FLE via Soergel bimodules

jt. w/ J. Campbell & S. Raskin

0. Introduction

Recall statement of LLC:

G & $\check{G}$ Langlands dual groups

$$\begin{array}{ccc} G & & \check{G} \\ \cup & & \cup \\ B & & \check{B} \\ \cup & & \cup \\ T & & \check{T} \end{array}$$

κ & $\check{\kappa}$ dual nondegenerate levels, i.e. invariant bilinear forms on $\mathfrak{g}$ & $\check{\mathfrak{g}}$.

$F = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$

$$\begin{tikzcd} G_F \arrow[d, rightsquigarrow] & \check{G}_F \arrow[d, rightsquigarrow] \\ D_\kappa(G_F) & D_{\check{\kappa}}(\check{G}_F) \end{tikzcd}$$

G_F, $\check{G}_F$ loop groups

convolution algebras of D-modules

$D_K(G_F)$-mod

2-categories of representations of $D_K(G_F)$, i.e. $\mathcal{C}$ w/ $D_K(G_F) \otimes \mathcal{C} \to \mathcal{C}$

<u>Conjecture</u> (Gaitsgory)
Equivalence of 2-categories:

$\mathbb{L}$: $D_K(G_F)$-mod $\xrightarrow{\sim} D_{\check{K}}(\check{G}_F)$-mod

$k \cong \mathbb{F}$

today's talk:

$I \quad \check{I}$ Iwahori subgps

<u>quasi-thm</u>

$$\left\{ \begin{array}{c} D_K(G_F)\text{-mod} \\ \text{generated by} \\ I\text{-invariant} \\ \text{vectors} \end{array} \right\} \simeq \left\{ \begin{array}{c} D_{\check{K}}(\check{G}_F) \\ \text{-mod} \\ \text{gen'd} \\ \text{by } \check{I} \\ \text{inv vectors} \end{array} \right.$$

$$\wr\!\wr \qquad\qquad\qquad\qquad \wr\!\wr$$

$$D_K\left({}_{I\text{-mon}}\backslash G_F / I\text{-mon}\right)\text{-mod} \qquad\qquad \text{" "}$$

one can relate these 2 affine Hecke algebras using Soergel bimodules, & one can relate rep's on both sides & verify low ramifications predictions LLC.

ex. LL: Whittaker

$\Updownarrow$

Kac-Moody invariants

i.e. $D_K(G_F/N_{F,\psi}))$

$$\updownarrow$$

$\hat{\mathfrak{g}}^\vee_K$-mod

$\mathbb{L}$: I invariants $\leftrightarrow$ I invariant

i.e.

$$\mathbb{L}: D_K(G_F/I) \simeq D_{\check{K}}(\check{G}_F/\check{I})$$

<u>conj</u> (Gaitsgory)

$$D_K(N_{F,\psi}\backslash G_F/I) \simeq \hat{\mathfrak{g}}^\vee_K\text{-mod}^{\check{I}}$$

proved for most levels w/
J. Campbell &
Raskin

<u>conj</u> (Gaitsgory) the above equivalence is compatible w/ Feigin-Frenkel for affine W-algebras

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1. Review/sketch of Soergel bimodules

(Soergel, Beilinson, Ginzburg, Bezrukavnikov, Yun, Elias, Williamson, Stroppel, Fiebig...)

1.1. motivating problem:

$\mathfrak{g}$ simple Lie alg

$\mathfrak{g}\text{-mod}_{\hat{0}}^{B}$ ~ regular block of Cat $\mathcal{O}$

L_w simple objects

M_w Verma

A_w dual Verma

T_w tilting, $w \in W$

convention: $w \leftrightarrow w \cdot -2\rho$

i.e. $L_e = M_e = A_e = T_e$

Kazhdan-Lusztig conj (at least) indicates that

$$K_0(\mathfrak{g}\text{-mod}_0^B)$$ is

controlled by (W, S)

weyl group

$\mathcal{C}$ $D(G)$ module

$\mathcal{C}^B \circlearrowright D({}_B\backslash G/_B)$

$D({}_B\backslash G/_B) \circlearrowright \mathfrak{g}\text{-mod}^B_{\hat{0}}$

$\uparrow$ $\qquad\qquad$ $\cong$

Cat of finite Hecke algebra $\qquad$ $D({}_B\backslash G/_{B\text{-mon}})$

Soergel: can reconstruct

$D({}_B\backslash G/_B), D({}_{B\text{-mon}}\backslash G/_B), \ldots$

from (W, S) & action on $\mathfrak{t}^*$.

