Relative Geometric Langlands (joint work w/ Yiannis Sakellaridis)

Plan:

1. Automorphic L-fns
2. Local geometric context
3. Bernstein asymptotics, statement of results

--- Break

4. Spherical Varieties
5. Zastava Space and semi-infinite orbits

Thm (Hecke, 1917)

$f : \mathbb{H} \longrightarrow \mathbb{C}$ modular cusp form, level 1

$$\int_0^\infty f(iy)\, y^{s-1}\, dy = L\left(\tfrac{1}{2}+s, f\right)$$ if f has first Fourier coeff $= 1$ [Whittaker]

Modern Reformulation

$G = PGL_2$ $\qquad H = \mathbb{G}_m = \begin{pmatrix} * & \\ & 1 \end{pmatrix} \hookrightarrow G$

$[G] = G(F)\backslash G(\mathbb{A})$ $\qquad F = \mathbb{Q}, \mathbb{F}_q(C)$

$f : [G]/K \longrightarrow \mathbb{C}$ unramified cusp. Hecke eigenform $\updownarrow$ $G(\mathbb{A}) \curvearrowright \pi$

$[G]/K = \mathrm{Bun}_G(\mathbb{F}_q)$

$$\underbrace{\int_{[H]} f(h)\, dh}_{\text{automorphic period integral}} \text{``}=\text{''} L\left(\tfrac{1}{2}, \pi, \mathrm{std}\right)$$ — f is Whittaker normalized

$\mathbb{C}^2 = \mathrm{std} \curvearrowleft \check{G} = SL_2(\mathbb{C})$

$$\begin{bmatrix} \mathrm{Bun}_H \longrightarrow pt \\ \downarrow \\ \mathrm{Bun}_G \end{bmatrix}$$

Relative Langlands: use subgp $H \subset G$ or (space $H\backslash G$) to study functoriality + L-fns

Caveat:

(1) Square period

e.g. Waldspurger's Formula $\qquad F'/F$ quadratic

$G = PGL_2/F \qquad H = (\mathrm{Res}_{F'/F}\mathbb{G}_m)/\mathbb{G}_m$

$$\left| \int_{[H]} f \right|^2 = \frac{L(\frac{1}{2}, \pi_F', \text{std})}{L(1, \pi, \underset{\check{G}}{\circlearrowleft} Ad)} \quad \text{if } \int_{[G]} |f|^2 = 1$$

Petersson inner prod

Famous Split Case
[Gan-Gross-Prasad Conjecture]

$H = SO_{2n} \hookrightarrow SO_{2n} \times SO_{2n+1} = G$

General Expectation (Sakellaridis–Venkatesh)

$$\left| \int_{[H]} f \right|^2 = \frac{L(\text{special value}, \pi', V_X \circlearrowleft \check{G}_X)}{L(1, \pi', Ad)}$$

Caveat ②: when H not red, integral incorrect

$Bun_H \subset \text{Maps}(C, \overline{H \backslash G}/G)$

$\downarrow$

Bun_G

if $H \backslash G$ has uniqueness

$\dim \text{Hom}_{H(F_v)}(\pi_v^{\circlearrowleft G(F_v)}, \mathbb{C}) \leq 1$

mult. 1 $\Leftarrow$ $H \backslash G$ spherical + technical

$$\left| \int_{[H]} f \, dh \right|^2 = \prod_{\text{prime } v} \left(\text{local computation} \underset{\text{Conj}}{=} \frac{L_v(s_0, \pi_v', \check{G}_X \circlearrowright V_X)}{L_v(1, \pi_v', Ad)} \right)$$

generalized Ichino-Ikeda conj (Sakellaridis-Venkatesh)

Thm (Sakellaridis-W) Prove Conj for spherical X with $\check{G}_X = \check{G}$ + some extra assump.

