C

C - dualizable category

$F : C \to C$

$\mathrm{Tr}(F, C) \in \mathrm{Vect}$

$$\mathrm{Vect} \xrightarrow{u_C} C \otimes C^\vee \xrightarrow{F \otimes \mathrm{Id}} C \otimes C^\vee \xrightarrow{ev_C} \mathrm{Vect}$$

$$\mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) = \mathrm{Autom.}$$

$C = \mathrm{QCoh}(Y)$

$C^\vee = \mathrm{QCoh}(Y)$

$$\downarrow$$

$\mathrm{QCoh}(Y) \otimes \mathrm{QCoh}(Y)$

$\shortparallel$

$$\mathrm{QCoh}(\mathcal{Y}\times\mathcal{Y})$$

$$\begin{tikzcd} \mathrm{QCoh}(\mathcal{Y}) \arrow[r, "\Gamma(\mathcal{Y},-)"] & \mathrm{Vect}. \\ \mathrm{QCoh}(\mathcal{Y}\times\mathcal{Y}) \arrow[u, "\Delta^*"] & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{QCoh}(\mathcal{Y}) \arrow[r, "\Delta_*"] & \mathrm{QCoh}(\mathcal{Y}\times\mathcal{Y}) \\ \mathrm{Vect} \arrow[u] & \end{tikzcd}$$

$$F = f_* \qquad f : \mathcal{Y} \to \mathcal{Y}.$$

$$\boxed{\mathrm{Tr}(f_*, \mathrm{QCoh}(\mathcal{Y})) = \Gamma(\mathcal{Y}^f, \mathcal{O}_{\mathcal{Y}^f})}$$

$$\begin{tikzcd} \mathcal{Y}^f \arrow[r] \arrow[d] & \mathcal{Y} \arrow[r] \arrow[d, "\Delta"] & \mathrm{pt} \\ \mathcal{Y} \arrow[r] & \mathcal{Y}\times\mathcal{Y} & \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[r, "\mathrm{Graph}_f"] \arrow[d] & Y \times Y \\ pt & \end{tikzcd}$$

$$\mathrm{Tr}(\mathrm{Fr}, \mathrm{IndCoh}(Y)) = \Gamma(Y^f, \omega_{Y^f})$$

$$\mathrm{IndCoh}(Y)^\vee = \mathrm{IndCoh}(Y)$$

$$\begin{tikzcd} \mathrm{IndCoh}(Y) \arrow[r] & pt \\ \mathrm{IndCoh}(Y \times Y) \arrow[u, "\Delta^!"] & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{IndCoh}(Y) \arrow[r, "\Delta_*"] & \mathrm{IndCoh}(Y \times Y) \\ pt \arrow[u] & \end{tikzcd}$$

$$\begin{tikzcd} \omega_Y \\ k \arrow[u] \end{tikzcd}$$

$\mathcal{Y}$ is smooth;

canonically Calabi-Yau $\iff \mathcal{O}_{\mathcal{Y}} = \omega_{\mathcal{Y}}$.

$\mathrm{Dmod}(\mathcal{Y})$

$$\mathrm{Dmod}(\mathcal{Y}) \simeq \mathrm{Dmod}(\mathcal{Y})^{\vee}.$$

$$\begin{tikzcd}
\mathrm{Dmod}(\mathcal{Y}) \arrow[r, "C_{dR}(\mathcal{Y}{,}-)"] & \mathrm{Vect} \\
\mathrm{Dmod}(\mathcal{Y} \times \mathcal{Y}) \arrow[u, "\Delta^!"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Dmod}(\mathcal{Y}) \arrow[r, "\Delta_*"] & \mathrm{Dmod}(\mathcal{Y} \times \mathcal{Y}) \\
\mathrm{Vect} \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
\omega_{\mathcal{Y}} \\
k \arrow[u, mapsto]
\end{tikzcd}$$

$$\mathrm{Tr}(f_*, \mathrm{Dl}(\mathcal{Y})) = C_\bullet(\mathcal{Y}^f, \omega_{\mathcal{Y}^f})$$

$$\begin{tikzcd}
\mathrm{Shv}^{all}(\mathcal{Y}) \arrow[r, "C_c(\mathcal{Y}{,}-)"] & \mathrm{Vect}. \\
\mathrm{Shv}^{all}(\mathcal{Y}\times\mathcal{Y}) \arrow[u, "\Delta^*"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Shv}^{all}(\mathcal{Y}) \arrow[r, "\Delta_!"] & \mathrm{Shv}^{all}(\mathcal{Y}\times\mathcal{Y}) \\
\mathrm{Vect} \arrow[u] &
\end{tikzcd}
\qquad e \mapsto e_{\mathcal{Y}}$$