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appearance of $\mathfrak{t}^*$?

$\mathfrak{g}\text{-mod}^{B}_{\hat{0}}$ ← generalized trivial cent character.

action of $Z\mathfrak{g}_{\hat{0}} \simeq \mathrm{Sym}\,\mathfrak{t}_{\hat{0}}$

on Cat (i.e. $\rightarrow$

---

how to use this?

tautologically,

$$\mathfrak{g}\text{-mod}^{B,C}_{\hat{0}} \simeq \text{mod-}\mathrm{End}\Big(\bigoplus_w P_w\Big)$$

Soergel: can reconstruct using much less.

<u>thm</u> (Soergel)

$$\mathbb{V} : \mathfrak{g}\text{-mod}^{B}_{0} \longrightarrow Z_a\text{-mod}$$

$$M \longrightarrow \mathrm{Hom}(T_{w_0}, M)$$

is fully faithful on tilting objects

<u>thm'</u> $\mathbb{V}'$:

$$\mathfrak{g}\text{-mod}^{B}_{\hat{0}} \to \text{mod-}\mathrm{End}(T_{w_0})$$

fully faithful on tiltings

<u>thm''</u>:

$$Z_{\hat{0}} \longrightarrow \mathrm{End}(T_{w_0})$$

$\mathrm{Sym}\, \mathfrak{t}_{\hat{o}} \longrightarrow \mathrm{Sym}\, \mathfrak{t} \,/\, (\mathrm{Sym}\, \mathfrak{t}_+^W)$

<u>rmk 1</u>: $\mathbb{V}$ far from fully faithful on all of $\mathfrak{g}\text{-mod}^B(\hat{\mathcal{O}}_0 : \mathcal{O}_0 :))$

$T_{w_0} \simeq P_e$

$\mathbb{V}' : \mathcal{O}_0 \longrightarrow \text{mod-}\mathrm{End}(T_{w_0})$

$\mathcal{O}_0 / \langle L_w, w \in e \rangle$

rmk 2: why fully faithful on tiltings?

soc $M_w \simeq L_e$

cosoc $A_w \simeq L_e$

$\mathrm{Hom}(\ker \mathbb{V}', \text{verma flag}) = 0$ vanish

$$\mathrm{Hom}\left(\begin{array}{c}\text{dual}\\ \text{verma}\\ \text{flag}\end{array},\ \ker \mathbb{V}'\right) \simeq 0$$

<u>rmk 3</u>

$$K^b(\text{Tiltings}) \xrightarrow{\sim} D^b(\mathcal{O}_0)$$

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to say what essential image is of $\mathbb{V}$, it's clarifying to pass to bi-monodromic picture:

$$\begin{tikzcd}
{}_B\backslash^{G}/_{B}\text{-mon} \arrow[rr, squiggly] \arrow[d, phantom, "\wr"] & & {}_{B\text{-mon}}\backslash^{G}/_{B\text{-mon}} & B\text{-mon} \\
& R := \mathrm{Sym}\, \mathfrak{t} \arrow[ul] \arrow[dl, "B"'] & & R \otimes R \arrow[ul] \arrow[u] \arrow[d, bend left, squiggly] \arrow[dl, bend left] \\
\mathfrak{g}\text{-mod}_{\hat{0}} \arrow[rr, squiggly] & & \mathfrak{g}\text{-mod}_{\hat{0}} & B\text{-mon}
\end{tikzcd}$$

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want analog of T_{w_0}:

again just projective cover of L_e

(pro-object).

tiltings $\rightsquigarrow$?

