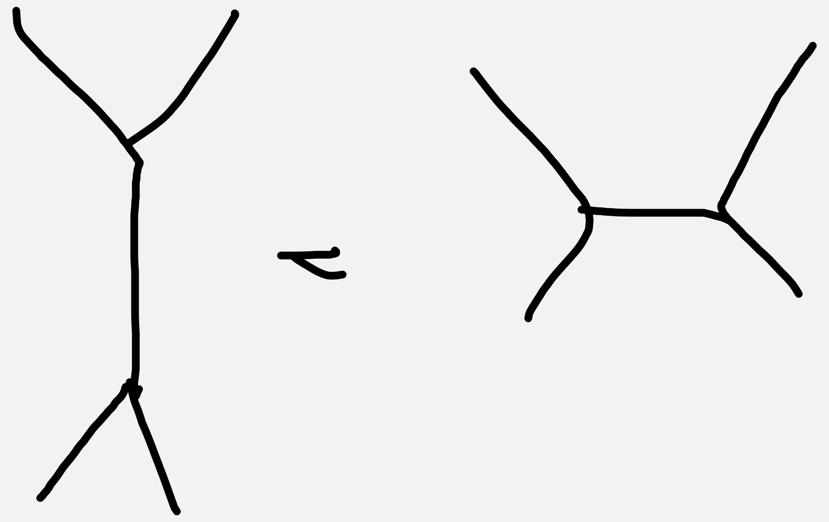
Σ_2-inv. $\qquad e_\alpha \otimes e_\beta \mapsto 0 \quad \alpha \neq \beta$



$V^{\otimes 3} \to V$
is Σ_3-inv.

Furthermore, assume we choose a cyclic covector $t \in V^*$ is nondegen.

$(a,b) := t(ab)$



Σ_4-invariant

a,b,c,d
$\to t(abcd)$
$= (a(bc), d)$

Now V ∞-dim space

functions on n-dim variety $\quad X = \mathbb{R}^n$

coordinates $x_1 \dots x_n$

"Continuous basis" $\quad f_{\lambda_1 \dots \lambda_n}(x_1, \dots, x_n)$

$$\left(f_{\lambda_1 \dots \lambda_n}(x_1 \dots x_n) := \exp\left(i\left(\lambda_1 x_1 + \dots + \lambda_n x_n\right)\right) \right)$$

$\dashrightarrow$ Multiplication $\quad V \otimes V \to V$

$$(f * g)(z_1 \dots z_n) = \underbrace{\int}_{2n\text{-dim}} \underbrace{K(x_1 \dots x_n, y_1 \dots y_n, z_1 \dots z_n)}_{3n\text{ variab}} f(x_1 \dots x_n)\, g(y_1 \dots y_n) \cdot dx_1 \dots dx_n\, dy_1 \dots dy_n$$

$$f^*_{\lambda_1 \dots \lambda_n}(x_1 \dots x_n)\, dx_1 \dots dx_n \; f^*_{\lambda_1 \dots \lambda_n}(y_1 \dots y_n)\, dy_1 \dots dy_n = \int K(\vec{x}, \vec{y}, \vec{z})\, f^*_{\lambda_1 \dots \lambda_n}(z_1 \dots z_n)\, dz_1 \dots dz_n$$

Geometric Langlands   Variety ~ $Bun_G(C)$

Hecke corresp.   family of comm. operators

$G = GL_n$ mult. one.   A question: how to

under normalization of Hecke eigenfunctions

$\cdots\to$ $\cdots$ kernel

Write an "explicit formula"?

Various meanings of "explicit formula" $X/_k$ alg.

$\left(X = (\mathrm{Bun}_G C)^3 \right)$

1) k is a local field

$Y \xrightarrow{\pi} X$ dominant $\mathrm{vol} \in \Gamma_{alg}(Y, K_{Y/X})$

$\dashrightarrow \pi_*(|\mathrm{vol}|)$ $\mathbb{R}$-valued function on $X(K)$ if $\int$ converges

e^x

$\cancel{z}e^x$

loose constant

Versions. $\mathrm{vol}^{\otimes N}$ is well defined

additive characters,

mult characters

$Y \to \mathbb{G}_a \times$

$\times$ alg torus

$K_{Y/X} \otimes K_X^{1/2}$

2) $\mathrm{char}(k)=0$ Holonomic D_X-module with a choice of a cyclic vector

3) $\forall$ char. : motivic "constructible" sheaf $\leftarrow$ motivic local system

4) char $= p > 0$ $\bar{\mathbb{Q}}^{CM}$-function on $X(\mathbb{F}_{q^n})$ $\mathrm{Trace}(\mathrm{Frob}_q)$

Quantum Hitchin system

char $= 0$ On $\mathrm{Bun}_G(\mathbb{C})$

$H_1, \ldots, H_n$ commuting diff. op. on n-dim variety

$f_{\lambda_1 \ldots \lambda_n}$ = solution of holon. system

$$(H_i - \lambda_i) f = 0$$

$i = 1 \ldots n$

normalized $f_{\lambda_1 \ldots \lambda_n}(\text{special pt}) = 1$

Semiclassical limit of K

$$\begin{array}{c} T^*\mathrm{Bun}_G = T^*X \\ \text{section} \uparrow \quad \downarrow \\ B \end{array}$$

$\Lambda_{\text{add}} \subset \overline{T^*X} \times \overline{T^*X} \times T^*X$

graph of addition

?

