Derived Satake via factorization III

$\mathrm{Rep}(G)$ symmetric monoidal

$\rightsquigarrow \mathrm{Rep}(G)_{X^I}$ factorization cat.

has a unique t-structure s.t.

$$\mathrm{Rep}(G)_{X^I} \xrightarrow{\mathrm{oblv}} D(X^I)$$

is t-exact

Proposition (Raskin) This t-structure is regular (in particular, coherent).

Likewise, the category

$$\mathcal{O}_G\text{-mod}^{fact}(\mathrm{Rep}(G\times G))_{X^I}$$

has a unique t-structure s.t. forgetful functor to $\mathrm{Rep}(G\times G)_{X^I}$ is t-exact.

Theorem (C.-Raskin) This t-structure is coherent, and the vacuum module $\mathcal{O}_{G,X^I}$ is "almost ULA."

S = smooth variety

$D(S) \curvearrowright \mathcal{C}$

$$\begin{tikzcd} \mathrm{QCoh}(S) \otimes_{D(S)} \mathcal{C} \arrow[r, shift left, "\mathrm{ind}_S"] & \mathcal{C} \arrow[l, shift left, "\mathrm{oblv}_S"] \end{tikzcd}$$

We call $c \in \mathcal{C}$ <u>ULA</u> if $\mathrm{oblv}_S(c)$ is compact.
If $\mathrm{QCoh}(S) \otimes_{D(S)} \mathcal{C}$ is compactly generated by objects $\mathcal{P} \otimes \mathrm{oblv}_S(c_i)$ where $\mathcal{P}$ is perfect on S, we call $\mathcal{C}$ <u>ULA generated</u>.

<u>Example</u> $\mathcal{C} = \mathrm{Rep}(G)_{X^I}$

$$\mathrm{Loc}_{X^I} : \mathrm{Rep}(G)^{\otimes I} \longrightarrow \mathrm{Rep}(G)_{X^I}$$

The objects $\mathrm{Loc}_{X^I}(\boxtimes_i V_i)$, V_i fin.-dim., are ULA generators.

Suppose $\mathcal{C}$ has a t-structure comp. w/ the $D(S)$-action (i.e. $\mathcal{C} \to D(S)^\vee \otimes \mathcal{C} \simeq D(S) \otimes \mathcal{C}$ is t-exact). Then $\mathrm{QCoh}(S) \otimes_{D(S)} \mathcal{C}$ has a unique t-structure s.t.

$$\begin{tikzcd} \mathrm{QCoh}(S) \otimes_{D(S)} \mathcal{C} \arrow[r, shift left, "\mathrm{ind}_S"] & \mathcal{C} \arrow[l, shift left, "\mathrm{oblv}_S"] \end{tikzcd}$$

is t-exact.

Call $c \in \mathcal{C}$ <u>almost ULA</u> if $\mathrm{oblv}_S(c)$ is almost compact. If $(\mathrm{QCoh}(S) \otimes_{D(S)} \mathcal{C})^{\geq 0}$ is compactly generated by objects $\tau^{\geq 0}(\mathcal{F} \otimes \mathrm{oblv}_S(c_i))$ w/ c_i almost ULA, we call $\mathcal{C}$ <u>almost ULA generated</u>.

For $\mathcal{C}$ coherent and almost ULA gen., we $D(S) \curvearrowright \mathcal{C}^{ren}$ and $\mathcal{C}^{ren}$ is ULA gen.

$$\begin{tikzcd}
\mathrm{Rep}(G)_{X^I} \cong \mathcal{O}_G\text{-mod}(\mathrm{Rep}(G \times G))_{X^I} \arrow[d, "\mathrm{oblv}^{com \to fact}"] & t\text{-exact} \\
\mathcal{O}_G\text{-mod}^{fact}(\mathrm{Rep}(G \times G))_{X^I} &
\end{tikzcd}$$

Theorem $\Rightarrow$ admits a $D(X^I)$-linear, monadic right adjoint (on bounded below objects).

$\Rightarrow \mathcal{O}_G\text{-mod}^{fact}(\mathrm{Rep}(G \times G))_{X^I}$ almost ULA gen.

$$\Rightarrow \mathrm{Sph}^{spec}_{\check{G}, X^I} := (\mathcal{O}_{\check{G}}\text{-mod}^{fact}(\mathrm{Rep}(\check{G} \times \check{G}))_{X^I})^{ren}$$

Over a point $x \in X(k)$:

$$\mathrm{Sph}^{spec}_{\check{G}, x} = \mathrm{IndCoh}(\mathcal{H}^{spec}_{\check{G}, x}) = \mathrm{IndCoh}(\{0\} \times_{\check{\mathfrak{g}}} \{0\})^{\check{G}}$$

$$\mathrm{oblv}^{com \to fact} = \Delta_* : \mathrm{IndCoh}(pt/\check{G}) \to \mathrm{IndCoh}((\{0\} \times_{\check{\mathfrak{g}}} \{0\})/\check{G})$$

