Sheaves on semi-infinite flags as fact. modules

k = alg. closed field of char. 0
X = smooth conn. curve $/ k$
G = conn. reductive group

The semi-infinite flag space $\mathrm{Fl}_G^{\frac{\infty}{2}}$ is supposed to have the following properties:

- factorization space $/X$
- alg. loop group $\mathcal{L}G \curvearrowright \mathrm{Fl}_G^{\frac{\infty}{2}}$
- $\mathrm{Fl}_G^{\frac{\infty}{2}}$, or rather its cat. of D-modules $D(\mathrm{Fl}_G^{\frac{\infty}{2}})$, deforms with the level

$Fl_G^{\frac{\infty}{2}}$ is supposed to be the correct space on which to localize Kac-Moody rep'ns at critical level.

$\chi : Z(U_{crit}(\hat{\mathfrak{g}})) \longrightarrow k$ reg. central char. ($\xleftrightarrow{F-F}$ reg. oper)

We hope to have an equivalence

$$Loc_\chi : \hat{\mathfrak{g}}_{crit}\text{-mod}_\chi \xrightarrow{\sim} D_{crit}(Fl_G^{\frac{\infty}{2}}).$$

$B = TN$ Borel subgroup

First guess: $Fl_G^{\frac{\infty}{2}} = \mathcal{L}G/\mathcal{L}B$

Feigin-Frenkel worked with analytic version.

Not the correct space: looking at $\mathcal{L}^+G$-equivariant objects.

At a point in X, ABG $\Rightarrow$

$$D_{crit}(Gr_G)^{\mathcal{L}B} \xrightarrow{\sim} \underset{\text{up to ren.}}{\mathrm{IndCoh}}((\check{n} \overset{\times}{\underset{\check{g}}{}} \{0\})/\check{N})$$

On the other hand, we want

$$D_{crit}(Fl_G^{\frac{\infty}{2}})^{\mathcal{L}^+G} \xrightarrow{\sim} \hat{g}_{crit}\text{-mod}_x^{\mathcal{L}^+G}.$$

According to a conjecture of F-G recently proved by Raskin-Yang, the RHS is equiv. to

$$D_{crit}(\mathcal{L}^+G\backslash \mathcal{L}G/\mathcal{L}^+G) \underset{Rep(\check{G})}{\otimes} \mathrm{Vect}$$

$$\downarrow\wr \ \text{derived Satake}$$

$$\mathrm{IndCoh}_{\mathcal{N}}(\{0\} \overset{\times}{\underset{\check{g}}{}} \{0\}/\check{G}) \underset{Rep(\check{G})}{\otimes} \mathrm{Vect}$$

$$\|$$

$$\mathrm{IndCoh}_{\mathcal{N}}(\{0\} \overset{\times}{\underset{\check{g}}{}} \{0\}).$$

The semi-infinite IC sheaf

Need to modify $D(\mathcal{L}G/\mathcal{L}B)$
$$\mathrel{\ddot{\shortparallel}}$$
$$D_{crit}(\mathcal{L}G/\mathcal{L}B).$$

We'll work with $D(\mathcal{L}G)^{\mathcal{L}N}$ and keep track of right $\mathcal{L}T$-action.

Recall that the orbits of $\mathcal{L}N \curvearrowright Gr_G$ are indexed by coweights
$$\Lambda := \mathrm{Hom}(\mathbb{G}_m, T).$$

The orbits S^λ, $\lambda \in \Lambda$, have infinite dim. and codim, but do have well-defined dimension rel. to one another.

$$j^\lambda : S^\lambda \hookrightarrow Gr_G$$

$\Delta^{\circ} := j_!^{\circ}\, \omega_{S^{\circ}}$ standard

$\nabla^{\circ} := j_*^{\circ}\, \omega_{S^{\circ}}$ costandard

These are fact. algebras, and have canonical map

$$\Delta^{\circ} \longrightarrow \nabla^{\circ}.$$

The category $D(Gr_G)^{\mathcal{L}N}$ has a natural t-structure: the usual t-structure on $D(Gr_G)$ vanishes on $D(Gr_G)^{\mathcal{L}N}$!

This t-structure makes sense factorizably, and as such Δ° and ∇° belong to the heart.

$$IC^{\frac{\infty}{2}} := \mathrm{im}(\Delta^{\circ} \to \nabla^{\circ}).$$

Warning: at a fixed point in X, Δ^0 is ~~a~~ simple object, and ∇^0 does not belong to the heart.

