n

$$\begin{tikzcd} \mathcal{O} \arrow[d, bend left, "C^{\cdot}(n,-)"] \\ \mathrm{Rep}(T) \arrow[u, bend left, "\mathrm{Ver}"] \end{tikzcd}$$

$$\Omega = C^{\cdot}(n,-) \circ \mathrm{Ver}$$

$$C^{\cdot}(n,-)^{\mathrm{enh}} : \mathcal{O} \xrightarrow{\sim} \Omega\text{-mod}(\mathrm{Rep}(T))$$

$$\begin{tikzcd} \mathcal{O} \arrow[d, bend left, "C^{\cdot}(n,-)"] \\ \mathrm{Rep}(T) \arrow[u, "\mathrm{Ver}^{\vee}"] \end{tikzcd}$$

$$C^{\cdot}(n,-) \circ \mathrm{Ver}^{\vee} = \mathrm{Id}$$

$$\mathrm{Ver}^{\vee} \circ C^{\cdot}(n,-) = ?$$

Claim $\exists$ family of functors over $\mathbb{A}^1/\mathbb{G}_m$ whose fiber at 0 is
whose fiber at 1 is Id

Proof · $\mathrm{Id} \longleftrightarrow \mathrm{Dist}(B) \in \mathrm{Dmod}(G)$

· $\mathrm{Ver}^\vee \circ C^\cdot(\mathfrak{n}, -)$

$$\updownarrow$$

$$\mathrm{Distr}(BN^- \times_T T) \in \mathrm{Dmod}\left(\frac{G/N^- \times G/N}{T}\right)$$

Wonderful degeneration degenerates

$$G \rightsquigarrow \frac{G/N^- \times G/N}{T}$$

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2 families over $\mathbb{A}^1/\mathbb{G}_m$ $\} \longleftrightarrow$ filtered vector spaces

Example $\mathrm{gr}_k(M) = M^k[-k]$

To have a filtered object with

This associated graded $\Leftarrow$

$$M^{k} \xrightarrow{\partial} M^{k+1} \xrightarrow{\partial} M^{k+2}$$

$$\mathrm{Ver}^{\vee} \circ C^{\cdot}(\mathfrak{n}, -)(V^{\lambda})$$

$$\underset{\substack{\ell(w) \\ \| \\ k}}{\bigoplus_{k} \bigoplus} M^{\vee, w\cdot\lambda}[-k].$$

Our construction defines a complex, i.e. introduces a differential s.t. the complex is quasi-isomorphic to V^{λ}

$$M \in \mathrm{ob}(C)^{gr} \rightsquigarrow$$

$$M_t \in \mathrm{ob}(C) \otimes \mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$$

$$k[\xi] \curvearrowright M$$

$\deg(\xi) = 1 \qquad \xi^2 = 0$

$$\Updownarrow$$

$(pt \times_{/\mathbb{A}^1} pt) \circlearrowright M$

$$M_t = \bigoplus_k M^k t^{-k} \cdot \mathbb{C}[t]$$

$$d_t = d + \xi$$

$$\begin{tikzcd} \mathrm{Rep}(T) \arrow[r, "\mathrm{Ver}^\vee"] & \mathcal{O} \\ \mathcal{O} \arrow[u, "C^\cdot(n, -)"] & \end{tikzcd}$$

$$\begin{tikzcd} \Omega\text{-mod}(\mathrm{Rep}(T)) \arrow[r] & \mathrm{Rep}(T) \arrow[r, "\mathrm{Ver}^\vee"] & \mathcal{O} \\ & & \mathbb{C}[\xi] \arrow[u, bend right] \end{tikzcd}$$

$$\left(\hat{\mathfrak{g}}_\kappa\text{-mod}_{N(K)}\right)^{T(O)} = \mathcal{O}^{\mathrm{aff}}$$