Philosophy from Physics (Ben-Zvi — Sakellaridis — Venkatesh)

mult 1

$G \circlearrowright X = H \backslash G$ $\rightsquigarrow$ $M = T^*X \longrightarrow \mathfrak{g}^*$

Hamiltonian G-space $\xrightarrow{\text{Gaiotto-Witten}}$ boundary conditions for super Yang-Mills for G

$\check{G}_X \longrightarrow \check{G}$

$V_X \in \text{Rep}(\check{G}_X)$ $\rightsquigarrow$ $\check{M} = V_X \times^{\check{G}_X} \check{G} \longrightarrow \check{\mathfrak{g}}^*$

Hamiltonian $\check{G}$-space $\longrightarrow$ boundary conditions for SYM $\check{G}$

S-duality

$M^\vee$ sometimes $T^*X^\vee$

- $\check{G}_X$ Gaitsgory-Nadler $\check{G}_X^{GN} \subset \check{G}$ (Tannakian)

SV: $\check{G}_X \longrightarrow \check{G}_X^{GN} \subset \check{G}$

Knop-Schalke

Symmetric Spaces (e.g. of spherical) : W_X : little Weyl gp

roots (up normalization) : restricted root system.

Geometric Version of Local Conj

"Quantization"

$F = k((t))$, $\mathcal{O} = k[[t]]$ $\quad k = \bar{\mathbb{F}}_q$ or $\mathbb{C}$

$X^\circ = H\backslash G$ quasiaffine, Choose $X^\circ \subset X$ affine

$X_F(k) = X(F)$, $X_F^\circ = X_F - (X - X^\circ)_F$

Conj (BZSV) $\exists$ monoidal equiv $M^\vee/\check{G}$

$$D^b_{G_{\mathcal{O}}}(X_F^\circ) \xrightarrow{\sim} \mathrm{Perf}\left(V_X^\wedge/\check{G}_X\right)$$

$\circlearrowleft \mathrm{Sat}_G$

$M^\vee \downarrow \check{\mathfrak{g}}^*$

$V_X \in \mathrm{Rep}(\check{G}_X)_{\mathbb{Z}\times\mathbb{Z}/2}$

graded, super $\updownarrow$ special value

- Left: Fusion, Right: $\otimes$

Example

Thm (Bez-Fink) $X = G \circlearrowleft G \times G$

$$D^b_{G_{\mathcal{O}}}(G_F/G_{\mathcal{O}}) \simeq \mathrm{Perf}\left(\check{\mathfrak{g}}^*[2]/\check{G}\right)$$

Function-theoretic analog:

- Unramified Hecke eigenfunctions on $X^\circ(F)$
- Plancherel decomposition on $X^\circ(F)$

Reduce to Bernstein asymptotics $\begin{cases} X = G & \text{Bezrukavnikov-Kazhdan} \\ X \text{ spherical}, & \text{SV} \end{cases}$

$X \rightsquigarrow X_0$

e.g. $X = SL_2$

$$\begin{tikzcd} SL_2 \arrow[r, hook] \arrow[d] & Mat_2 \arrow[d, "\det"] & \{\text{rank} \le 1\} \arrow[l, hook'] \arrow[d] \\ 1 \arrow[r] & \mathbb{A}^1 & 0 \arrow[l] \end{tikzcd}$$

Keyword: $X = G$ $\quad X \hookrightarrow$ Vinberg semigroup

X spherical, $X_0^\bullet$ = open G-orbit of normal bundle in $\overline{X} \supset X$

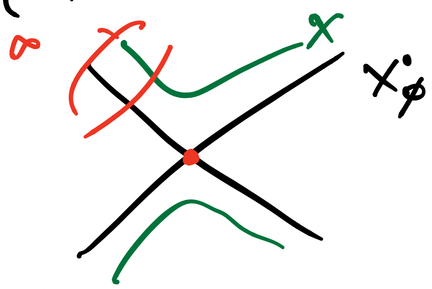
$\uparrow$

de Concini–Procesi wonderful compact

Thm (BK, SV)