By construction, $IC^{\frac{\infty}{2}}$ is a fact. alg in $D(Gr_G)^{\mathcal{L}N}$ such that $j^{0,!} IC^{\frac{\infty}{2}} = \omega_{S^0}$.

Recall that $\mathcal{L}G$ has a structure of factorization module space over the Gr_G, and hence ~~D(LG)~~ $D(\mathcal{L}G)^{\mathcal{L}N}$ has a structure of fact. module cat. over $D(Gr_G)^{\mathcal{L}N}$.

Proposal (Dennis):

$$D(Fl_G^{\frac{\infty}{2}}) := IC^{\frac{\infty}{2}}\text{-mod}^{fact}(D(\mathcal{L}G)^{\mathcal{L}N})$$

I'll check this proposal on $\mathcal{L}^+G$-invariants.

<u>Theorem</u> (in progress, vague) There is a canonical equiv. of fact. cat.'s

$$\mathrm{IC}^{\frac{\infty}{2}}\text{-mod}^{\mathrm{fact}}\left(D_{\mathrm{crit}}(\mathrm{Gr}_G)^{\mathcal{L}N}\right)$$

$$\xrightarrow{\sim} \mathrm{IndCoh}\left(\mathrm{LS}_{\check{T}}(\overset{\circ}{D}) \times_{\mathrm{LS}_{\check{G}}(\overset{\circ}{D})} \mathrm{LS}_{\check{G}}(D)\right)$$

At a point:

$$\mathrm{LS}_{\check{T}}(\overset{\circ}{D}) \times_{\mathrm{LS}_{\check{G}}(\overset{\circ}{D})} \mathrm{LS}_{\check{G}}(D) \cong \left(\check{\mathfrak{t}} \times_{\check{\mathfrak{g}}} \{0\}\right)/\check{T}$$

This equiv. is compatible w/ actions

$$D_{\mathrm{crit}}(\mathcal{L}^+G\backslash\mathcal{L}G/\mathcal{L}G)^{\mathrm{ren}} \xrightarrow{\substack{\text{derived}\\ \text{Satake}}} \mathrm{IndCoh}\left(\mathrm{LS}_{\check{G}}(D) \times_{\mathrm{LS}_{\check{G}}(\overset{\circ}{D})} \mathrm{LS}_{\check{G}}(D)\right)$$

and

$$D(\mathcal{L}T) \xrightarrow[\sim]{\mathrm{LCFT}} \mathrm{QCoh}(\mathrm{LS}_{\check{T}}(\overset{\circ}{D})).$$

Q: how to make sense of IndCoh here?

$\mathcal{Y} \to X_{dR}$ crystal of Artin stacks

$\rightsquigarrow$ factorization spaces

$\mathcal{Y}(D)$ = horizontal jets

and

$\mathcal{Y}(\overset{\circ}{D})$ = meromorphic horizontal jets

$$\mathcal{Y}(D) \underset{\mathrm{Ran}_{dR}}{\times} X_{dR} = \mathcal{Y}$$

For $\mathcal{Y} = pt/H \times X_{dR}$ we have

$\mathcal{Y}(D) = LS_H(D)$ and $\mathcal{Y}(\overset{\circ}{D}) = LS_H(\overset{\circ}{D})$.

$f : \mathcal{Y} \to \mathcal{Z}$ quasi-affine map over X_{dR}

Heuristic: as fact. cat.'s

$$\mathrm{QCoh}(\mathcal{Y}(\overset{\circ}{D}) \underset{\mathcal{Z}(\overset{\circ}{D})}{\times} \mathcal{Z}(D))$$

$$\simeq f_* \mathcal{O}_{\mathcal{Y}(D)}\text{-mod}^{fact}(\mathrm{QCoh}(\mathcal{Z}(D))).$$

The point is that the RHS is a little better-behaved.

Apply this to $\mathcal{Y} = pt/\check{T} \times X_{dR}$ and $\mathcal{Z} = pt/\check{G} \times X_{dR}$ to get

$$\mathrm{QCoh}(LS_{\check{T}}(\overset{\circ}{D}) \underset{LS_{\check{G}}(\overset{\circ}{D})}{\times} LS_{\check{G}}(D))$$

$$\simeq \mathcal{O}_{\check{G}/\check{T}}\text{-mod}^{fact}(\mathrm{Rep}(\check{G})).$$

We now explain how to construct

$$IC^{\frac{\infty}{2}}\text{-mod}^{fact}(D(Gr_G)^{\mathfrak{L}N}) \to \mathcal{O}_{\check{G}/\check{T}}\text{-mod}^{fact}(\mathrm{Rep}(\check{G})).$$

