Recall

$\mathrm{IB} \subseteq MT \subseteq \mathrm{LagCorr}^{1\text{-shifted}}_{\mathrm{Bun}}$

$\circlearrowright S$

$\mathrm{EOM}_B : MT \longrightarrow \mathrm{LagCorr}^{1\text{-shifted}}$

$\mathrm{EOM}_B := \mathrm{Maps}(C_{dR}, -)$

(-1)-shifted! will need grading shear!

$\mathrm{EOM}_A(C) : MT \longrightarrow \mathrm{LagCorr}^{1\text{-shifted}}$

$\mathrm{EOM}_A := \mathrm{Sect}(C, (-)_{K_C^{1/2}})_{dR}$

Ginzburg–Rozenblyum

Conjecture

There exists a functor

$$GQ : \mathrm{Lag\,Corr}^{1\text{-shifted}}_{\substack{\text{pre-quantized} \\ \text{polarized}}} \longrightarrow \mathrm{dg\,Cat}$$

making the following diagram commute

$$\begin{tikzcd} \mathrm{IIB} \arrow[r, "GQ \circ EOM_A"] \arrow[d, "S"'] & \mathrm{Langlands}_{\mathrm{Aut}} \subseteq \mathrm{dg\,Cat} \arrow[d, "\mathbb{L}"'] \\ \mathrm{IIB} \arrow[r, "GQ \circ EOM_B"] & \mathrm{Langlands}_{\mathrm{Spec}} \subseteq \mathrm{dg\,Cat} \end{tikzcd}$$

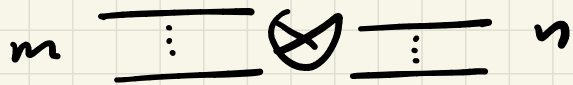
<u>Observation</u> Don't need full GQ. Every object in $\mathbb{II}B$ is a (twisted) 1-shifted conormal bundle

<u>Def</u> Let $f: X \to Y$ and let $\omega : X \to T^*[n]X$ be a closed n-shifted 1-form. Then the n-shifted ω-twisted conormal of f is the composition

$$N^*_\omega[n]f := \begin{tikzcd} & X \arrow[dl] \arrow[dr, "\omega"] & & f^*T^*[n]Y \arrow[dl] \arrow[dr] & \\ pt & & T^*[n]X & & T^*[n]Y \end{tikzcd}$$

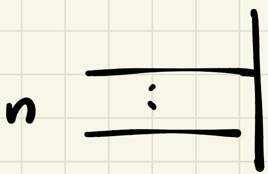
<u>Note</u> In $\mathbb{II}B$ all our 1-forms will be actually be exact

<u>Examples</u>



$$\parallel$$

$$N^{+}[1]\left(\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)/GL_n \times GL_m \to BGL_n \times BGL_m \right)$$



$$:= N^{*}_{d\chi}[1]\left(BU \to BGL_n \right)$$

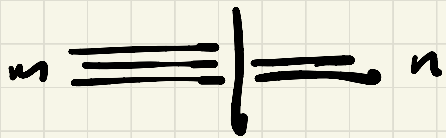
<u>Lemma</u> additive character χ of U

$$\Longrightarrow$$

closed 1-shifted 1-form $d\chi$ on BU

Spectral Whittaker Coefficients

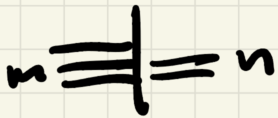
B-type / spectral objects corresponding to



$$\| $$

$$N^{\vee}_{dx}[1] \left(B(GL_m \ltimes U_{n,m}) \longrightarrow BGL_n \times BGL_m \right)$$

Apply the EOM_B

A 

sAS

$$\begin{array}{c} SL_2 \\ \cup \\ \mathbb{C}^\times \end{array}$$

$$\begin{tikzcd}
& \mathbb{G}_a[-1] & \\
& \mathrm{Maps}(C_{dR}, B\mathbb{G}_a) \arrow[u, "{[C]}"'] & \\
& \mathrm{Maps}(C_{dR}, GL_n \times U_{n,m}) \arrow[u, "{\mathrm{Maps}(C_{dR}, \chi)}"'] \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathrm{Flat}_{GL_n}(C) & & \mathrm{Flat}_{GL_m}(C)
\end{tikzcd}$$

$$p_*\left(\tilde{\chi}^!(sAS) \overset{!}{\otimes} q^!(-)\right)$$

To derive the spectral Artin-Schreier sheaf study the Lagrangian

$$d\varepsilon \hookrightarrow T^*[-1]\,\mathbb{C}[-1] \qquad \text{where} \qquad \mathbb{C}[-1] = \operatorname{Spec} \mathbb{C}[\varepsilon]$$