$x + y = z$

$\pi(x) = \pi(y) = \pi(z)$

Weinstein "cat": Ob = symplectic mfd

$\otimes = \times$

$\mathrm{Hom}(X, Y) =$ Lagr. subvar. in $\overline{X} \times Y$

$(X, -\omega)$

Classical $\int$ble system "$\Longleftrightarrow$" comm. monoid.

\+ section

$1 \longrightarrow$ Sympl.

Example: $n=1$ $\quad X = \mathbb{A}^1 - 0$

$$\left(\left(x\frac{d}{dx}\right)^2 - 2\left(x^2+\frac{1}{x^2}\right) - \lambda\right) f_\lambda = 0$$

$$\left(x\frac{d}{dx}\right)^2 - 2\left(x+\frac{1}{x}\right) - \lambda$$

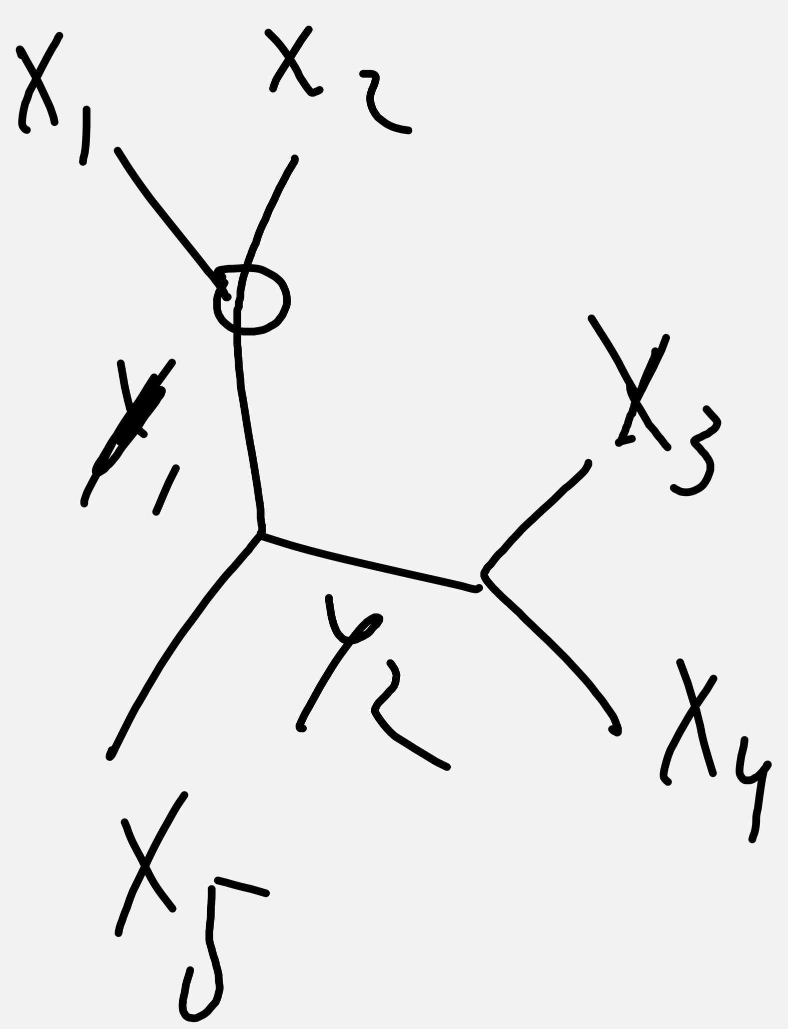
$$\frac{a+b+c+abc}{\sqrt{abc}}$$

$$\Phi(a,b,c) = abc + \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}$$

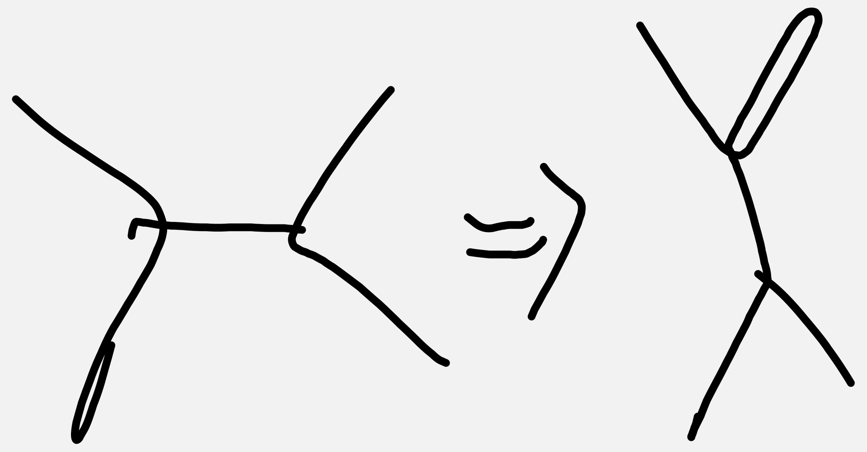
Kernel $\quad f_\lambda(x_1)\, f_\lambda(x_2) = \int \frac{dx_3}{x_3} \underbrace{\exp(\Phi(x_1,x_2,x_3))}_{K_3} f_\lambda(x_3)$

$$K_n(x_1 \dots x_n) = \int\cdots\int \exp\Big(\sum_{\text{vertices}} \Phi(\cdots)\Big) \wedge \frac{dy_i}{y_i}$$