<u>ex.</u> single stratum:

$$\begin{array}{c} B\backslash BwB/_{B\text{-mon}} \simeq T/_{T\text{-mon}} \end{array}$$

log of monodromy $\to$

$$R_{\hat{O}}\text{-mod}^{\mathrm{fd.}}$$
$$\wr\!\!\cong$$
$$\mathrm{Fun}(\check{T})^{\wedge}$$

projective object:

$$R_{\hat{o}} \longleftarrow\!\!| \text{ `free monodromic local system'}$$

$$j_{w,!} \rightsquigarrow \hat{j}_{w,!}$$

$$j_{w,*} \rightsquigarrow \hat{j}_{w,*}$$

$$T_w \rightsquigarrow \hat{T}_w, \quad w \in W$$

Thm (Soergel, Bezrukavnikov–Yun)

$$\mathbb{V}^\wedge : \quad \mathfrak{g}\text{-mod}_{\hat{0}}^{B\text{-mon}} \longrightarrow R\otimes R\text{-mod}$$

$$\wr\wr$$

$$B\text{-mon}\backslash G/B\text{-mon}$$

$$M \longmapsto \mathrm{Hom}(T^\wedge_{w_0}, M)$$

1. is fully faithful on free-mon filtrings

2. monoidal

(conv on LHS,

$- \underset{R}{\otimes} -$ on RHS)

<u>describe essential image:</u>

(W, S) $\quad s \in S \rightsquigarrow \hat{T}_s$

every $\hat{T}_y \oplus \hat{T}_{s_1} * \cdots * \hat{T}_{s_\ell}$

"enough" to determine

images of $\hat{T}_s$.

$$\hat{J}_{w!} \overset{\mathbb{V}^\wedge}{\longmapsto} R_w$$

$$R_w \cong \mathrm{Fun}\, \Gamma_w \subset \mathfrak{t}^* \times \mathfrak{t}^*$$

$$t \overset{BwB}{\rightsquigarrow} w^{-1} t w$$

$$\Rightarrow \mathbb{V}^\wedge(\hat{T}_s) \simeq R \otimes_{R^s} R$$

defn

subalgebra of R fixed by s.

$\parallel$

$(\mathrm{Sym}\, \mathfrak{t})^s$

Soergel bimodules are the minimal additive subcat of $R \otimes R$-mod containing R, $R \otimes_{R^s} R$, $s \in S$, closed under $\oplus$, $-\underset{R}{\otimes}-$, and $\mathbb{C}^{\oplus}$.

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i.e.

$$\mathrm{Tilting}^{\wedge}\left({}_{B\text{-mon}}\backslash G /_{B\text{-mon}}\right) \xrightarrow{\sim} \mathrm{SBim}^{\wedge}$$

$$\mathrm{Tiltings}\left({}_{B}\backslash G /_{B\text{-mon}}\right) \xrightarrow{\sim} \mathrm{Smod}$$

(Soergel modules minimal

additive, including $\oplus$, full subcat of R-mod containing k & $(k \otimes_R R')$

closed under $- \otimes_R -$ $\nwarrow$ SBim

Some applications of Soergel bimodules

<u>ex 0.</u> $U \backslash G / B \simeq \check{U} \backslash \check{G} / \check{B}$

$w_0 t^* \simeq t$

<u>ex1.</u>

$V^\wedge(T^\wedge_w)$

$C \oplus \; \underline{R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_l}} R}$

in particular

$V^\wedge(T^\wedge_{w_0}) \simeq R \otimes_{R^W} R$

$$H^*(\check{B}\backslash\check{G}/\check{B}) \xrightarrow{\sim}$$

$$H^*(\check{B}\backslash\check{P}_{s_1}\overset{\check{B}}{\times}\cdots\check{P}_{s_\ell}/\check{B})$$

$\rightsquigarrow$ Koszul duality

(Beilinson–Ginzburg)

<u>thm</u> (BGS, Bez-Yun)

$$D^{mix}\left({}_{B\text{-mon}}\backslash G/_{B\text{-mon}}\right)$$

$$\wr$$

$$D^{mix}\left(\check{B}\backslash\check{G}/\check{B}\right)$$

R ← graded alg w/ t in deg 2

__ex 2__

one can use SBim to give a presentation of Hecke category as a diagrammatic category

⟶ latter diag category

still reconstruct Hecke category w/ modular coefficients

(used by Riche–Williamson in mod rep theory)

ex 2½ Bez–Yun for Kac–Moody groups

ex 3 thesis of Chris Dodd, Soergel bimodule approach to 2 realizations of

affine Hecke category.
$(k=0)$

ex 4 SBim $\Rightarrow$ reprove
KL conjecture
(avoiding decomposition thm)
by Elias-Williamson.

