

Monoidal Structures on $GL(2)$ -Modules and Abstractly Automorphic Representations

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Introduction

- L-functions are important objects.
- When the L-functions are equal: get deep connections across mathematics.
 - Modularity Theorem \implies Fermat's Last Theorem.
- In this talk: only automorphic L-functions.

Introduction - cont.

- L-functions are built as GCDs of classes of zeta integrals.
- Two common ways to associate L-functions to representation of $GL(2)$:
 - Godement–Jacquet zeta integrals – uses pairs of vectors,
 - Jacquet–Langlands zeta integrals – uses single vectors.
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 - This talk: also give same class of zeta integrals!
- ⇒ Exotic multiplicative structure on p -adic and automorphic representations.
- Part of PhD thesis, under supervision of J. Bernstein.
Relevant part on arXiv as [Dor20] (and a bit of [Dor21]).

Overview

Introduction

p -adic representations

Godement–Jacquet vs. Jacquet–Langlands

Symmetric monoidal structure

Apology

Automorphic representations

Multiplicative structure

Abstract automorphicity

Notation

- Let F be a local field, $\text{char} F \neq 2$. Let $G = \text{GL}_2(F)$.
- Category $\text{Mod}(G)$ of *smooth* G -modules:
 - For $V \in \text{Mod}(G)$, each $v \in V$ is *smooth*, i.e. fixed by some neighborhood of unity.
 - Contragredient duality $V \mapsto \tilde{V}$: smooth vectors in dual vector space.
- An irreducible G -module is generic iff it has a Kirillov model.

L-functions à la Godement–Jacquet

- Recipe:

1. Matrix coefficient β :

$$\beta(g) = \langle v', \pi(g) \cdot v \rangle.$$

2. Test function $\Psi \in S(M_2(F))$.
3. Integrate:

$$Z_{\text{GJ}}(\Psi, \beta, s) = \int_{\text{GL}_2(F)} \Psi(g) \beta(g) |\det(g)|^{s+\frac{1}{2}} d^\times g.$$

- Get meromorphic zeta integral. Use GCD.

L-functions à la Jacquet–Langlands

- Recipe:
 1. Kirillov model $W : V \times F \rightarrow \mathbb{C}$.
 - Unique up to scalar if it exists.
 2. Test vector $v \in V$.
 3. Integrate:

$$Z_{\text{JL}}(W, v, s) = \int_{F^\times} W_v(y) |y|^{s-\frac{1}{2}} d^\times y.$$

- Get meromorphic zeta integral. Use GCD.

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 - Data: $v' \in \tilde{V}$, $v \in V$, $\Psi \in S(M_2(F))$, some redundancy.

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- Space of JL zeta integrals:
 - Data: Kirillov model, $v \in V$.

V

$$GJ = JL$$

Claim

For generic irreducible V , the two spaces are canonically isomorphic:

$$\tilde{V} \otimes_G S(M_2(F) \times F^\times) \otimes_G V \cong V.$$

- (Up to choice of Kirillov model.)
- Moreover – isomorphism respects zeta integrals.
- Remarkable – non-linear in V !

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- So $Y = S(\mathrm{M}_2(F) \times F^\times)$ should have a third, *hidden* G -action.

Claim

Hidden action exists: Y is a $G \times G \times G$ -module.

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- Choose symplectic space $U = F \cdot e_1 \oplus F \cdot e_2$ of $\dim U = 2$.
- Let

$$\begin{aligned} \mathrm{GL}_2(F)^{3, \det=1} &= \{g_1, g_2, g_3 \in \mathrm{GL}_2(F) \mid \det(g_1 g_2 g_3) = 1\} \\ &\hookrightarrow \mathrm{Sp}(U \otimes U \otimes U). \end{aligned}$$

- Have Weil representation $S(U \otimes U \otimes e_2) = S(\mathrm{M}_2(F))$.
- Compact induction of $S(\mathrm{M}_2(F))$ from $\mathrm{GL}_2(F)^{3, \det=1}$ to $\mathrm{GL}_2(F)^3$ gives Y .

From tri-modules to functors

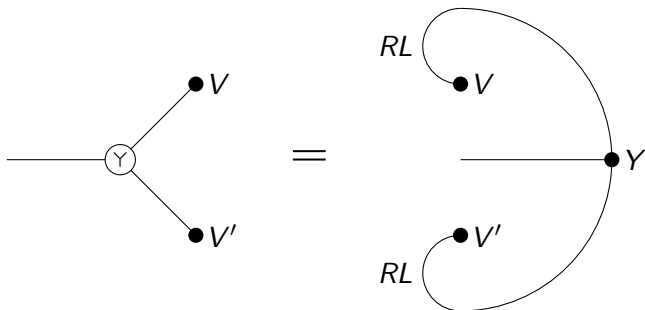
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From tri-modules to functors

- Tri-modules are strange. How can we make sense of them?
Where have we seen them before?
- Think of it as a bi-functor:

$$V \otimes_Y V' = V \otimes_G Y \otimes_G V'$$

- Saw stuff like this before: tensor products.



Additional structure

Claim

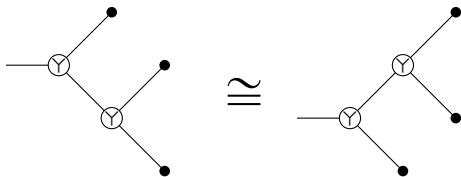
There is an essentially unique extension of \otimes to a symmetric monoidal structure on $\text{Mod}(G)$.

