Derived Satake via factorization I

$k = \bar{k}$, char $k = 0$
G = reductive group $/k$
X = smooth curve $/k$

$x \in X(k)$

The non-derived geometric Satake equiv. is a monoidal equiv. of abelian cat.'s

$$\mathrm{Sat}_G^\heartsuit : \underbrace{\mathrm{Rep}(\check{G})^\heartsuit}_{\text{alg. rep'ns}} \xrightarrow{\sim} \overbrace{D(\mathcal{H}_{G,x})^\heartsuit}^{\text{alg. D-modules}}$$

where

$$\mathcal{H}_{G,x} := \mathcal{L}^+G \backslash \mathcal{L}G / \mathcal{L}^+G$$
$$= \{(P_G, P_G', \alpha) \mid P_G, P_G' = G\text{-bundles on on } D_x = \hat{X}_x \text{ and } \alpha : P_G|_{\mathring{D}_x} \xrightarrow{\sim} P_G'|_{\mathring{D}_x}\}$$

is the Hecke stack at x.

$\mathrm{Rep}(\check{G})^\heartsuit \subset \mathrm{Rep}(\check{G})$ = DG cat. of complexes w/ $\check{G}$-action
$D(\mathcal{H}_{G,x})^\heartsuit \subset D(\mathcal{H}_{G,x}) =: \mathrm{Sph}_{G,x}$ = equivariant DG cat. $D(\mathrm{Gr}_{G,x})^{\mathcal{L}^+G}$

Observation: $\mathrm{Rep}(\check{G})$ is the derived category of $\mathrm{Rep}(\check{G})^\heartsuit$. Thus $\mathrm{Sat}_G^\heartsuit$ extends uniquely to a t-exact monoidal functor

$$\mathrm{Sat}_G^{\text{naive}} : \mathrm{Rep}(\check{G}) \longrightarrow \mathrm{Sph}_{G,x}.$$

This functor is <u>not</u> an equivalence. For example, we have

$$\begin{aligned} \mathrm{REnd}_{\mathrm{Sph}_{G,x}}(\delta_1) &= H^\bullet_{\mathcal{L}^+G}(pt) = H^\bullet_G(pt) \\ &\cong \mathrm{Sym}(\mathfrak{g}^*[-2])^G \quad \text{(Chern-Weil)} \end{aligned}$$

whereas $\mathrm{REnd}_{\mathrm{Rep}(\check{G})}(k) = k$.

"Derived Satake" means modifying $\mathrm{Rep}(\check{G})$ in order to upgrade $\mathrm{Sat}_G^{\text{naive}}$ to an equiv.

<u>Theorem</u> (Bezrukavnikov-Finkelberg) There is a canonical monoidal equiv.

$$\begin{tikzcd} \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}(\mathrm{Rep}(\check{G})) \arrow[r, "\sim"] & \mathrm{Sph}_{G,x} \\ \mathrm{Rep}(\check{G}) \arrow[u, "\mathrm{ind}"] \arrow[ur, "\mathrm{Sat}_G^{\text{naive}}"'] & \end{tikzcd}$$

Note that

$$\begin{aligned}
&\mathrm{REnd}_{\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}(\mathrm{Rep}(\check{G}))}(\mathrm{Sym}(\check{\mathfrak{g}}[-2])) \\
&\quad \cong \mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}} \cong \mathrm{Sym}(\check{\mathfrak{t}}[-2])^{W} \\
&\quad \cong \mathrm{Sym}(\mathfrak{t}^*[-2])^{W} \cong \mathrm{Sym}(\mathfrak{g}^*[-2])^{G}
\end{aligned}$$

NB: $\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}(\mathrm{Rep}(\check{G}))$ is symm. mon., but $\mathrm{Sph}_{G,x}$ is <u>not</u> in a natural way.

Namely, the assignment $x \mapsto \mathrm{Sph}_{G,x}$ upgrades to the structure of factorization category.

I = nonempty finite set

$x_I = (x_i)_{i\in I} \in X^I$

$$x_I \longmapsto \mathrm{Sph}_{G,x_I} := D(\mathcal{H}_{G,x_I})$$

$$\varphi : I \twoheadrightarrow J \rightsquigarrow \Delta_\varphi : X^J \to X^I$$

$$\mathrm{Sph}_{G,x_J} \cong \mathrm{Sph}_{G,\Delta_\varphi(x_I)}$$

That is, Sph_G naturally fibers over the Ran space $\mathrm{Ran} = \underset{I}{\mathrm{colim}}\, X^I$ parameterizing nonempty finite subsets of X.

The factorization structure ~~the~~ consists of a homotopy-coherent collection of ~~isomor~~ equivalences

$$\mathrm{Sph}_{G, x_I} \cong \bigotimes_{i \in I} \mathrm{Sph}_{G, x_i}$$

-Lurie tensor product

whenever the x_i are pairwise distinct.

Factorization categories assemble into a symm. mon. (∞-)category by applying Lurie tensor product fiberwise over Ran.