At a fixed point

$$\begin{tikzcd} \left(\hat{\mathfrak{g}}_\kappa\text{-mod}_{N(K)}\right)^{T(O)} \arrow[d, equal] \\ \hat{\mathfrak{g}}_\kappa\text{-mod}^{I} \end{tikzcd}$$

$$\begin{tikzcd} \left(\hat{\mathfrak{g}}_\kappa\text{-mod}_{N(K)}\right)^{T(O)} \arrow[d, "\mathrm{BRST}(\mathfrak{n}(K),-)"] \\ \mathrm{KL}(T,\kappa) \end{tikzcd}$$

$$\mathrm{BRST}\left(1_{\hat{\mathfrak{g}}\text{-mod}}\right) = \Omega^{\mathrm{Lus}} \in \mathrm{FactAlg}\left(\mathrm{KL}(T,\kappa)\right)$$

$$1_{\hat{\mathfrak{g}}\text{-mod}} = \mathrm{Vac}_\kappa$$

$$\mathrm{BRST}^{\mathrm{enh}} : \mathcal{O}^{\mathrm{aff}} \xrightarrow{\sim} \Omega\text{-mod}^{\mathrm{fact}}\left(\mathrm{KL}(T,\kappa)\right)$$

Conj / Thm

(Lin Chen - Charles Fu)

$KL(T, k) = \Lambda$-graded vector spaces

Λ - weight lattice of T

As a factorization category

$KL(T, k) = D\text{-mod}_k(\mathrm{Conf}(X, \Lambda))$

$\Omega \in \mathrm{FactAlg}(D\text{-mod}_k(\mathrm{Conf}(X, \Lambda)))$

Ω_{Betti} - $\mathrm{Mod}^{Fact}(\mathrm{Shv}(\mathrm{Conf}(X, \Lambda)))$

$\quad \shortparallel \leftarrow$ [BFS]

$\mathrm{Rep}_q(G)^{mxd}$ Lurie

$\mathrm{Fact}^{\lambda}(\Omega) \in D\text{-mod}_k(X^{\lambda})$

$\lambda \in \Lambda^{neg} - 0 \qquad \lambda = \sum n_i(-\alpha_i)$

$$X^\lambda = \prod_i X^{(n_i)}$$

$$\mathrm{Fact}^{-\alpha_i}(\Omega) = \mathbb{C}_X$$

$$\mathrm{Fact}^{-\alpha_i}(\Omega^{Heis}) = J^{-\alpha_i}$$

$$\Omega = \mathrm{BRST}(n(K), V_{act})$$

weird $-\alpha_i$-component is

$$J^{-\alpha_i}[-1]$$

$$\begin{tikzcd} \mathcal{O}^{aff} = (\hat{g}_\kappa\text{-}\mathrm{ind}_{n(K)})^{T(O)} \arrow[d, shift left, "\mathrm{BRST}"] \\ \mathrm{KL}(T, \kappa) \arrow[u, shift left, "\mathrm{Wak}"] \end{tikzcd}$$

$$\mathrm{IWak}^{\frac{\infty}{2}} : KL(T, \kappa) \longrightarrow \hat{\mathfrak{g}}_\kappa\text{-mod}^{N(K)T(O)}$$

$$\langle \mathrm{IWak}^{\frac{\infty}{2}}(\mathcal{M}_1), \mathcal{M}_2 \rangle_{\hat{\mathfrak{g}}} = \langle \mathcal{M}_1, \mathrm{BRST}(\mathcal{M}_2) \rangle_{\hat{\mathfrak{t}}}$$

$$\mathcal{M}_2 \in \hat{\mathfrak{g}}\text{-}\kappa\text{-mod}$$

$$\mathrm{Wak} = \mathrm{Av}_*^{N(O)} \circ \mathrm{Wak}^{\frac{\infty}{2}}$$

$$\underline{\mathrm{Ver}^\vee} = \mathrm{Av}_*^N \circ \mathrm{Ver}^-$$

$$\Omega^{\mathrm{IWak}} = \underset{\displaystyle \mathrm{FactAlg}\left(\hat{\mathfrak{g}}_\kappa\text{-mod}^{T(O)N(O)}\right)}{\underset{\Cap}{\mathrm{IWak}(\Omega)}}$$

$$\mathrm{IWak} : \Omega\text{-mod}(KL(T,\kappa)) \longrightarrow \Omega^{\mathrm{Wak}}\text{-mod}^{\mathrm{Fact}}\left(\hat{\mathfrak{g}}_\kappa\text{-mod}^{N(O)T(O)}\right)$$