Remarks

i) $\mathrm{oblv}^{\mathrm{com}\to\mathrm{fact}} : (\mathrm{Rep}(G)_{X^I})^{\heartsuit} \xrightarrow{\sim} (\mathrm{Sph}^{\mathrm{spec}}_{G,X^I})^{\heartsuit}$

ii) monadicity of right adjoint of $\mathrm{oblv}^{\mathrm{com}\to\mathrm{fact}} \Rightarrow \mathrm{Sph}^{\mathrm{spec}}_{G,X^I}$ strictly factorizes (a priori only lax)

Construction of equivalence

Following Gaitsgory-Lurie:

$$\mathrm{Sph}_{G,X^I} \curvearrowright \underset{\underset{\mathcal{W}_{X^I}}{\rotatebox{90}{$\in$}}}{D(\mathfrak{L}^+G\backslash\mathfrak{L}G/\mathfrak{L}N^-,\psi)_{X^I}}$$

nondegenerate Whittaker

$\mathcal{W}_{X^I}$ Whittaker vacuum

Naive Satake factorizes (Mirkovic-Vilonen, Raskin)

$$\mathrm{Rep}(\check{G})_{X^I} \longrightarrow \mathrm{Sph}_{G,X^I}$$

monoidal fact cat.

Theorem (Frenkel-Gaitsgory-Vilonen, Raskin)

~~$\mathrm{Rep}(\check{G})_{X^I} \xrightarrow{(-)*\mathcal{W}_{X^I}} D(\mathfrak{L}^+G\backslash\mathfrak{L}G/\mathfrak{L}N^-,\psi)$~~

$$\begin{tikzcd}
\mathrm{Rep}(\check{G})_{X^I} \arrow[r] \arrow[dr, "\sim"'] & \mathrm{Sph}_{G,X^I} \arrow[d, "(-)*\mathcal{W}_{X^I}"] \\
& D(\mathfrak{L}^+G\backslash\mathfrak{L}G/\mathfrak{L}N^-,\psi)_{X^I}
\end{tikzcd}$$

$$\mathrm{Sph}_{G,X^I} \curvearrowright D(\mathfrak{L}^+G\backslash \mathfrak{L}G/\mathfrak{L}N^-,\psi)_{X^I} \overset{FGV}{\simeq} \mathrm{Rep}(\check{G})_{X^I}$$

$$\rightsquigarrow \mathrm{Sph}_{G,X^I} \longrightarrow \mathrm{End}(\mathrm{Rep}(\check{G}))_{X^I} \simeq \mathrm{Rep}(\check{G}\times\check{G})_{X^I}$$

monoidal fact. functor

$$\rightsquigarrow \mathrm{Sph}_{G,X^I} \simeq \delta_1\text{-mod}^{fact}(\mathrm{Sph}_{G,X^I})$$

$$\longrightarrow \mathcal{O}_{\check{G}}\text{-mod}^{fact}(\mathrm{Rep}(\check{G}\times\check{G}))_{X^I}$$

$$\overset{\text{ren.}}{\rightsquigarrow} \quad \mathrm{Sph}_{G,X^I} \xrightarrow{\mathrm{Sat}_G} \mathrm{Sph}^{spec}_{\check{G},X^I}$$

monoidal fact. functor

Since Sat_G factorizes and preserves ULA objects, it suffices to check that it's an equivalence over $x \in X(k)$.

$$\begin{tikzcd}
\mathrm{Sat}_{G,x} : \mathrm{Sph}_{G,x} \arrow[r] \arrow[d, phantom, "\wr\!\!\mid\!\!\wr"] & \mathrm{Sph}^{spec}_{\check{G},x} \arrow[d, phantom, "\wr\!\!\mid\!\!\wr"] \\
\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}(\mathrm{Rep}\check{G}) \arrow[r, phantom, "\simeq"] & \mathrm{IndCoh}(\mathcal{H}^{spec}_{\check{G},x})
\end{tikzcd}$$

Koszul duality equiv.

It's far from clear that this square commutes.

Since $\mathrm{Sat}_{G,x}$ is $\mathrm{Rep}(\check{G})$-equivariant, it suffices to show that

$$\mathrm{Sym}(\check{\mathfrak{g}}[-2]) \cong \underline{\mathrm{End}}_{\mathrm{Rep}(\check{G})}(\delta_1) \xrightarrow{\mathrm{Sat}_{G,x}} \underline{\mathrm{End}}_{\mathrm{Rep}(\check{G})}(\mathcal{O}_{\check{G}}) \cong \mathrm{Sym}(\check{\mathfrak{g}}[-2])$$

is the identity.

~~NB: Bla Sym$(\check{\mathfrak{g}}[-2])^{\check{G}}$ generates Sym$(\check{\mathfrak{g}}[-2])$ as $\check{G}$-equivariant algebra, so it suffices to check that~~

~~Sym$(\check{\mathfrak{g}}[-2])$~~

In fact, it suffices to check that

$$\mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}} \cong \mathrm{End}(\delta_1) \xrightarrow{\mathrm{Sat}_{G,x}} \mathrm{End}(\mathcal{O}_{\check{G}}) \cong \mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}}$$

We proceed by reduction to the case $G=T$.

