Goal understand the role of quiver/bow varieties in the geometric Langlands correspondence for GL_n

Output A bunch of new conjectures about the behavior of Whittaker coefficients

joint with Philsang Yoo

based physical work of Gaiotto-Witten

Moore-Tachikawa category MT^{naive}

<u>objects</u> algebraic groups

<u>morphisms</u> $\mathrm{Hom}(G, H) =$ symplectic varieties M with an action of $\mathbb{C}^\times \times G \times H$ such that the $G \times H$ action is Hamiltonian and $wt_{\mathbb{C}^\times}(\omega_M) = 2$

<u>composition</u>

$$\begin{array}{ccccc} & M & & N & \\ G & \longrightarrow & H & \longrightarrow & K \end{array}$$

$$M \circ N = M \times N /\!\!/ \Delta H \quad \circlearrowleft \Delta \mathbb{C}^\times$$

Thm (Calaque–Safronov)

$$MT^{\text{naive}} \subseteq \mathrm{Lag}^{1\text{-shifted}}$$

$$G \longmapsto T^*[1]BG := \mathrm{Spec}_{BG}\, \mathrm{Sym}(\mathbb{T}_{BG}[-1])$$

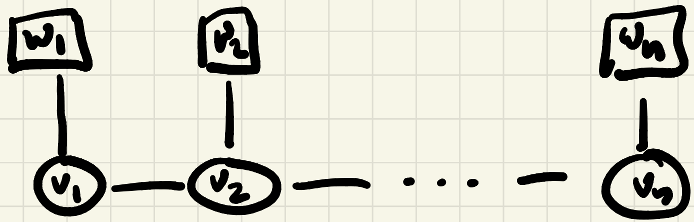
$$= \mathfrak{g}^*/G$$

$$G \xrightarrow{M} H \longmapsto \begin{array}{c} M/G\times H \\ \swarrow \quad \searrow \\ \mathfrak{g}^*/G \qquad \mathfrak{h}^*/H \end{array}$$

$$\begin{array}{c} M \\ \swarrow \quad \searrow \\ pt \qquad pt \end{array}$$

derived stack

Finite Type A quiver varieties



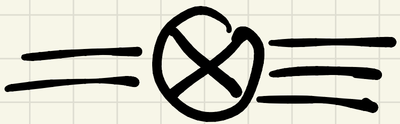
$$T^*\left(\mathrm{Hom}(\mathbb{C}^{v_1},\mathbb{C}^{v_2}) \times \mathrm{Hom}(\mathbb{C}^{v_2},\mathbb{C}^{v_3}) \cdots \mathrm{Hom}(\mathbb{C}^{v_{n-1}},\mathbb{C}^{v_n})\right)$$

$$\times$$

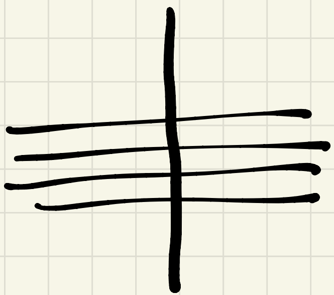
$$\prod \quad T^* \mathrm{Hom}(v_i, w_i) \;/\!\!/\; GL(v)$$