Construction of the functor

Begin with the functor

$$F : D(Gr_G)^{\mathfrak{L}N} \longrightarrow D(Gr_G) \longrightarrow D(Gr_G)_{\mathfrak{L}N^-, \underset{\text{nondeg.}}{\psi}}$$

$$\xrightarrow[\sim]{\text{Rashkin}} D(Gr_G)^{\mathfrak{L}N^-, \psi} \xrightarrow[\sim]{\text{FGV}} Rep(\check{G}).$$

Claim: $F(IC^{\frac{\infty}{2}}) \cong \mathcal{O}_{\check{G}/\check{T}}$.

functoriality of fact modules

$\rightsquigarrow$ desired functor

Need another construction of $IC^{\frac{\infty}{2}}$.

$\mathcal{C} = Rep(\check{T} \times \check{G})$ - module category

Def'n (Raskin) A **Drinfeld-Plücker structure** on an object c in $\mathcal{C}$ is an $\mathcal{O}_{\overline{\check{N} \backslash \check{G}}}$ -module structure.

This structure amounts to a collection of maps

$$k^\lambda \star c \to c \star V^\lambda \quad (\lambda \in \Lambda^+)$$

satisfying some relations.

<u>Example</u> $\mathrm{Rep}(\check{T} \times \check{G}) \curvearrowright D(\mathrm{Gr}_G)^{\mathcal{L}N \mathcal{L}^+T}$ via geom. Satake (w/ some shifts in the $\mathrm{Rep}(\check{T})$-action). The standard Δ^0 has a canonical Drinfeld-Plücker structure: since t^λ belongs to Gr_G^λ for $\lambda \in \Lambda^+$, we have

$$IC_{\overline{\mathrm{Gr}}_G^\lambda}\big|^!_{t^\lambda} \cong k[-\langle \lambda, 2\check{\rho} \rangle].$$

$$\rightsquigarrow k^\lambda \star \delta_1[-\langle \lambda, 2\check{\rho}\rangle] = \delta_{t^\lambda}[-\langle \lambda, 2\check{\rho}\rangle]$$

$$\to IC_{\overline{\mathrm{Gr}}_G^\lambda} = \delta_1 \star V^\lambda.$$

Now apply $\mathrm{Av}_!^{\mathcal{L}N}$ to get

$$k^\lambda \star \Delta^0[-\langle \lambda, 2\check{\rho}\rangle] \to \Delta^0 \star V^\lambda.$$

<u>Example</u> $\mathcal{C} = \mathrm{QCoh}\left(LS_{\check T}(D) \underset{LS_{\check T}(\mathring{D})}{\times} LS_{\check B}(\mathring{D}) \underset{LS_{\check G}(\mathring{D})}{\times} LS_{\check G}(D)\right)$

w/ obvious action of $\mathrm{Rep}(\check T \times \check G)$.

$$LS_{\check B}(D) \xrightarrow{p} \mathcal{Y}$$

trivial $\check B$-rep'n

Then $p_* \mathcal{O} = p_* k$ has a canonical Drinfeld-Plücker structure

$$k^\lambda \otimes p_* k \cong p_* k^\lambda \longrightarrow p_* V^\lambda$$
$$\cong p_*(k) \otimes V^\lambda.$$

$\mathrm{Rep}(\check T \times \check G) \curvearrowright \mathcal{C}$

<u>Def'n</u> A <u>Hecke structure</u> on an object c in $\mathcal{C}$ is an action of $\mathcal{O}_{\check G}$ as an algebra in $\mathrm{Rep}(\check T \times \check G)$.

Explicitly, this amounts to a collection of isomorphisms

$$c \star V^\lambda \xrightarrow{\sim} V^\lambda|_{\check T} \star c \quad (\lambda \in \Lambda^+).$$

<u>Example</u> $\mathrm{Rep}(\check{T} \times \check{G}) \curvearrowright \mathrm{Rep}(\check{B})$

Then $\mathcal{O}_{\check{N}}$ has a canonical Hecke structure.

<u>Example</u> $\mathcal{Y} = LS_{\check{T}}(D) \times_{LS_{\check{T}}(\mathring{D})} LS_{\check{B}}(\mathring{D}) \times_{LS_{\check{G}}(\mathring{D})} LS_{\check{G}}(D)$

$\mathcal{C} = \mathrm{QCoh}(\mathcal{Y})$

$$\begin{tikzcd}
LS_{\check{T}}(D) \arrow[r] \arrow[dr, "r"'] & LS_{\check{B}}(D) \arrow[d, "p"] \\
& \mathcal{Y}
\end{tikzcd}$$

Above example $\Rightarrow r_* \mathcal{O} = p_* \mathcal{O}_{\check{N}}$ has a canonical Hecke structure.