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2. local quantum Langlands w/ tame ramification

& unipotent monodromy

$$D_K\left({}_{I\text{-mon}}\backslash G_F / I\text{-mon}\right)^\circ \leftarrow \text{neutral block}$$

<u>thm</u> $(D, - \ldots)$

$$D_K\left({}_{I\text{-mon}}\backslash G_F / {}_{I\text{-mon}}\right)^\circ$$

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$$D_K\left({}_{K}\backslash \check{G}_F / K \quad \right)^\circ$$

$k(I\text{-mon}\backslash \quad / I\text{-mon})$

<u>rmk</u> version of "local Shimura correspondence" of G. Savin

<u>point</u>: Coxeter groups controlling both sides are canonically equivalent

$$W_{aff,G} : \quad W_f \ltimes \check{L}$$

$$\cup$$

$$W_{aff} \quad W \ltimes \check{Q}$$

$$\cup$$

$$W \ltimes \check{Q}_K$$

$$\check{Q}_K = \{ \check{\lambda} \in \check{Q} : \; K(\check{\lambda}) \in Q \}$$

$$W \ltimes \check{Q} \simeq W \ltimes Q$$

$W_f^n \otimes_K \cong W_f^n \otimes_K$

Coxeter groups controlling LHS & RHS.

<u>rmk</u> extension to non-neutral blocks should be possible

(following work of Lusztig–Yun in finite type).

finite type

2.2

thm (Campbell–D.–Raskin)

$$D_\kappa\left(N_{F,\psi}\backslash G_F / I\text{-mon}\right)$$

$$\wr\!\!\mid\!\!2$$

$$\hat{\mathfrak{g}}^\vee_{\kappa^\vee}\text{-mod}^{\check{I}}$$

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rmk finite type analogy:

$\mathfrak{g}\text{-mod}^B_{\hat\lambda}$ possibly singular block of $\mathcal{O}$

$W_\lambda \subset W$ parabolic subgroup

$\psi_\lambda : N \to \mathbb{G}_a$

$$\psi|_{U_{\alpha_s}} \neq 0 \longleftrightarrow \langle \alpha_s, \lambda+\rho\rangle = 0.$$

ex. λ reg $\longleftrightarrow$ $\psi = 0$

$\lambda = -\rho \longleftrightarrow \psi$ nondeg char

Thm (Milicic–Soergel):

$$D\left(N,\psi_\lambda \backslash G/B\right)$$

$$\mathfrak{g}\text{-mod}_{\hat{\lambda}}^{B}$$

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thm: (Campbell–D.)

$$\mathrm{Loc}: \mathfrak{g}\text{-mod}_{\hat{\lambda}} \hookrightarrow D(G/(N, \psi_{\lambda}))$$

$L \subset G$ levi corresponding to w_λ

thm'':

$\mathfrak{g}\text{-mod}_{\hat\lambda}$

$\cong$

$\mathrm{pind}_L^G \quad L\text{-mod}_{\hat\lambda}$

<u>cor</u> $\quad P = LU$

$$D(N,\psi\backslash G/B) \simeq \mathfrak{g}\text{-mod}_{\hat\lambda}^{B}$$

$$\cong \qquad\qquad \cong$$

$$D(B\backslash G/B) \otimes_{D(B\backslash P/N,\psi)}$$

$$\mathcal{D}(B \backslash P / B)$$

$$\uparrow$$

K_0: $\qquad \mathcal{H} \otimes \mathrm{sgn}$

$\mathcal{H}_L$

$$\mathcal{D}(G/N_1,\psi) \underset{\mathrm{loc}}{\overset{\Gamma}{\rightleftarrows}} \mathfrak{g}\text{-mod}_{\hat\lambda}$$

$$\delta_{N_1\psi} \mapsto \mathrm{ind}_{N_1}^{\mathfrak{g}} \mathbb{C}_\psi \qquad \mathfrak{g}\text{-mod}_{\hat\lambda}$$