$n=5$



$y_1 \quad y_{n-3}$



$$\Phi(x_1,x_2,y_1) + \Phi(y_2,x_3,x_4) + \Phi(y_1,y_2,x_5)$$

Question.

$\mathcal{H} / PSL_2(\mathbb{Z})$

$\forall$ weight k

holom parabolic forms of weight

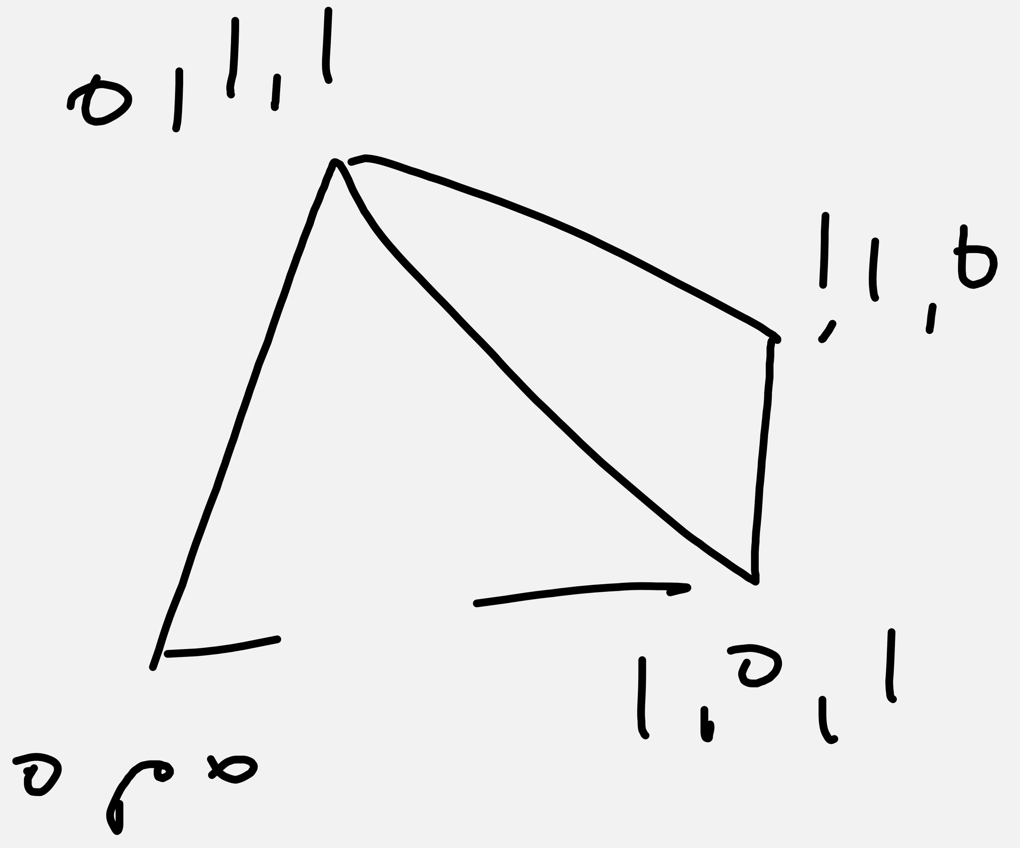
Hecke eigenform $\quad f_\alpha(q_1) = q + \cdots \quad \bar{\mathbb{Q}}^{CM}$ coeffs

$$\alpha = 1 \ldots \left\lceil \frac{k}{12} \right\rceil$$

$$\sum_{\substack{\alpha, \\ k \geq 1}} f_\alpha(q_1) \cdot f_\alpha(q_2) \cdot f_\alpha(q_3) \cdot t^k \in \mathbb{Q}[[q_1, q_2, q_3, t]]$$

$X = [0,1]$

$K : X^3 \to \mathbb{R}$ char. function of



$\forall\, n \geq 1$ $\quad A_n = \mathbb{Q}^{X \cap \frac{1}{n}\mathbb{Z}} = \mathbb{Q}^{\{\frac{0}{n}, \frac{1}{n}, \dots, \frac{n}{n}\}} \simeq \mathbb{Q}^{n+1}$

basis e_x $\quad x \in X \cap \frac{1}{n}\mathbb{Z}$

$$e_{x_1} \cdot e_{x_2} = \sum_{x_3} K(x_1, x_2, x_3)\, e_{x_3}$$

comm assoc alg.

Proof: uniform in n

Verlinde alg for sl_2 and level $\leq n$.

X^4 $\quad PL_{\mathbb{Z}}$ transfs