Have a canonical map

$$\mathcal{O}_{\overline{\check{N}\backslash\check{G}}} \longrightarrow \mathcal{O}_{\check{G}}$$

$$\rightsquigarrow \mathcal{O}_{\overline{\check{N}\backslash\check{G}}}\text{-mod}(\mathcal{C}) \rightleftarrows \mathcal{O}_{\check{G}}\text{-mod}(\mathcal{C})$$

<u>Proposition</u> (Dennis) $\Delta^{\circ}\, \mathcal{O}_{\overline{\check{N}\backslash\check{G}}} \otimes \mathcal{O}_{\check{G}} \cong IC^{\frac{\infty}{2}}$

<u>Corollary</u> $IC^{\frac{\infty}{2}}$ has a canonical Hecke structure.

Note that

$$p_*(\mathcal{O}) \underset{\mathcal{O}_{\check{N}\backslash\check{G}}}{\otimes} \mathcal{O}_{\check{G}} \cong r_* \mathcal{O}_{LS_{\check{T}}(D)}.$$

$$r_* \mathcal{O}_{LS_{\check{T}}(D)} = p_* \mathcal{O}_{\check{N}}$$

Recall Sam's "geometric Chevalley functor":

$$\begin{aligned}
&\mathrm{Rep}(\check{G}) \otimes D(\mathrm{Gr}_G)^{\mathfrak{L}N\mathfrak{L}^+T} \\
&\cong D(\mathfrak{L}N^-,\psi \backslash \mathfrak{L}G/\mathfrak{L}^+G) \otimes D(\mathfrak{L}^+G\backslash \mathfrak{L}G/\mathfrak{L}N\mathfrak{L}^+T) \\
&\xrightarrow{\mathrm{conv}} D(\mathfrak{L}N^-,\psi\backslash \mathfrak{L}G/\mathfrak{L}N\mathfrak{L}^+T) \\
&\xrightarrow{\mathrm{oblv}} D(\mathfrak{L}G/\mathfrak{L}N\mathfrak{L}^+T) \xrightarrow{i^!_{\mathfrak{L}B}} D(\mathfrak{L}B/\mathfrak{L}N\mathfrak{L}^+T) \\
&\cong D(\mathrm{Gr}_T) \cong \mathrm{Rep}(\check{T})
\end{aligned}$$

$$\rightsquigarrow D(\mathrm{Gr}_G)^{\mathfrak{L}N\mathfrak{L}^+T} \longrightarrow \mathrm{Fun}(\mathrm{Rep}(\check{G}), \mathrm{Rep}(\check{T})) \cong \mathrm{Rep}(\check{T}\times\check{G}). \qquad (*)$$

Theorem (Raskin) $(*)$ sends

$$\Delta^0 \mapsto RT(\check{N}\backslash\check{G}, \mathcal{O}),$$

and the induced Drinfeld-Plücker structure on $RT(\check{N}\backslash\check{G}, \mathcal{O})$ is induced by $\mathcal{O}_{\overline{\check{N}\backslash\check{G}}} \to RT(\check{N}\backslash\check{G}, \mathcal{O})$.

Now the claim, hence the construction of the functor, follows from the commutative square

$$\begin{tikzcd} D(Gr_G)^{\mathcal{L}N\mathcal{L}^+T} \arrow[r, "(*)"] \arrow[d, "oblv"'] & Rep(\check{T} \times \check{G}) \arrow[d, "inv_{\check{T}}"] \\ D(Gr_G)^{\mathcal{L}N} \arrow[r, "F"] & Rep(\check{G}). \end{tikzcd}$$

Namely, Raskin's thm.

$\Rightarrow$ $(*)$ sends

$$IC^{\frac{\infty}{2}} = \Delta^0 \underset{\mathcal{O}_{\overline{\check{N}\backslash\check{G}}}}{\otimes} \mathcal{O}_{\check{G}} \mapsto RT(\check{N}\backslash\check{G}, \mathcal{O}) \underset{\mathcal{O}_{\overline{\check{N}\backslash\check{G}}}}{\otimes} \mathcal{O}_{\check{G}} = \mathcal{O}_{\check{G}}.$$