Want to treat this as 1-shifted so need to introduce invertible degree 2 parameter β

$$d(\varepsilon\beta) \hookrightarrow T^*[1]\,\mathbb{C}[-1]\,((\beta))$$

This should be telling us to deform the dualizing sheaf of $\mathbb{C}[-1]$ by $\varepsilon\beta$ to get

$$SAS = \left(\omega_{\mathbb{C}[-1]}((\beta)),\ -\varepsilon\beta\right)$$

experimentally verified

Recall

KD: $IC(\mathbb{C}[-1])_{per} \longrightarrow (\mathbb{C}[u]\text{-mod})_{per}$

$(V, d) \longmapsto (V[u], d + \varepsilon u)$

so get

$$KD(sAs) = (V((\beta))[u], (u-\beta)\varepsilon)$$

which is roughly

$$\Delta\mathbb{C}^\times \subseteq \mathbb{C}_u \times \mathbb{C}^\times_\beta$$

<u>Claim</u> working over $\mathbb{C}((\beta))$ we get a character sheaf on $\mathbb{C}[-1]$

B]

$$
\begin{tikzcd}
 & \mathrm{Sect}\left(C, \left(\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)/(GL_n \times GL_m)\right)_{K_C^{1/2}}\right) \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathrm{Bun}_{GL_n}(C) & & \mathrm{Bun}_{GL_m}(C)
\end{tikzcd}
$$

$$\pi_*\left(p^!_{ren}(-)\right)$$

A] $_m \equiv\!|\!=_n$

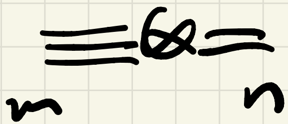
$$\begin{tikzcd}
& \mathbb{A}^1 & e^X \\
& \mathrm{Bun}_{GL_n \ltimes U_{n,m}}^{p(K_C^{1/2})} \arrow[u, "\chi_{m,N}"] \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathrm{Bun}_{GL_n} & & \mathrm{Bun}_{GL_m}
\end{tikzcd}$$

$n = m$

$\left[\mathrm{Sect}\left(C, \, T^*GL_n \times \mathbb{C}^n \, /GL_n \, /GL_m\right)\right]$

$$q_*\left(\left(\chi_{m,N}^! e^X\right) \overset{!}{\otimes} p^!(-)\right)$$

B|



$$
\begin{tikzcd}
 & \mathrm{Maps}\left(C_{dR}, \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n) / GL_m \times GL_n\right) \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathrm{Maps}(C_{dR}, BGL_m) & & \mathrm{Maps}(C_{dR}, BGL_n)
\end{tikzcd}
$$

$$q_*\left(p^!(-)\right)$$

In the case $C = \mathbb{B} = \mathbb{D} \cup_{\mathbb{D}^\times} \mathbb{D}$ then

much of this was constructed BeF, BFN
Lan

B-side

$$\operatorname{Hom}_{MT}(1, G) \xrightarrow{\ GQ \circ F_{\mathbb{B}}\ } \operatorname{Hom}(\mathrm{Vect}, \operatorname{IndCoh}(\operatorname{Flat}_G(\mathbb{B})))$$

$$\cong$$

$$\operatorname{IndCoh}(\operatorname{Flat}_G(\mathbb{B}))$$

no analogue for other C

$$\cong$$

$$\operatorname{Mod}(\operatorname{Sym} \mathfrak{g}[-2])^G \otimes \mathrm{sVect}$$

$$\cong$$

$$\begin{array}{c} G \\ \times \\ \mathbb{C}^\times \end{array} \circlearrowright \begin{array}{c} M \\ \downarrow \mu \\ \mathfrak{g}^* \end{array} \longmapsto \mu_* \mathcal{O}_M[\sqrt{\beta}, \sqrt{\beta}^{-1}] \in \left(\operatorname{QCoh}(\mathfrak{g}^*/G)[\sqrt{\beta}, \sqrt{\beta}^{-1}]\right)^{\mathbb{G}_m}$$

$|\beta| = (2, -2)$

A-side (BFN)

$$\mathrm{Hom}(1, G) \xrightarrow{\ GQ \circ F_A\ } D\text{-mod}(\mathrm{Bun}_G(\mathbb{B}))$$

$$G \circlearrowright T^* V \longmapsto \pi_* \, \omega^{\mathrm{ren}}_{\mathrm{Maps}(\mathbb{B}, V/G)}$$

$\mathbb{C}^\times \circlearrowright$

$$\mathrm{Maps}(\mathbb{B}, V/G) \xrightarrow{\ \pi\ } \mathrm{Bun}_G(\mathbb{B})$$

<u>Note</u> I want to use the dimension theory which differs from the one in BFN by $\sum_x \dim V_x \langle x, \lambda \rangle$ on the component labelled by $\lambda \in \pi_1(G)$.

Applying our constructions to $C = \mathbb{B}$ recovers their results and extends them to the case of twisted coefficients.

In particular the fact that the sAd is a character sheaf lets us verify that we are producing a ring objects.