$$GL(v) = \prod GL(v_i)$$

A Moore-Tachikawa perspective on Quiver varieties

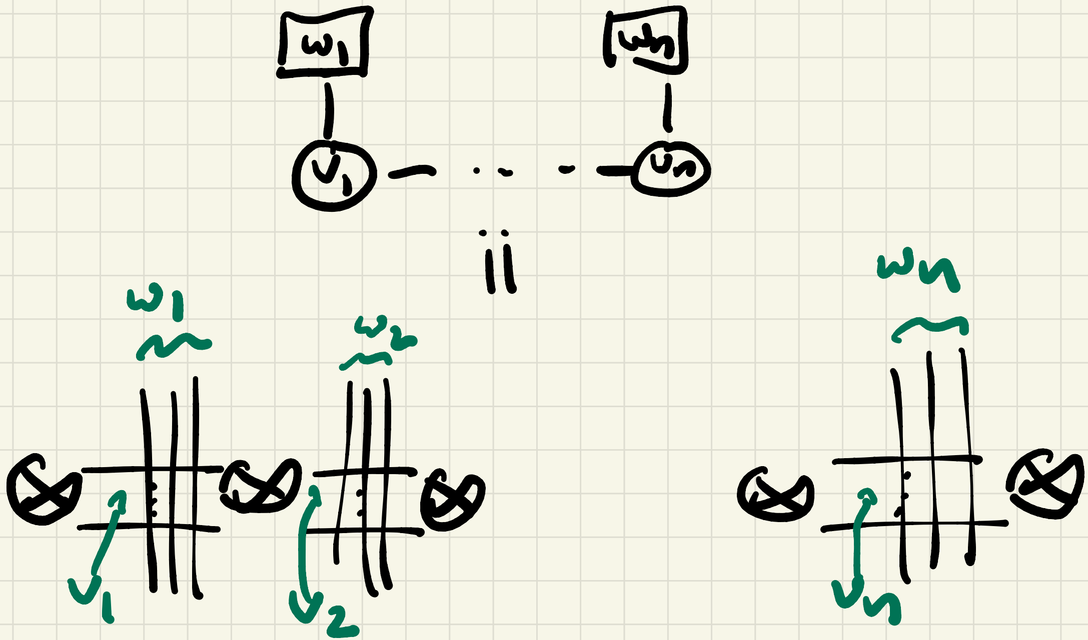
n  m $:=$

$$GL_m \times GL_n \times \mathbb{C}^\times$$
$$\circlearrowright$$
$$T^*\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$$

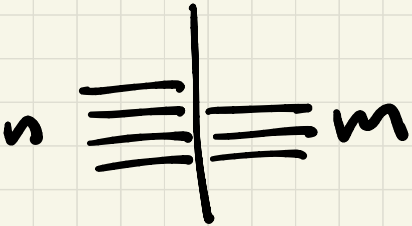
n  n $:=$

$$GL_n \times GL_n \times \mathbb{C}^\times$$
$$\circlearrowright$$
$$T^*(GL_n \times \mathbb{C}^n)$$

$$(g_1, g_2)\cdot(u, v) = (g_1 u g_2^{-1}, g_1 v)$$


w1
wn
v1
vn
w1
w2
wn
v1
v2
vn

For bow variety need one more piece

 $=$

$$Gl_n \times Gl_m \times \mathbb{C}^\times$$

$$Gl_n \times S(\rho) \subseteq T^*Gl_n$$

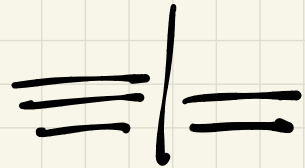
$$Gl_n \times S(\rho) \cong T^*Gl_n /\!\!/_f \, U(\rho), \qquad T^*Gl_n = Gl_n \times gl_n$$

$$\rho : SL_2 \longrightarrow GL_n$$

$$e \longmapsto \begin{pmatrix} \underbrace{\begin{matrix} 0 & 1 \\ & 0 \end{matrix}}_{n-m} & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

