Derived Satake via factorization II

Recall from last time: we want to extend

$$\mathrm{Sph}^{spec}_{\check{G},x} = \mathrm{IndCoh}(\mathcal{H}^{spec}_{\check{G},x}) \simeq \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}(\mathrm{Rep}(\check{G}))$$

to a factorization category, and to lift B-F to an equivalence of monoidal fact categories $\mathrm{Sph}_G \simeq \mathrm{Sph}^{spec}_{\check{G}}$ $(\mathrm{Sph}_G := D(\mathcal{H}_G)^{ren})$.

Heuristic: $f: \mathcal{Y} \to \mathcal{Z}$ affine morphism

$$\mathrm{QCoh}(\mathcal{Y}(\overset{\circ}{D}_{dR}) \underset{\mathcal{Z}(\overset{\circ}{D}_{dR})}{\times} \mathcal{Z}(D_{dR})) \simeq f_* \mathcal{O}_{\mathcal{Y}}\text{-mod}^{fact}(\mathrm{QCoh}(\mathcal{Z}(D_{dR})))$$

<u>Theorem</u> (Raskin) For reasonable $\mathcal{Z}$ (e.g. classifying stacks), this is literally true at a point.

Aside (Betti setting): A = comm. alg.

$$A\text{-mod}^{\mathbb{E}_n} \xrightarrow{\mathrm{oblv}} \mathrm{Vect}$$

admits a left adjoint.

<u>Proposition</u> The monad

$$\mathrm{Vect} \underset{\mathrm{oblv}}{\overset{\dashrightarrow}{\longleftarrow}} A\text{-mod}^{\mathbb{E}_n}$$

is given the assoc alg. $S^{n-1} \otimes A$.

$$\Rightarrow A\text{-mod}^{\mathbb{E}_n} \xrightarrow{\sim} S^{n-1} \otimes A\text{-mod}^{\ell, \text{left}}$$

Note that $\mathrm{Spec}(S^{n-1} \otimes A) = \underline{\mathrm{Map}}(S^{n-1}, \mathrm{Spec}\, A)$.

For $n=2$ we get a Betti version of the heuristic:

$$A\text{-mod}^{\mathbb{E}_2} \xrightarrow{\sim} \mathrm{QCoh}(\underline{\mathrm{Map}}(S^1, \mathrm{Spec}\, A))$$

Our case: $\mathcal{Y} = pt/G$, $\mathcal{Z} = pt/G \times pt/G$, $f = \Delta$

$$\mathcal{Y}(\mathring{D}_{dR}) \underset{\mathcal{Z}(\mathring{D}_{dR})}{\times} \mathcal{Z}(D_{dR}) = LS_G(\mathring{D}) \underset{LS_G(\mathring{D}) \times LS_G(\mathring{D})}{\times} (LS_G(D) \times LS_G(D))$$

$$= LS_G(D) \underset{LS_G(\mathring{D})}{\times} LS_G(D) = \mathcal{H}_G^{spec}$$

$$\Rightarrow \mathrm{QCoh}(\mathcal{H}^{spec}_{G,x}) \cong \mathcal{O}_G\text{-mod}^{fact}(\mathrm{Rep}(G \times G))_x$$

I = nonempty finite set

$\rightsquigarrow \mathcal{O}_G\text{-mod}^{\text{fact}}(\mathrm{Rep}(G\times G))_{X^I}$

$J = I \sqcup J_0, \quad x_J = \{x_j\} \in X^J \rightsquigarrow M_{x_J} \in \cancel{\mathrm{Vect}}\ \mathrm{Rep}(G\times G)_{x_J}$

comp. as J varies, w/ iso.'s

$$M_{x_J} \cong M_{x_I} \otimes \bigotimes_{j\in J_0} \mathcal{O}_G$$

when $x_j \neq x_k$ for $j \neq k \in J_0$, and $x_i \neq x_j$ for $i \in I$, $j \in J_0$

Renormalization

Why renormalize?

- to get IndCoh (as opposed to QCoh) at a point, needed for B-F;
- to get a better category, e.g. compactly generated
- to get a "ULA generated" category: once the derived Satake functor is constructed, can check that it is an equivalence at a point.

$\mathcal{C}$ = DG cat. w/ t-structure

Standing assumptions:

- $\mathcal{C}^{\geq 0}$ stable under filtered colimits;
- (right completeness) $\operatorname{colim}_n \tau^{\leq n} c \xrightarrow{\sim} c$ for any $c \in \mathcal{C}$.

An object $c \in \mathcal{C}$ is called <u>almost compact</u> if $\tau^{\geq n} c$ is compact in $\mathcal{C}^{\geq n}$ for all n.
Call c <u>coherent</u> if it is almost compact and eventually coconnective (almost compact $\Rightarrow$ eventually connective by right completeness).

E.g. $\mathcal{C} = \mathrm{QCoh}(Y)$, Y = almost finite type scheme
$\mathcal{F} \in \mathrm{QCoh}(Y)$ is coherent in this sense
$\Leftrightarrow$ coherent in usual sense.

We say that the t-structure on $\mathcal{C}$ is <u>coherent</u> if (in addition to standing assumptions)

- $\mathcal{C}^{\geq 0}$ is compactly generated;
- $\mathcal{C}^{\geq 1} \hookrightarrow \mathcal{C}^{\geq 0}$ preserves compact objects ($\Leftrightarrow$ coherent objects preserved by truncation)

Remark: $\mathcal{C}$ coherent $\Rightarrow (\mathcal{C}^\heartsuit)^c \subset \mathcal{C}^\heartsuit$ is abelian

Definitions are from appendix to "Integral transforms for coherent sheaves" by Ben-Zvi, Nadler, and Preygel.

If $\mathcal{C}$ is coherent and is compactly generated by $\mathcal{C}^{coh}$:= coherent objects, we call $\mathcal{C}$ <u>regular</u>.

NB: coherence is a property of $\mathcal{C}^+$ (not so for regularity).

If $\mathcal{C}$ is coherent, then it admits a canonical renormalization

$$\underbrace{\mathcal{C}^{ren}}_{\text{regular}} := \mathrm{Ind}(\mathcal{C}^{coh}) \xrightarrow{\text{t-exact}} \mathcal{C}$$

such that

$$(\mathcal{C}^{ren})^+ \xrightarrow{\sim} \mathcal{C}^+.$$

E.g. $\mathcal{C} = \mathrm{QCoh}(Y)$, Y almost finite type
$\mathcal{C}$ coherent and $\mathrm{QCoh}(Y)^{ren} = \mathrm{IndCoh}(Y)$.