$$\mathrm{Tr}(f_!, \mathrm{Shv}^{all}(\mathcal{Y})) = C_c(\mathcal{Y}^f, e_{\mathcal{Y}^f})$$

$$\mathrm{Shv}^{all}(\mathcal{Y}_1) \otimes \mathrm{Shv}^{all}(\mathcal{Y}_2) \xrightarrow{\sim} \mathrm{Shv}^{all}(\mathcal{Y}_1 \times \mathcal{Y}_2)$$

$\mathrm{Shv}(\mathcal{Y})$

$$(\mathrm{Shv}(\mathcal{Y}))^\vee \simeq \mathrm{Shv}(\mathcal{Y})$$

$\mathrm{Shv}(\mathcal{Y})$ - compactly generated.

$$C_1 \simeq C_2^\vee \Longleftrightarrow C_1^c \simeq (C_2^c)^{op}$$

$$\mathbb{D} : (\mathrm{Shv}(\mathcal{Y})^c)^{op} \longrightarrow \mathrm{Shv}(\mathcal{Y})^c$$

Take $\mathbb{D}$ to be the Verdier duality.

$$\begin{tikzcd} & \mathrm{Shv}(\mathcal{Y}) \arrow[r, "C(\mathcal{Y},-)"] & \mathrm{Vect} \\ \mathrm{Sh}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \arrow[r, "\boxtimes"] & \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}) \arrow[u, "\Delta^!"] & \end{tikzcd}$$

$$\mathcal{U}_{\mathcal{Y}} := \Delta_*(\omega_{\mathcal{Y}}) \in \mathrm{Shv}(\mathcal{Y}\times\mathcal{Y})$$

$$\begin{array}{c} \mathcal{U}_{\mathrm{Shv}(\mathcal{Y})} := (\boxtimes^R)(\mathcal{U}_{\mathcal{Y}}) \\ \cap \\ \mathrm{Shv}(\mathcal{Y}) \boxtimes \mathrm{Shv}(\mathcal{Y}) \end{array}$$

$$\mathrm{Tr}(f_*, \mathrm{Shv}(\mathcal{Y})) \neq C(\mathcal{Y}^f, \omega_{\mathcal{Y}^f})$$

$\mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{Y}))$ will be explain next time.

$$\begin{array}{ccc} & \mathrm{Shv}(\mathcal{Y}) & \xrightarrow{C_c(\mathcal{Y},-)} \mathrm{pt} \\ & \downarrow{\Delta^*} & \\ \mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \xrightarrow{\boxtimes} & \mathrm{Shv}(\mathcal{Y}\times\mathcal{Y}) & \end{array}$$

not a perfect pairings.

"would be unit"

$\boxtimes^L(\mathcal{U}_{\mathcal{Y}})$ - pro-object

Thm

$$\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \xrightarrow{ev^*_{\mathrm{Bun}_G}} \mathrm{Vect}$$

is a perfect pairing.

$$ev^!_Y(\mathcal{F}_1, \mathcal{F}_2) = C(Y, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2)$$

$$ev^*_Y(\mathcal{F}_1, \mathcal{F}_2) = C_c(Y, \mathcal{F}_1 \overset{*}{\otimes} \mathcal{F}_2)$$

Example Y; $W \subseteq T^*Y$

$$\mathrm{Shv}_W(Y) \otimes \mathrm{Shv}_W(Y) \xrightarrow{ev^*_Y} \mathrm{Vect}$$

Under what conditions is it perfect?

- Y is smooth and proper variety.

  $W = \{0\}$

- $\mathcal{Y}$ has finitely many isom. classes of points. $\mathcal{W} = T^*\mathcal{Y}$
- $\mathcal{Y} = N \backslash G / B$; $U \in \mathrm{Bun}_G(\mathbb{P}^1)$.