fixed under $t^{-2} \rho(t)$

$$S(\rho) = e + C_{gl_n}(f) \subseteq gl_n$$

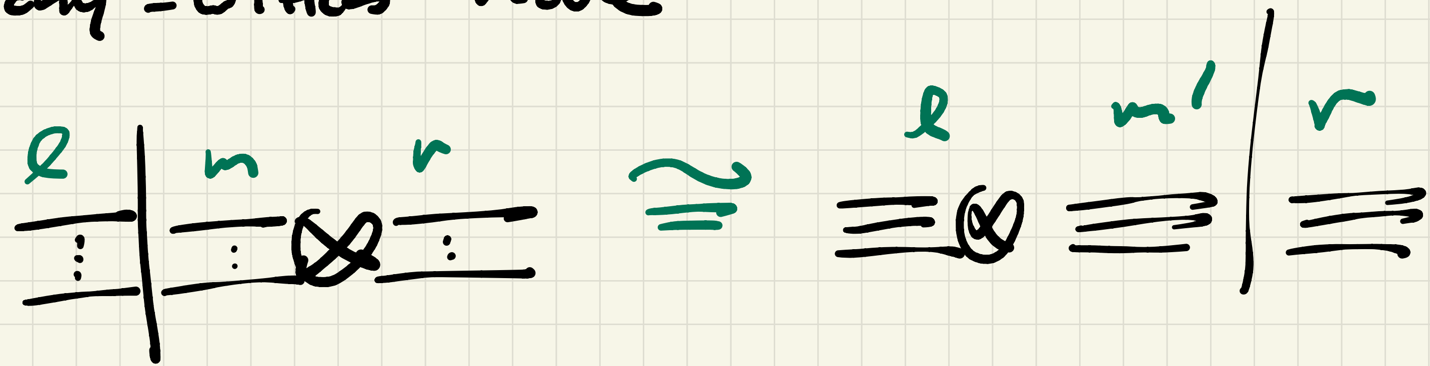
Def A bow variety is any $\mathrm{Hom}_{MT}(1,1)$ that is a composition of  , 

Ex  $:= \left(U \backslash\!\backslash_f T^*GL \times T^*GL_n /\!/_f U \right) /\!/_{\Delta GL_n}$

$$= U \backslash\!\backslash_f T^*GL_n /\!/_f U$$

There is a relation in MT

Hanany-Witten move



when $l, r, m, m' \geq 0$ and

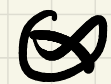
$$m + m' = l + r + 1$$

The purpose of brane diagrams is to describe moduli spaces of vacua in IIB string theory in $\mathbb{R}^{10}$

D3-branes

carry Gl_n SYM

D5-branes



NS5 branes

	0	1	2	3	4	5	6	7	8	9
D3	x	x	x	x						
D5	x	x	x		x	x	x			
NS5	x	x	x					x	x	x

A symplectic view on geometric Langlands [Elliott–Yoo]

$$\mathrm{EOM}_B(\Sigma \times C) \cong \mathrm{Maps}(\Sigma_B \times C_{dR}, T^*[1]BG)$$
$$\cong \mathrm{Maps}(\Sigma_B, \mathrm{Maps}(C_{dR}, T^*[1]BG))$$

only $\mathbb{Z}/2\mathbb{Z}$ graded $\longrightarrow$

$$\cong \mathrm{Maps}(\Sigma_B, T^*[-1]\mathrm{Flat}_G(C))$$

$$\mathrm{EOM}_A(\Sigma \times C) \cong \mathrm{Sect}(\Sigma_B \times C, (T^*[1]BG)_{K_C^{1/2}})_{dR}$$
$$\cong \mathrm{Sect}(\Sigma_B, \mathrm{Sect}(C, (T^*[1]BG)_{K_C^{1/2}})_{dR})$$
$$\cong \mathrm{Maps}(\Sigma_B, T^*[1]\mathrm{Bun}_G(C)_{dR})$$

Get Functors

F_A, F_B

Geometric quantization

$MT \longrightarrow \text{Lag Cor} \dashrightarrow \text{dg Cat}$

$F_B = \text{Maps}(C_{dR}, -)$

$F_A = \text{Sect}(C, (-)_{K_C^{1/2}})_{dR}$

Ex M/G

$\downarrow$

$T^*[1]BG$

$\text{Maps}(C_{dR}, M/G)$

$\downarrow$

$\text{Maps}(C_{dR}, T^*[1]BG)$

$\wr\Vert$

$T^*[-1]\ \text{Flat}_G(C)$

Axioms for geometric quantization

$$GQ(T^*[1]X) = \mathrm{IndCoh}(X)$$

$$GQ(\mathrm{Crit}(f)) = MF(f)$$

$$GQ(N^*_X[1]Y) = \mathrm{IndCoh}(X) \xrightarrow{f_*} \mathrm{IndCoh}(Y)$$

But otherwise it is a black art.