(same things hold

mutatis mutandis for affine Lie algebras ($\kappa \notin \mathbb{Q}$)

$$\mathfrak{g}\text{-mod}_\lambda \rightsquigarrow \hat{\mathfrak{g}}\text{-mod}_\kappa^{\hat{\lambda}}$$

$$\hat{\mathfrak{g}}\text{-mod}_\kappa^{\hat{\lambda}} \hookleftarrow D_\kappa(G_F) \text{ submodule generated by } M_\lambda$$

$W_\kappa := W_F \ltimes \check{Q}_\kappa$

$$\left({}^{D}_{K}N_{F(\psi)} \Big\backslash {}^{G}F \Big/ I\text{-mon} \right)^{12}_{\mathfrak{g}'} \cong W_{F} \times Q_{\check{K}}$$

$$\frac{\check{\lambda}' \circlearrowleft W_{K}}{\mathfrak{g}\text{-mod}^{\check{I}\text{-mon}}_{\check{K}}} \qquad \check{\lambda} \circlearrowleft W_{K}$$

each block on each

side just is:

$\mathcal{H}^0_K \leftarrow$ identified neutral blocks of

$\underline{D_K(I\text{-mon} \backslash^{GF}/I\text{-mon})}^0 \mathrel{\dot{\varepsilon}} D_K(\ldots)^c$

$\mathcal{H}^0_{\alpha K} \otimes {}^{v}\mathrm{sgn}$

$\mathcal{H}^{0,\lambda}$

$\uparrow$

sub monodromic Hecke

category corresponding to stab of λ

S_{λ} <=> nondegenerate Whittaker

2.3 Compatibility w/ Feigin-Frenkel duality.

$\mathfrak{g}, \kappa \rightsquigarrow W_{\mathfrak{g},\kappa}$

W-algebra

Thm (Feigin–Frenkel)

$$\mathcal{W}_{\mathfrak{g},\kappa} \simeq \mathcal{W}_{\mathfrak{g}^\vee,\kappa^\vee}$$

$\Psi : \hat{\mathfrak{g}}_\kappa\text{-mod} \longrightarrow \mathcal{W}_{\mathfrak{g},\kappa}\text{-mod}$

Thm (Raskin) induced map

$$(\hat{\mathfrak{g}}_\kappa\text{-mod})_{N_F,\psi} \xrightarrow[\sim]{\Psi} \mathcal{W}_\kappa\text{-mod}$$

Consider that map:

$$D_K\left({}_{N_{F,\psi}}\backslash^{G_F}/_{I} \right) \otimes \hat{g}\text{-mod}_K^I$$
$$\downarrow$$
$$\hat{g}\text{-mod}_K^{N_{F,\psi}} \simeq W\text{-mod}_K$$

<u>conj</u> (Gaitsgory) K not positive.

then:

$$D_K\left(N_{F,\psi}\backslash G_F/I\right) \otimes \hat{g}\text{-mod}^I \to W\text{-mod}_{g,K}$$

$\wr$ FLE $\otimes$ FLE

FF

$$\hat{\mathfrak{g}}\text{-mod}^{\check{I}} \otimes D(\check{N}_{F,\psi} \backslash \check{G}_F / \check{I})$$

$$- \otimes -$$

$$W_{\check{\mathfrak{g}},\check{\kappa}}\text{-mod}$$

commutes up to natural isomorphism.

<u>rmk</u> for $G_{\mathcal{O}}$ (instead of I) w/ κ generic, proved by Arakawa–Frenkel.

