A - associative algebra

$$\begin{tikzcd} A \arrow[r] & \mathrm{Der}(A,A) \arrow[r] & Z_{E_2}(A)[1] \\ & \mathbb{C}[\xi] \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

$$0 \to L_A \to A \otimes A \to A \to 0$$

Fiber sequence of Lie algebras

B is an E_n-algebra $\Rightarrow$ $B[n-1]$ is a Lie algebra.

$\mathbb{C}[\xi]$ $\quad \deg(\xi) = 1$

$\left(pt \times_{\mathbb{A}^1} pt\right) \circlearrowright A \iff$ deformation of A over $(\mathbb{A}^1)^{\wedge}_0$

$\nearrow$ is a deformation of A-mod over $(\mathbb{A}^1)^{\wedge}_0$

Interested in: deformations of A s.t. the deformation of A-mod is trivialized

$$\Updownarrow$$

$$\mathbb{C}[\varepsilon] \longrightarrow A \quad (\text{MC elements})$$

$$\begin{tikzcd}
A[n-1] \arrow[r] & \mathrm{Der}(A,A) \arrow[r] & Z_{E_{n+1}}(A)[n] \\
& \mathbb{C}[\varepsilon] \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

$\deg(\varepsilon) = 1$

$(\mathrm{pt} \times_{\mathbb{A}^1} \mathrm{pt}) \circlearrowright A$ as a E_1-algebra.

for deformations of $(A\text{-mod}^c)\text{-mod}^c$ over $(\mathbb{A}^1)^{\wedge}_0$

A-col $= Z_{E_2}((A$-mod $)$-mod $)$

$$U^{E_n}_{Lie}(\mathbb{C}\cdot\xi) \longrightarrow A$$

$$\mathbb{C}[\xi_n] \simeq U^{E_n}_{Lie}(\mathbb{C}\cdot\xi)$$

$\deg(\xi_n) = n$

$n = 2$

$$\left\{\begin{array}{l} \underline{\mathbb{C}}_X \cdot \xi \longrightarrow A^{ch} \text{ as Lie-}* \\ \mathbb{C}[\xi_n] \otimes \omega_X \longrightarrow A^{ch} \text{ as chiral} \end{array}\right.$$

$\mathrm{Fact}(A^{ch}) \in \mathrm{D\text{-}mod}(\mathrm{Ran})$

$\mathrm{Fact}(A^{ch})|_{X^n} := A^{ch,n} \in \mathrm{Dmod}(X^n)$

$A^{ch,1} \in \mathrm{Sh}(X^1)$

$\shortparallel$

$A^{ch}[1]$

In the associative setting.

$\deg(\xi) = 1$

$$A\text{-mod}^{gr} \longrightarrow \tilde{A}\text{-mod}$$

$$A \longrightarrow \tilde{A}$$ - associative algebra over $\mathbb{C}[t]$

Chiral algebras graded by $\mathbb{Z}^{\geq 0}$

chiral algebras $\Longleftrightarrow$ factorization algebras on Ran

A <u>graded</u> chiral algebra $\Longleftrightarrow$ collection of $A^{(d)} \in \mathrm{Shv}(X^{(d)})$ + factorization structure

$$\mathbb{C}[\xi_z] \otimes \omega_x \xrightarrow[n]{} \mathcal{A}^{ch}$$

$$\Downarrow$$

$$\mathbb{C}_{X^{(1)}} \longrightarrow \mathcal{A}^{(1)} \text{ on } X^{(1)}$$

compatible w/ factorization

$$\Downarrow$$

$$s^d \in C^0_{dR}(X, \mathcal{A}^{(1)})$$

$$\left(\mathcal{A}\text{-mod}^{\mathrm{Fact}}\right)^{gr} \longrightarrow \tilde{\mathcal{A}}\text{-mod}^{\mathrm{Fact}}$$

$$A_0 \subseteq A$$

require that $\}$ commutes with A_0

$$\begin{array}{c} \tilde{A}\text{-mod} \\ \downarrow \\ A_0\text{-mod} \end{array}$$

$$\mathcal{A}_0 \subseteq \mathcal{A}$$

$s^a \in \mathrm{HOM}(\mathrm{Unit}[-a], \mathcal{M}^r)$
$(A_0\text{-mod}^{fact})(X^a)$

$$\begin{array}{c} \tilde{A}\text{-mod}^{fact} \\ \downarrow \\ A_0\text{-mod}^{fact} \end{array}$$