Describe the unit object:

$$U_{\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)} \in \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$$

$$\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \underset{i^R}{\overset{i}{\rightleftarrows}} \mathrm{Shv}(\mathrm{Bun}_G)$$

$i \circ i^R : \mathrm{Shv}(\mathrm{Bun}_G)$

<u>Thm</u> $\uparrow$ is given by an integral Hecke functor. P

$$\mathrm{Shv}(\mathrm{Bun}_G) \xrightarrow{H(V,-)} \mathrm{Shv}(\mathrm{Bun}_G \times X^I)$$

$V \in \mathrm{Rep}(\check{G})^{\otimes I}$

$M \in \mathrm{Shv}(X^I)$

$$\begin{tikzcd} {} \arrow[d, "\otimes M"] \\ \mathrm{Shv}(\mathrm{Bun}_G \times X^I) \arrow[d] \\ \mathrm{Shv}(\mathrm{Bun}_G) \end{tikzcd}$$

$P \otimes P : \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G) \circlearrowleft$

the projector onto

$\mathrm{Shv}_{\mathrm{Nilp} \times \mathrm{Nilp}}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$

<u>Cor</u>

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$
$$\downarrow \wr$$
$$\mathrm{Shv}_{\mathrm{Nilp} \times \mathrm{Nilp}}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$$

• Y - ~~proper~~ smooth proper scheme

$$\mathrm{Shv}_{2\mathcal{O}}(Y) \otimes \mathrm{Shv}_{\mathcal{W}}(Z) \xrightarrow{\sim} \mathrm{Shv}_{2\mathcal{O}_Y \times \mathcal{W}}(Y \times Z)$$

For constructible over $\mathbb{C}$,

Y doesn't have to be proper.

$$\boxed{\begin{gathered}\mathrm{Shv}_{\mathcal{W}_1}(Y_1) \otimes \mathrm{Shv}_{\mathcal{W}_2}(Y_2) \xrightarrow{\sim} \mathrm{Shv}_{\mathcal{W}_1 \times \mathcal{W}_2}(Y_1 \times Y_2) \\ \text{dim. } 0. \quad e^{\frac{x}{y}}\end{gathered}}$$

• Y - have finitely many classes of points.

$$\mathrm{Shv}(Y) \otimes \mathrm{Shv}_{\mathcal{W}}(Z) \xrightarrow{\sim} \mathrm{Shv}_{T^*Y \times \mathcal{W}}(Y \times Z)$$

• $$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathcal{W}}(Z) \xrightarrow{\sim} \mathrm{Shv}_{\mathrm{Nilp} \times \mathcal{W}}(\mathrm{Bun}_G \times Z).$$

<u>Fact</u> all objects in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ have reg. singularities.

The unit is given as follows:

$$\Delta_!(\underline{e}_{\mathrm{Bun}_G}) \in \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$$

$$(P \otimes P)(\Delta_!(\underline{e}_{\mathrm{Bun}_G})) \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

ev^{e} satisfies the duality axiom against $\Delta_!(\underline{e}_{\mathrm{Bun}_G})$

$\mathrm{Shv}(\mathrm{Bun}_G)_{co}$

$$\mathrm{Shv}(\mathrm{Bun}_G) = \lim_{\overleftarrow{U \subseteq \mathrm{Bun}_G}} \mathrm{Shv}(U)$$

$$U_1 \xhookrightarrow{j} U_2 \qquad j^*$$

$$\mathrm{Shv}(\mathrm{Bun}_G)_{co} = \underset{\overrightarrow{U}}{\mathrm{colim}}\ \mathrm{Shv}(U) \qquad j_*$$

A priori,

$$\mathrm{Shv}(\mathrm{Bun}_G)^\vee = \mathrm{Shv}(\mathrm{Bun}_G)_{co}$$

$$\mathrm{Shv}(\mathrm{Bun}_G)_{co} \xrightarrow{\mathrm{Mir}_{\mathrm{Bun}_G}} \mathrm{Shv}(\mathrm{Bun}_G)$$

$$u_y := \Delta_*(\omega_y)$$
$$(ps\text{-}u)_y := \Delta_!(\underline{e}_y).$$

$$\mathrm{Mir}_{\mathcal{Y}}(\mathcal{F}) = p_{2*}\left(p_1^{*}(\mathcal{F}) \overset{!}{\otimes} \omega_{\mathrm{Sch},\mathcal{Y}} \right)$$

$$\begin{tikzcd}
 & \mathcal{Y} \times \mathcal{Y} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathcal{Y} & & \mathcal{Y}
\end{tikzcd}$$

<u>Def</u> $\mathcal{Y}$ is miraculous if $\mathrm{Mir}_{\mathcal{Y}}$ is an equivalence.

<u>Thm</u> Bun_G is Miraculous.