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fix $\underline{\check{\lambda}}$, $\underline{\lambda}$

$$\underline{D_\kappa\left(N_{F,\psi}\backslash G_F/I\right)}_{\check{\lambda}} \otimes \underline{\mathfrak{g}\text{-mod}^I}_{\lambda}$$

$$\mathrm{sgn} \otimes \mathcal{H}^0_K \otimes \mathrm{sgn}$$

$$\mathcal{H}_{K,\lambda^\vee} \qquad \mathcal{H}_{K,\lambda} \hookrightarrow W_{\mathfrak{g},K}\text{-mod}$$

"Gelfand" Graev monodromy

"Harish-Chandra monodromy"

$$(\mathrm{Sym}\,\mathfrak{t})^{W_{K,\lambda^\vee}}_{\hat{\lambda}^\vee} \qquad (\mathrm{Sym}\,\mathfrak{t})^{W_{K,\lambda}}_{\hat{\lambda}}$$

2-sided sgn quotient should be describable in terms of

"singular Soergel" bimodules

(developed by G. Williamson)

rmk $\ \ {}_{\mathrm{sgn}\otimes H_1} H \otimes \mathrm{sgn}_{H_2}$

Koszul dual to

$$D(P_1 \backslash G / P_2)$$

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match 2 embeddings into $\mathcal{W}$-mod, suffices to:

(i) match h.w.

if resulting

W-modules

(ii) check that Gelfand-Graev & Harish-Chandra monodromy are exchanged

$\boxed{5}$ —

check on (monodromic) Wakimotos

$$[\hat{j}_{w!}, \hat{j}_{y!}] \simeq 0 \quad \text{if } w \neq y$$

$$\Rightarrow \begin{array}{c} \mathrm{End}(T) \\ \downarrow \\ \mathrm{End}(\mathrm{gr}^{\cdot} T) \\ \uparrow \\ \underline{\text{Verma flag}} \end{array}$$

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thanks!

$$D(N_\psi\backslash G/B)\otimes_{D(B\backslash G/B)} D(B\backslash G/N_\psi)$$

$$\hookdownarrow$$

$$D(N_{\psi_1}\backslash G/N,\psi_2)$$

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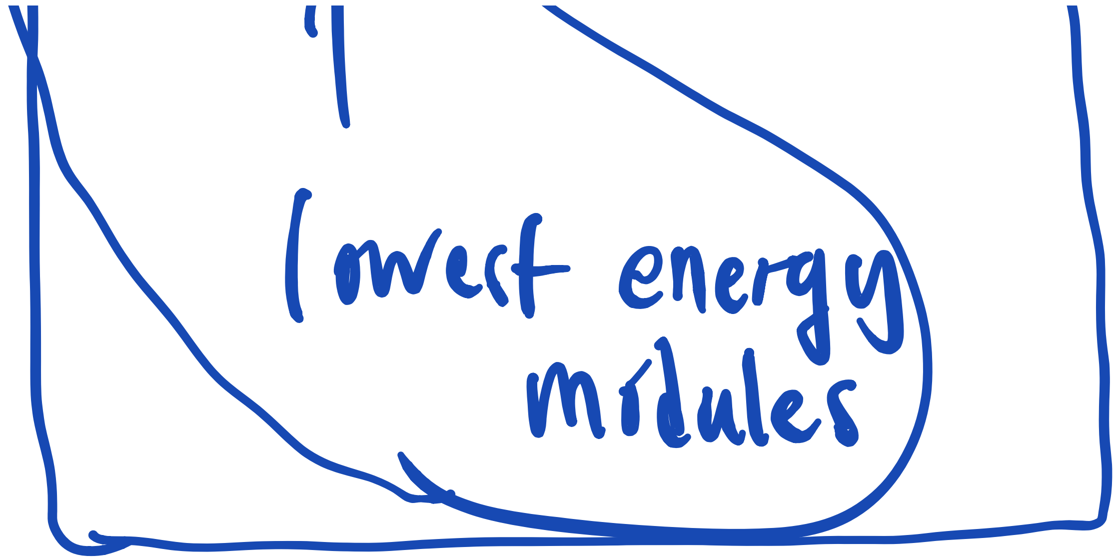
fg

$\mathcal{W}$-mod $\subset$ $\mathcal{W}$-mod


lowest energy
modules