A_0 - chiral algebra
$\mathcal{M}^r$ - chiral A_0-module

$$j_* j^* (A_0 \boxtimes \mathcal{M}^r) \longrightarrow \Delta_* \mathcal{M}^r$$

$s \in \Gamma(X, \mathcal{M}^r)$ is a screening if the following happens

$$[a, s] \in \Delta_* \mathcal{M}^r$$
$$\shortparallel$$
$$\phi(a)(0, \partial_t)$$

$\Delta_* \mathcal{M}^r = \mathcal{M}^r \otimes \mathcal{D}_X$

$$\omega_X \mathcal{M} = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

- right multiplication corresponds to $(\xi, 0)$
- the diagonal action corresponds to $(0, \xi)$

$$\mathcal{M}^{\ell} \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{d} \mathcal{M}^{r} \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \mathcal{M}$$

$$[a, s] = d(\phi(a))$$

$$\phi : \mathcal{A}_0 \longrightarrow \mathcal{M}^{\ell} \otimes_{\mathcal{O}_X} \mathcal{D}$$

is a map of right D-modules.

ϕ satisfies a cocycle condition

<u>Lemma 1</u> A screening elt in $\mathcal{M} \Rightarrow$

$$0 \to \mathcal{M} \longrightarrow ? \longrightarrow \mathcal{A}_0 \to 0$$

with a given splitting over ω_X

$$j_! j^! (\mathcal{M} \boxtimes \mathcal{N}_1) \longrightarrow \Delta_* \mathcal{N}_2$$

<u>Lemma</u>? Given a screening section of $\mathcal{M} \Longrightarrow \mathcal{N}_1 \longrightarrow \mathcal{N}_2$.

$$\begin{array}{c} \mathcal{A} \\ \cup \\ \mathcal{A}_0 \end{array} \quad ; \quad \mathcal{A}_+ \text{ - augmentation ideal}$$

We'll be looking for a MC in $\mathcal{A}$ w.r.t. to $\mathcal{A}_0$

$$s^d \in \mathrm{Ext}^{2d}_{\mathcal{A}_0\text{-mod}^{\mathrm{Fact}}(X^{(d)})} (\mathcal{A}_0, \mathcal{A}^{(d)})$$

satisfying factorization.

Suppose that $\mathcal{A}^{(d)} \in \mathrm{Dmod}(X^{(d)})$.

$$\mathcal{A}^{(1)} = \mathcal{A}_1[1]$$

$$S^1 \in \mathrm{Ext}^1_{\mathcal{A}_0\text{-mod}^{fd}}(\mathcal{A}_0, \mathcal{A}_1)$$

<u>Defn</u> S^1 satisfies "quantum Serre relations" if $(S^1)^{(d)}$

$$\rotatebox{90}{$\in$}$$

$$\mathrm{Ext}^{2d}_{X^{(d)}-\Delta}(\mathcal{A}_0, \mathcal{A}^{(d)})$$

extends to an elemt of Ext^{2d} <u>over</u> $X^{(d)}$

$$KL(T,k)$$

$$\circlearrowleft$$

$$\Omega \text{ - grade by } \Lambda^{neg.}$$

$$\mathrm{Wak}: KL(T,k) \longrightarrow \hat{\mathfrak{g}}\text{-mod}_\kappa^{N(O)T(O)}$$

$$\Omega^{\mathrm{Wak}} \in \mathrm{FactAlg}(\mathfrak{g}\text{-mod}_\kappa \ldots)$$

$$\Omega^{\mathrm{Wak}} \text{ graded by } \Lambda^{\mathrm{adj.}}$$

$$\mathcal{A} = (\Omega^{\mathrm{Wak}})_0 = \mathbb{W}\mathrm{ak}^0$$

$$\cup$$

$$\mathcal{A}_0 = \mathcal{A}_{\mathfrak{g},\kappa} \text{ - Kac-Moody chiral algebra.}$$