<u>Lemma</u> $\mathrm{Mir}_{\mathrm{Bun}_G}$ sends

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_{co} \xrightarrow{\sim} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

<u>Thm</u> The following diagram

commutes:

$$\begin{tikzcd}
\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \arrow[dr, "ev^*_{\mathrm{Bun}_G}"] & \\
& \mathrm{Vect} \\
\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)_{co} \arrow[uu, "\mathrm{id}\otimes \mathrm{Mir}_{\mathrm{Bun}_G}"] \arrow[ur, "ev^!_{\mathrm{Bun}_G}"'] &
\end{tikzcd}$$

$\mathcal{Y}_1 \times \mathcal{Y}_2$.

$$Q \in \mathrm{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2)$$

$$\rightsquigarrow$$

$$\mathrm{Shv}(\mathcal{Y}_1)_{co} \longrightarrow \mathrm{Shv}(\mathcal{Y}_2)$$

$\mathrm{Char} \subseteq \mathrm{Shv}(G/\mathrm{Ad}\, G)$

$G/\mathrm{Ad}\, G = \left(\mathrm{Bun}_G(\mathbb{P}^1)^{0,\infty}\right)^{s.s.}$

$\mathcal{C}$ - compactly generated:

Def $\mathcal{C}$ is proper if $\forall c_1, c_2 \in \mathcal{C}^c$,

$\mathrm{Hom}(c_1, c_2)$ is fin. dim.

$$\Updownarrow$$

$$\mathrm{Vect} \xrightarrow{u_{\mathcal{C}}} \mathcal{C} \otimes \mathcal{C}^\vee$$

$u_{\mathcal{C}}$ admits a left adjoint.

$$\begin{tikzcd}
\mathcal{C} \otimes \mathcal{C}^\vee \arrow[rr, "(u_{\mathcal{C}})^L"] \arrow[dr, "Se_{\mathcal{C}} \otimes id"'] & & \mathrm{Vect} \\
& \mathcal{C} \otimes \mathcal{C}^\vee \arrow[ur, "ev_{\mathcal{C}}"'] &
\end{tikzcd}$$

Defn $\mathcal{C}$ is Serre if $Se_{\mathcal{C}}$ is an equivalence.

In this case, the unit of the

new pairing is

$$\mathbb{D}^{op} \xrightarrow{\mathbb{D}} \mathbb{D}^{\vee}$$

$d = \mathrm{colim}\, d_i \quad \mathbb{D}(d) := \lim_i \mathbb{D}(d_i)$

$\mathbb{D} = C \otimes C^{\vee}$

$\mathbb{D}^{\vee} = C^{\vee} \otimes C$.

$(ps\text{-}u)_C := \mathbb{D}(u_C) \in C \otimes C^{\vee}$

Lemma If C is free $\Rightarrow$ the unit of new pairing is given by $(ps\text{-}u)_C$.

$$Ps\text{-}Id_C : C \longrightarrow C$$

Lemma If C is Serre then Se_C is the inverse of $P_S - I_d$ C.

$$Shv_\omega(Y) \otimes Shv_\omega(Y) \xrightarrow{ev_Y^!} Vect$$

$u_{Shv_\omega(Y)}$ is the value of the right adjoint to

$$Shv_\omega(Y) \otimes Shv_\omega(Y) \xrightarrow{(\boxtimes \circ i \times i)} Shv(Y \times Y)$$

on $u_Y := \Delta_*(\omega_Y)$.

$$(u_{Shv_\omega(Y)})^L = ev_Y^*.$$

Defi (Y, N) is Serre if

$(ev_Y^*, (\boxtimes \circ (i \times i))^R (Ps - Id)_Y))$

$(\mathrm{ps}\text{-}u)_Y := \Delta_!(\omega_Y)$ define a duality

between $\mathrm{Shv}_{\mathcal{N}}(Y)^\vee \simeq \mathrm{Shv}_{\mathcal{N}}(Y)$.

Lemma If $(Y, \mathcal{N})$ is Serre, then $\mathrm{Shv}_{\mathcal{N}}(Y)$ is Serre, and $\left(\mathrm{Se}_{\mathrm{Shv}_{\mathcal{N}}(Y)}\right)^{-1}$ is Miraculous functor.

$$\boxed{eu^{*}_{\mathrm{Bun}_G} = eu^{!}_{\mathrm{Bun}_G}}$$

Arinkin, Kazhdan, Raskin, Rozenblyum, Varshavsky.