Additional structure

Claim

There is an essentially unique extension of \mathbb{Y} to a symmetric monoidal structure on $\text{Mod}(G)$.

- Means:
 - there is a unit $\mathbb{1}_Y$,
 - \mathbb{Y} is symmetric,
 - \mathbb{Y} is associative.



The unit

- Unit is given by Whittaker space:

$$\mathbb{1}_Y = \left\{ f : \mathrm{GL}_2(F) \rightarrow \mathbb{C} \left| \begin{array}{l} f \text{ is locally const and} \\ \text{compact supp mod } U_2(F), \\ f \left(\left(\begin{array}{cc} 1 & u \\ & 1 \end{array} \right) g \right) = e(u) \cdot f(g) \right. \right\}.$$

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- Important takeaway:
 - Irreducible V has $\dim \mathrm{Hom}(\mathbb{1}_Y, V) = 0, 1$, exactly if V is generic.
 - Choice of map $\mathbb{1}_Y \rightarrow V$ is same data as Kirillov model.

Representations as algebras

- GJ vs JL is now

$$V \otimes \mathrm{Hom}(\mathbb{1}_Y, V) = V \otimes \mathbb{1}_Y \otimes \mathrm{Hom}(\mathbb{1}_Y, V) \rightarrow V \otimes V$$

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is an isomorphism.

- Turns out generic V are commutative algebras (in fact, idempotents).
- Follows because $\mathbb{1}_\gamma \twoheadrightarrow V$ is surjective.

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- This product respects the irreducible representations of G .
- New product \otimes behaves like second case, despite $\mathrm{GL}_2(F)$ not being commutative.

Global theory

Notation

- Let F be a global function field, $\text{char} F \neq 2$. Let $\mathbb{A} = \mathbb{A}_F$, $G = \text{GL}_2(\mathbb{A})$.
- Category $\text{Mod}(G)$ of smooth G -modules.

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- Category $\text{Mod}(G)$ of smooth G -modules.
 - Warning: the category is huge, mostly unimportant!
 - Good only as an ambient space for the automorphic representations to live in.
- An irreducible $(\pi, V) \in \text{Mod}(G)$ is *automorphic* iff it is a subquotient of $\tilde{\mathcal{S}}$, where

$$\mathcal{S} = S(\text{GL}_2(F) \backslash \text{GL}_2(\mathbb{A})).$$

Monoidal structure

- Fix $e: \mathbb{A}/F \rightarrow \mathbb{C}^\times$.
- Space $Y = \mathcal{S}(M_2(\mathbb{A}) \times \mathbb{A}^\times)$ is a $G \times G \times G$ -module via Weil representation.
- Have global symmetric monoidal \otimes_Y induced from the local theory, given by

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- Unit is global Whittaker space,

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- We will use the theta functional to construct a multiplicative structure on $\mathcal{S} = S(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}))$.
- Hope to turn it into a commutative algebra!
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 - Some kind of convolution product – despite GL_2 not being commutative.
- Lots of research questions to answer: what are the modules of this algebra? What are its properties? ...

Theta functional

- The theta functional is a map:

$$\begin{aligned} \mu: Y &\rightarrow \mathbb{C} \\ \Psi(g, y) &\mapsto \sum_{(\xi, q) \in M_2(F) \times F^\times} \Psi(\xi, q). \end{aligned}$$

- Theta functional is invariant to $GL_2(F)^3$ (essentially the Poisson summation formula).

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- Map m sends Godement–Jacquet test data to Jacquet–Langlands test data. Get the same zeta integral!
- Philosophy: $\mathrm{GJ} = \mathrm{JL} \implies$ multiplicative structure.

Algebra of automorphic functions

- Let

$$\mathcal{J} \subseteq \mathcal{S} = S(\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A}))$$

be the space of smooth compactly supported functions, orthogonal to all characters $\chi(\det(g))$ (“generic automorphic functions”).

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Remark

Unit map $\mathbb{1}_Y \rightarrow \mathcal{J}$ is contragredient to sending an automorphic form to its Whittaker function.

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- Have a category of \mathcal{J} -modules in the symmetric monoidal category $\text{Mod}(G)$:

$$\text{Mod}^{\text{aut}}(G),$$

“abstractly automorphic representations”.

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- Decomposes into cuspidal and Eisenstein components, the same way p -adic representations do.
- ... And many more properties!

Questions?



References



Gal Dor.

Exotic Monoidal Structures and Abstractly Automorphic Representations for $GL(2)$.

arXiv e-prints, page [arXiv:2011.03313](#), November 2020.



Gal Dor.

Global Parabolic Induction and Abstract Automorphicity.

arXiv e-prints, page [arXiv:2102.11820](#), February 2021.



Abstract

Consider the function field F of a smooth curve over \mathbb{F}_q , with $q \neq 2$. L-functions of automorphic representations of $GL(2)$ over F are important objects for studying the arithmetic properties of the field F . Unfortunately, they can be defined in two different ways: one by Godement–Jacquet, and one by Jacquet–Langlands. Classically, one shows that the resulting L-functions coincide using a complicated computation.

Each of these L-functions is the GCD of a family of zeta integrals associated to test data. I will categorify the question, by showing that there is a correspondence between the two families of zeta integrals, instead of just their L-functions. The resulting comparison of test data will induce an exotic symmetric monoidal structure on the category of representations of $GL(2)$.

It turns out that an appropriate space of automorphic functions is a commutative algebra with respect to this symmetric monoidal structure. I will outline this construction, and show how it can be used to construct a category of automorphic representations.