Want: MC in Ω^{Wak} relative to $\mathcal{A}_0$.

$$\mathcal{A}_1 = \bigoplus_i \mathbb{W}^{-\alpha_i}[-1]$$

$$\mathcal{A}^{(1)} = \bigoplus_i \mathbb{W}^{-\alpha_i}$$

Main construction:

$$\exists \; s_i \in \Gamma(X, \mathbb{W}^{-\alpha_i})$$

Need to replace $G(\mathcal{O})$ by

its twist by Γ using the

T-tensor $g(\omega_x)$

Without the twist, the highest weight like $W^{-\alpha_i}$ is ω.

With the twist it's 0.

Call this section h_{α_i}

$$s_i = e_i(-1) \cdot h_{\alpha_i}$$

<u>Thm</u> These s_i's satisfy the quantum Serre relations in the above sense.

$$X^{(\lambda)}$$

<u>wt</u>$(\lambda) = s_i s_j(g) - g$

[illegible] $\widetilde{\text{Sch}}$ [illegible] fact

$$
\begin{tikzcd}
\Omega^{\mathrm{wk}}\text{-mod} \arrow[r] & \mathrm{Sh} \arrow[d] \\
\Omega\text{-mod}^{\mathrm{Fact}}(\mathrm{KL}(T,\kappa)) \arrow[u, "\mathrm{unlink}"] \arrow[r] & A_{g,\kappa\text{-mod}}^{\mathrm{fin}} \arrow[d] \\
& \left((\hat{\mathfrak{g}}\text{-mod})_{N(a)}\right)^{T(O)}
\end{tikzcd}
$$

$\underline{\text{Thm}}$ $\left(\widetilde{\Omega^{\mathrm{wk}}}\right),$

$$
\begin{tikzcd}
\widetilde{\Omega^{\mathrm{wk}}} \\
A_{g,\kappa} \arrow[u, "S"]
\end{tikzcd}
$$

S_{cr}

$\underline{\text{Conj}}$ The composite functor is the inverse of

$\mathrm{BRST} : (\hat{\mathfrak{g}}\text{-mod}_{N(a)})^{T(O)} \longrightarrow \Omega\text{-mod}^{\mathrm{Fact}}(\mathrm{KL}(T,\kappa))$

Easy: $\mathrm{BRST} \circ S_{cr} \simeq \mathrm{Id}$

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i_λ :

$(\lambda \cdot x) \mid x \hookrightarrow \mathrm{Gr}_{\mathrm{inf}}(X - x) \to \mathrm{Gr}_{\mathrm{inf}}(X)_{\infty \cdot x}$

$$i_{\lambda *}(\mathbb{C} \boxtimes \Omega) \to i_{\mu *}(\mathbb{C} \boxtimes \Omega)$$

$$\Updownarrow$$

$$i_\mu^* \, i_{\lambda *}(\mathbb{C} \boxtimes \Omega) \to \mathbb{C}$$

cohomologically graded vector space

$$\mathbb{W}\mathrm{ak}^\lambda \longrightarrow \mathbb{W}\mathrm{ak}^\mu$$

$$i_{\Delta *}(\mathrm{Wak}^\lambda \boxtimes \Omega^{\mathrm{Wak}}) \to i_{\mu *}(\mathrm{Wak}^\mu)$$

$S^{\lambda\#}$

map between chiral $A_{g,\kappa}$-modules on $\mathrm{Gr}(X)$

$$j_* j^*(\mathcal{M} \boxtimes \mathcal{N}_1) \to \mathcal{N}_2 \rightsquigarrow$$

$$\mathcal{N}_1 \to \mathcal{N}_2.$$

FF:

$$\mathbb{M}_q^{\nu,\lambda} \longrightarrow \mathbb{M}_{q[t]}^{\nu,\mu} \longrightarrow$$

$$\mathbb{W}^{\lambda} \longrightarrow \mathbb{W}^{\mu}[k]$$