$\exists$ $\quad \mathrm{Asymp} : C^\infty(X^\bullet(F)) \longrightarrow C^\infty(X^\bullet_\emptyset(F))$

• $G(F)$-equiv



Thm (Sakellaridis–W) Assume X affine spherical $\check{G}_X = \check{G}$, no type N roots

(A) $\quad \mathrm{Asymp} : C^\infty_c(X(\mathcal{O}) \cap X^\bullet(F)) \longrightarrow C^\infty(X_0(\mathcal{O}) \cap X_0^\bullet(F))$

is trace of Frob of $\Psi : \mathrm{Perv}(\mathcal{M}_X)_{G_\mathcal{O}} \longrightarrow \mathrm{Perv}(\mathcal{M}_{X_0})_{G_\mathcal{O}}$

• $\mathcal{M}_X$ f.d. model of $X_\mathcal{O}$.

(B) $\quad IC_{X(\mathcal{O})}$ = IC fn of $X_\mathcal{O}$

$\mathrm{Asymp}(IC_{X(\mathcal{O})}) \in C^\infty(X_0^\bullet(F))^{G(\mathcal{O})} \qquad X_0^\bullet = N^- \backslash G$

$$X_0^\bullet(F) = N^-(F) \backslash G(F) / G(\mathcal{O}) = \check{\Lambda}$$

$$\mathrm{Asymp}(IC_{X(\mathcal{O})}) = \frac{\prod_{\text{pos. coroot}} (1 - q^{-1} e^{\check{\alpha}})}{\prod \quad (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})}$$

$\check\lambda \in wt(V_X^+)$

$e^{\check\lambda}$ = indicator fn for $\check\lambda$

$$\frac{1}{1-q^{-\frac12}e^{\check\lambda}} = \sum_{n\geq 0} \left(q^{-\frac12}e^{\check\lambda}\right)^n$$

Mellin transf. $\widehat{\mathrm{Asymp}}(IC_{X(O)})(\chi) = \dfrac{L(\frac12, \chi, V_X^+)}{L(1,\chi,\check n)}$

$\chi \in \check T(\mathbb{C})$

$L(1,\chi,\check t) = \zeta(1)$?

$=$ "half" of $\boxed{\dfrac{L(\frac12, \chi, V_X)}{L(1,\chi, \check{\mathfrak g}/\check{\mathfrak t})}}$

$\boxed{\text{Local Computation} = \left|\mathrm{Asymp}(IC_{X(O)})\right|^2}$

- $V_X^+ \in \mathrm{Rep}(\check T)$
- $V_X = V_X^+ \oplus (V_X^+)^*$ has $(SL_2)_\alpha$ $\forall$ simple root
  - <u>Conj</u> $V_X \in \mathrm{Rep}(\check G)$
- Assuming $V_X \in \mathrm{Rep}(\check G)$, determine highest wt in terms of X.

<u>Thm</u> (Mirabolic Satake) Braverman-Finkelberg-Ginzburg - Travkin. $\curvearrowright$ $\mathrm{Perf}\left(V_X^{odd}[1]/\check G\right)$

- $\check G_X = \check G$, no type N roots (avoid $O_n \backslash GL_n$)

Jacquet, Mao: $O_n \backslash GL_n = (N,\psi)\backslash \widetilde{GL_n(F)}$

Concretely: X has open $\underset{\cup}{X^\circ}$ B-orbit

B

- $P_\alpha / R(P_\alpha) = PGL_2$
- $X^\circ P_\alpha / R(P_\alpha) = \mathbb{G}_m \backslash PGL_2$

$X = G \circlearrowleft G \times G$

$\check{G}_X = \check{G} \neq \check{G} \times \check{G}$

- X similar $\underline{(N, \psi)\backslash G}$
- Mirabolic : $X = \mathbb{A}^n \times GL_n \circlearrowleft GL_n \times GL_n$

(Rankin-Selberg convolution)

[BFGT]

$\not= X^\circ = H \backslash G = (\mathbb{A}^n - 0) \times GL_n$

$H =$ mirabolic diag.