In particular, we can speak of monoidal factorization cat.'s = associative algebras in factorization categories. Key example: Sph_G with its convolution product.

An aside: in the Betti setting, factorization categories (on $X = \mathbb{A}^1$, satisfying some condition) are equivalent to $\mathbb{E}_2$-monoidal categories, i.e. braided monoidal categories in the derived sense. A monoidal fact. cat. has an additional commuting $\mathbb{E}_1$-monoidal structure, making it an $\mathbb{E}_1 + \mathbb{E}_2 = \mathbb{E}_3$-monoidal ~~structure~~. category.

Dunn additivity

This explains (heuristically) why $\mathrm{Sph}^{\heartsuit}_{G,x}$ is symm. mon.: an $\mathbb{E}_3$-algebra in (1,1)-categories is already symmetric.

A standard construction attaches a fact. cat. to any symm. mon. DG cat. One can show that Sph_G does <u>not</u> arise in this way.

Goal: construct a monoidal fact. cat. $\mathrm{Sph}^{spec}_{\check{G}}$ such that

$$\mathrm{Sph}^{spec}_{\check{G},x} \cong \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}(\mathrm{Rep}(\check{G}))$$

for $x \in X(k)$, and extend the B-F equiv. to the factorization setting.

Following Arinkin-Gaitsgory, we can define $\mathrm{Sph}^{spec}_{\check{G},x}$ to be the "true" category of Hecke operators at x. Recall the global geometric Langlands conjecture:

$$\underset{\text{renormalized}}{D}(\mathrm{Bun}_G(X)) \cong \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}(X))$$

moduli of flat $\check{G}$-bundles on X

<u>derived</u>

(here X is assumed projective)

$\mathcal{H}_{G,x} \curvearrowright \mathrm{Bun}_G(X)$

Likewise, the spectral Hecke stack

$$\mathcal{H}^{spec}_{\check{G},x} := \{(E_{\check{G}}, E'_{\check{G}}, \alpha) \mid E_{\check{G}}, E'_{\check{G}} = \text{flat } \check{G}\text{-bundles on } D_x,\ \alpha : E_{\check{G}}|_{\mathring{D}_x} \xrightarrow{\sim} E'_{\check{G}}|_{\mathring{D}_x}\}$$

acts on $LS_{\check{G}}(X)$.

Thus it seems reasonable to put

$$\mathrm{Sph}^{spec}_{\check{G},x} := \mathrm{IndCoh}(\mathcal{H}^{spec}_{\check{G},x}),$$

and indeed we have

$$\mathrm{IndCoh}(\mathcal{H}^{spec}_{\check{G},x}) \simeq \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}(\mathrm{Rep}(\check{G})).$$

Namely: a choice of coordinate at x determine an isomorphism of groupoids

$$\mathcal{H}^{spec}_{\check{G},x} \cong LS_{\check{G}}(D) \times_{LS_{\check{G}}(\mathring{D})} LS_{\check{G}}(D)$$

$$\cong \{0\}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \{0\}/\check{G} \cong (\{0\} \times_{\check{\mathfrak{g}}} \{0\})/\check{G}.$$

Now $\{0\} \times_{\check{\mathfrak{g}}} \{0\} = \mathrm{Spec}\,\mathrm{Sym}(\check{\mathfrak{g}}^*[1])$, and $\mathrm{IndCoh}(\{0\} \times_{\check{\mathfrak{g}}} \{0\})$ is compactly generated by the skyscraper/unit object δ_0, corresponding the ~~aug~~ augmentation k.

Thus

$$\mathrm{IndCoh}(\{0\} \underset{\check{\mathfrak{g}}}{\times} \{0\}) \cong \mathrm{REnd}(\delta_0)\text{-mod}$$
$$\cong \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}.$$

Turning $\check{G}$-equivariance back on, we obtain the desired equivalence.

The stack $\mathcal{H}^{spec}_{\check{G},x}$ factorizes: a natural candidate for the fact. cat. would be

$$\mathrm{Sph}^{spec}_{\check{G},X^I} := \mathrm{IndCoh}(\mathcal{H}^{spec}_{\check{G},X^I}).$$

Problem: $\mathcal{H}^{spec}_{\check{G},X^I}$ is not locally of finite type! So defining coherent sheaves (as opposed to quasicoherent) is problematic.

A toy model: $A = k[x, y, t, u, \frac{t-u}{x-y}, \frac{t-u}{(x-y)^2}, \dots]$

$$\begin{tikzcd}
\mathbb{A}^1 \times \mathbb{A}^1 \arrow[r, hook] \arrow[d] & \mathrm{Spec}\, A \arrow[d] & \mathbb{A}^2 \times (\mathbb{A}^2 \setminus \Delta) \arrow[l, hook'] \arrow[d] \\
\mathbb{A}^1 \arrow[r, hook] & \mathbb{A}^2 = \mathrm{Spec}\, k[x,y] & \mathbb{A}^2 \setminus \Delta \arrow[l, hook']
\end{tikzcd}$$