Want: a $\mathbb{A}^1/\mathbb{G}_m$ - family of functors $\qquad F_t$

$$\Omega^{Wak}\text{-mod}^{fact}\left(\hat{g}_\kappa\text{-mod}^{N(O)T(O)}\right) \longrightarrow \hat{g}_\kappa\text{-mod}^{N(O)T(O)}$$

$$\begin{tikzcd}
\Omega\text{-mod}(KL(T,\kappa)) \arrow[r, "\mathrm{Wak}"] & \Omega^{Wak}\text{-mod}^{fact}\left(\hat{g}_\kappa\text{-mod}^{N(O)T(O)}\right) \arrow[d, "F_t"] \\
 & \hat{g}_\kappa\text{-mod}^{N(O)T(O)} \arrow[d] \\
 & \left(\hat{g}_\kappa\text{-mod}_{N(K)}\right)^{T(O)}
\end{tikzcd}$$

F_0 : forgetful functor

$$\Omega^{Wak}\text{-mod}^{fact}\left(\hat{g}_\kappa\text{-mod}^{N(O)T(O)}\right) \longrightarrow \hat{g}_\kappa\text{-mod}^{N(O)T(O)}$$

F_1

$$BRST^{enh} \circ F_1 = Id$$

The way we'll produce F_t

is by having an action of

<u>$\mathbb{C}[\xi]$ on F_0</u>

$\mathbb{C}[\xi] \circlearrowright M \rightsquigarrow M_t$

$(\mathrm{pt} \times \mathrm{pt} / \mathbb{A}^1) \circlearrowright A \rightsquigarrow A_t$

$\Downarrow$

$$\mathbb{C}[\eta] \xrightarrow[\text{as } E_2\text{-algebras}]{} Z(A)$$

$\deg(\eta) = 2$

If we want the category A_t and not to deform (have a connection along $\mathbb{A}^1$)

$S: \mathbb{C}[\xi] \longrightarrow A$

as associative algebras,

$\mathbb{C}\cdot\xi \longrightarrow A$

as Lie algebras

Maurer-Cartan element

A_t lives over $\mathbb{A}^1/\mathbb{G}_m$.

$$A\text{-mod} \longrightarrow A_t\text{-mod}$$
$$M \longmapsto M_t$$

$$A_t\text{-mod} \longrightarrow QCoh(\mathbb{A}^1)$$

$$A_0 \subseteq A$$

$$A_t\text{-mod} \longrightarrow QCoh(\mathbb{A}^1)\otimes A_0\text{-mod}.$$

need S to commute with A_0

A $\quad E_k$-algebra

$$\begin{gathered} \mathbb{C}\cdot\xi \longrightarrow A[1-k] \\ \Updownarrow \\ \mathrm{ind}_{Lie}^{E_k}(\mathbb{C}\cdot\xi) \longrightarrow A \\ \| \\ \mathbb{C}[\xi], \quad \deg(\xi) = k \end{gathered}$$

$\mathbb{C}_L[k]$; $\deg L(\alpha) = k$

X - space ($X = \mathbb{R}^k$)

A E_n-algebra

$\forall n$ $\quad \mathrm{Fact}^n(A) \in \mathrm{Shv}(X^n)$

The screening data therefore is a map of factorization algebras

$$\underline{\mathbb{C}}_{X^n} \xrightarrow{s^n} \mathrm{Fact}^n(A) \quad \text{on } X^n \ \forall n$$

$$\Updownarrow$$

$$s^n \in C^\bullet_{dR}(X^n, \mathrm{Fact}^n(A))$$

that behaves factorizably

$\mathcal{A}$ - chiral algebra $\rightsquigarrow$

$\mathrm{Fact}^n(\mathcal{A}) \in \mathrm{D\text{-}mod}(X^n)$

$s^n \in C^\bullet_{dR}(X^n, \mathrm{Fact}^n(A)) \in$ factorizable