- Orthosymplectic : $X = SO_{2n} \circlearrowleft SO_{2n} \times SO_{2n+1}$

(Gross-Prasad)

[BFT]

$\check{G}_X \to \check{G}$

- ψ, Whittaker : J. Campbell
- $X = G \circlearrowleft G \times G$, Ψ : S, Schieder, Lin Chen.
- $X \supset G \circlearrowleft G \times G$, $IC_{X(O)}$: Bouthier-Ngo-Sakellaridis

monoid L-monoid.

$\to \Psi$

- X_O scary [Bouthier-Kazhdan-Varshavsky]
- f.d. model (Grinberg-Kazhdan thm)

Drinfeld's proof :

2 Model :

- $\mathcal{M}_X = \mathrm{Maps}_{gen}(C, X/G \supset X^\circ/G)$

$$\bullet \quad \mathcal{Y}_X = \mathrm{Maps}_{gen}\big(C, X/B \supset \underset{\substack{\| \\ pt}}{X^\circ/B}\big)$$

$$= \prod{}' \, X(\mathcal{O}_v) \cap X^\circ(F_v) \big/ B(\mathcal{O}_v)$$

$X = \overline{N^- \backslash G}$: Finkelberg-Mirkovic : Zastava space.

$$\begin{tikzcd}
\mathcal{Y}_X \arrow[r, hook] \arrow[d, "\pi"'] & \overline{\mathcal{Y}}_X \arrow[dl, "\overline{\pi}"] \\
A_X &
\end{tikzcd}$$

$$A_X = \mathrm{Map}(C, \underset{\sim}{X /\!\!/ N} / T)$$

<u>Thm</u> (S-W) Under our assumptions

$\overline{\pi}$ stratified semi-small.

Decomposition + factorization property:

$$\mathrm{Asymp}(IC_{X(\mathcal{O})}) \simeq \overline{\pi}_! IC_{\overline{\mathcal{Y}}}$$

$$\text{function}: \quad \frac{1}{\prod_{B_X^+} (1 - q^{-\frac{1}{2}} e^{\lambda})}$$

$B_X^+ =$ fibers of $\overline{\pi}$ with s.small equality.

• Roughly speaking: reduce to consider all affine embeddings of $\mathbb{G}_m \backslash GL_2 \times (\text{torus})$

<u>Ex</u> $X = \mathbb{G}_m \backslash PGL_2$

$$\begin{tikzcd}
\mathcal{Y}_X = \mathrm{Sym}\, C \overset{\circ}{\times} \mathrm{Sym}\, C \arrow[r, hook] \arrow[d, "add"] & \mathrm{Sym}\, C \times \mathrm{Sym}\, C \arrow[dl] \\
A_X = \mathrm{Sym}\, C = \coprod C^{(n)} &
\end{tikzcd}$$

<u>Prop</u> (Contraction Principle / Braden's Thm)

$$s^! \Psi(IC_{X_\sigma}) = s^! i^! j_! (IC_{X_\sigma} \otimes \text{Jordan Block})$$

$$\Big\Vert$$

$$\boxed{\pi_{0!} \Psi(IC_{X_\sigma}) \underset{\text{Contraction}}{\overset{=}{\uparrow}} \pi_! IC_{X_\sigma}}$$

$$\begin{tikzcd}
Y_X \arrow[rr, squiggly, "\Psi"] \arrow[dr, "\pi"'] & & Y_{X_0} \arrow[dl, "\pi_0"] \\
& A_X &
\end{tikzcd}$$

Radon Transform

$$C^\infty(N^- \backslash G) \xrightarrow{\text{Intertwiner}} C^\infty(N^- \backslash G)$$

Intertwining operators $\circ$ Asymp = Radon