That case follows drom:

<u>Theorem</u> (Beilinson)

$$\begin{tikzcd}
D(\mathcal{L}T)_{X^I} \arrow[r, "\sim"] & \mathrm{QCoh}(\mathrm{LS}_{\check{T}}(\mathring{D}))_{X^I} \\
D(\mathrm{Gr}_T)_{X^I} \arrow[u] \arrow[r, "\sim"] & \mathrm{QCoh}(\mathrm{LS}_{\check{T}}(D))_{X^I} \arrow[u] \cong \mathrm{Rep}(\check{T})_{X^I}
\end{tikzcd}$$

This is much stronger than derived Satake for T!

For the general case: use a certain $(\mathrm{Sph}_G, \mathrm{Sph}_T)$-bimodule

$$\mathrm{Sph}_{G,B} := \left(D(\mathfrak{L}^+G\backslash\mathfrak{L}G)_{\mathfrak{L}N\mathfrak{L}^+T}\right)^{\mathrm{ren}}$$

Counterpart on the spectral side:

there is a factorization algebra

$$\Upsilon(\check{n}, \mathcal{O}_{\check{G}}) \in \mathrm{Rep}(\check{G}\times\check{T})$$

s.t. $\Upsilon(\check{n}, \mathcal{O}_{\check{G}})\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}\times\check{T}))$ is coherent, etc. and

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\check{B}} := \left(\Upsilon(\check{n}, \mathcal{O}_{\check{G}})\text{-mod}^{\mathrm{fact}}(\mathrm{Rep}(\check{G}\times\check{T}))\right)^{\mathrm{ren}}$$

~~on~~ satisfies

$$\mathrm{Sph}^{\mathrm{spec}}_{\check{G},\check{B},x} \simeq \mathrm{IndCoh}\left(\left(\{0\}\underset{\check{\mathfrak{g}}}{\times}\check{n}\right)/\hat{\check{B}}_{\check{T}}\right)$$

<u>Theorem</u> (Rashin) There is a factorization functor

$$\begin{aligned} D(\mathfrak{L}^+G\backslash\mathfrak{L}G)_{\mathfrak{L}N\mathfrak{L}^+T} &\longrightarrow \mathrm{Rep}(\check{G}\times\check{T}) \\ \underset{\underset{\text{vacuum}}{\shortparallel}}{\Delta^0} &\longmapsto \Upsilon(\check{n}, \mathcal{O}_{\check{G}}) \end{aligned}$$

$\rightsquigarrow \mathrm{Sat}_{G,B} : \mathrm{Sph}_{G,B} \longrightarrow \mathrm{Sph}^{spec}_{\check{G},\check{B}}$

"factorizable ABG functor"

Consider the following diagram:

$$\begin{tikzcd}
\mathrm{Sph}_{G,x} \arrow[r, "\mathrm{Sat}_{G,x}"] \arrow[d, "(-)*\Delta^0"'] & \mathrm{Sph}^{spec}_{\check{G},x} \arrow[d, "(-)*\Upsilon(\check{n},\mathcal{O}_{\check{G}})"] \\
\mathrm{Sph}_{G,B,x} \arrow[r, "\mathrm{Sat}_{G,B}"] & \mathrm{Sph}^{spec}_{\check{G},\check{B},x} \\
\mathrm{Sph}_{T,x} \arrow[r, "\mathrm{Sat}_T", "\sim"'] \arrow[u, "\Delta^0*(-)"] & \mathrm{Sph}^{spec}_{\check{T},x} \arrow[u, "\Upsilon(\check{n},\mathcal{O}_{\check{G}})*(-)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}} = \mathrm{End}_{\mathrm{Sph}_G}(\delta_1) \arrow[r, "\sim"] \arrow[d, hook] \arrow[dr, phantom, "\text{Chevalley}"] & \mathrm{End}_{\mathrm{Sph}^{spec}_{\check{G}}}(\mathcal{O}_{\check{G}}) = \mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}} \arrow[d, hook] & \\
\mathrm{Sym}(\check{\mathfrak{t}}[-2]) = \mathrm{End}_{\mathrm{Sph}_{G,B}}(\Delta^0) \arrow[r, "\sim"] & \mathrm{End}_{\mathrm{Sph}^{spec}_{\check{G},\check{B}}}(\Upsilon(\check{n},\mathcal{O}_{\check{G}})) \arrow[r, equal] & \mathrm{Sym}(\check{\mathfrak{t}}[-2]) \\
\mathrm{End}_{\mathrm{Sph}_T}(\delta_1) \arrow[r, "\mathrm{Sat}_T"] \arrow[u] \arrow[u, shift left, equal] \arrow[d, equal] & \mathrm{End}_{\mathrm{Sph}^{spec}_{\check{T}}}(\mathcal{O}_{\check{T}}) \arrow[u] \arrow[u, shift left, equal] \arrow[d, equal] & \\
\mathrm{Sym}(\check{\mathfrak{t}}[-2]) \arrow[r, equal] & \mathrm{Sym}(\check{\mathfrak{t}}[-2]) &
\end{tikzcd}$$