$$GQ\left(\begin{array}{c} L \\ \downarrow \\ T^*[1]X \end{array}\right)$$

fibers are all 0-shifted symplectic stacks

Build a coherent sheaf whose stalks are $GQ(L \cap T^*_xX)$

Expectations with geometric Langlands

MT

$\cup\!|$

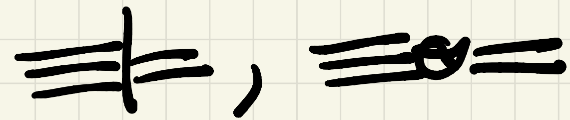
IIB

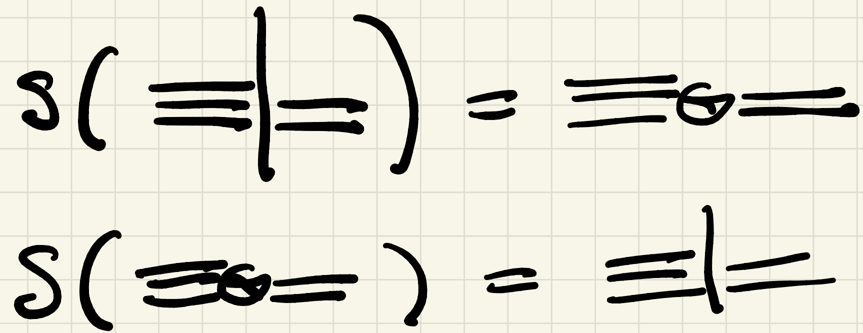
$\circlearrowleft$

S

<u>objects</u> $T^*\Pi\, BGL_n$

<u>morphisms</u>





Expectation

$$
\begin{tikzcd}
\mathrm{IIB} \arrow[r, "F_A"] \arrow[d, "S"'] & \text{Lag Corr} \arrow[r, "GQ"] \arrow[d, phantom, "\circlearrowright"] & \text{Langlands} \subseteq \text{dg Cat} \arrow[d, "\mathbb{L}"', "\cong"] \\
\mathrm{IIB} \arrow[r, "F_B"] & \text{Lag Corr} \arrow[r, "GQ"] & \text{Langlands} \subseteq \text{dg Cat}
\end{tikzcd}
$$

<u>Note</u> S extends to a partially defined functor on MT sending G to G^L

In the case $C = \mathbb{B} = \mathbb{D} \cup_{\mathbb{D}^\times} \mathbb{D}$ then much of this was constructed BeF, BFN

<u>B-side</u>

$$\mathrm{Hom}_{MT}(1, G) \xrightarrow{GQ \circ F_{\mathbb{B}}} \mathrm{Hom}(\mathrm{Vect}, \mathrm{IndCoh}(\mathrm{Flat}_G(\mathbb{B})))$$

$$\cong$$

$$\mathrm{IndCoh}(\mathrm{Flat}_G(\mathbb{B}))$$

no analogue for other C

$$\cong$$

$$\mathrm{Perf}(\mathrm{Sym}\, \mathfrak{g}[2])^G$$

$$\cong$$

$$G \circlearrowright \begin{array}{c} M \\ \downarrow \mu \\ \mathfrak{g}^* \end{array} \longmapsto \mu_* \mathcal{O}_M[\beta, \beta^{-1}] \in (\mathrm{QCoh}(\mathfrak{g}^*/G)[\beta, \beta^{-1}])^{\mathbb{G}_m}$$

$|\beta| = (1, -1)$

A-side (BFN)

$$\mathrm{Hom}(1,G) \xrightarrow{\;GQ \circ F_A\;} D\text{-mod}(\mathrm{Bun}_G(\mathbb{B}))$$

$$G \circlearrowright T^*V \longmapsto \pi_* \, \omega_{\mathrm{Maps}(\mathbb{B}, V/G)}$$

$\mathbb{C}^\times \circlearrowright$ fiber

$$\mathrm{Maps}(\mathbb{B}, V/G) \xrightarrow{\;\pi\;} \mathrm{Maps}(\mathbb{B}, V/G)$$

<u>Note</u> to get $\mathbb{C}^\times$ acting on both V, V^* with weight one need to shift ω on each component of $\mathrm{Bun}_G(\mathbb{B})$.