

Chiral algebra

~~Gauge theory~~ tools for the
Analytic Geometric Langlands program II

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Structure of the talk

- An intertwining kernel for Gaudin Hamiltonians
- The $SL(2)$ addition kernel $K(a,b,c)$
- A Vertex Algebra interpretation of $K(a,b,c)$
- Hitchin Hamiltonians intertwining properties
- Hecke operators intertwining properties
- Arakawa's Vertex Algebras and addition (multiplication) kernels
- Other generalizations

Preliminary facts

The magic of $2 \times 2 \times 2=8$

- Hamiltonian action of $\mathrm{SL}(2) \times \mathrm{SL}(2) \times \mathrm{SL}(2)$ on \mathbb{C}^8
- Quadratic moment maps μ_a, μ_b, μ_c
- Unique invariant quartic polynomial: $\mathrm{Tr} \mu_a^2 = \mathrm{Tr} \mu_b^2 = \mathrm{Tr} \mu_c^2$
- Notation: coordinates $s^{\pm\pm\pm}$
- Nice Lagrangian submanifold: $s^{\pm\pm\pm} = a^\pm b^\pm c^\pm$
 - Vanishing moment maps
 - Useful quantization $\delta^{(2)} \left(\det_{\beta\gamma} s^{+\beta\gamma} \right) \simeq \delta^{(2)} \left(\det_{\alpha\gamma} s^{\alpha+\gamma} \right) \simeq \delta^{(2)} \left(\det_{\alpha\beta} s^{\alpha\beta+} \right)$

Trifundamental Symplectic Bosons

Aka Weyl Vertex Algebra, see Lafforgue's talk

- Symplectic boson VOA attached to symplectic vector space \mathbb{C}^8

$$Z^{\alpha\beta\gamma}(z)Z^{\alpha'\beta'\gamma'}(w) \sim \frac{\epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}\epsilon^{\gamma\gamma'}}{z-w} \quad [Z_n^{\alpha\beta\gamma}, Z_m^{\alpha'\beta',\gamma'}] = \delta_{n+m,0}\epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}\epsilon^{\gamma\gamma'}$$

- $\widehat{\mathfrak{sl}}(2)$ Kac-Moody subalgebras at critical level

$$J_a = \mu_a[Z], J_b = \mu_b[Z], J_c = \mu_c[Z]$$

- Centers of Kac-Moody subalgebras coincide!

$$\mathrm{Tr} \mu_a^2[Z] = \mathrm{Tr} \mu_b^2[Z] = \mathrm{Tr} \mu_c^2[Z]$$

The non-chiral symplectic boson

A free theory

$$\mathcal{Z}(E_a, E_b, E_c) = \int DZ D\bar{Z} e^{\int_C (Z, \bar{\partial}Z) - (\bar{Z}, \partial\bar{Z})} = \frac{1}{|\det \bar{\partial}_{E_a \otimes E_b \otimes E_c \otimes K^{\frac{1}{2}}}|}$$

$$\left\langle Z^{\alpha\beta\gamma}(z) Z^{\alpha'\beta'\gamma'}(w) \right\rangle_{E_a, E_b, E_c} = \int DZ D\bar{Z} e^{\int_C (Z, \bar{\partial}Z) - (\bar{Z}, \partial\bar{Z})} Z^{\alpha\beta\gamma}(z) Z^{\alpha'\beta'\gamma'}(w) = \frac{\mathcal{G}^{\alpha,\beta,\gamma;\alpha',\beta',\gamma'}(z, w)}{|\det \bar{\partial}_{E_a \otimes E_b \otimes E_c \otimes K^{\frac{1}{2}}}|}$$

- Z is a section of $E_a \otimes E_b \otimes E_c \otimes K^{\frac{1}{2}}$
- $(\ , \)$ is symplectic pairing
- Partition function is an half-density on $\text{Bun} \times \text{Bun} \times \text{Bun}$

Sugawara vectors and Hitchin Hamiltonians

A central point

$$H_a(z) \cdot \mathcal{Z}[E_a, E_b, E_c] = \langle \text{Tr} \mu_a^2[Z] \rangle$$

$$H_a(z) \cdot \mathcal{Z}[E_a, E_b, E_c] = H_b(z) \cdot \mathcal{Z}[E_a, E_b, E_c] = H_c(z) \cdot \mathcal{Z}[E_a, E_b, E_c]$$

- $K(a,b,c) = Z(a,b,c)$ intertwines Hitchin's Hamiltonians!
- Easy to generalize to real bundles: chiral algebra on surface with involution

Regular punctures

A nice singularity

$$V(z, \bar{z}) = \int dv d\bar{v} e^{vZ^{+++}(z) - \bar{v}\bar{Z}^{+++}(\bar{z})}$$

$$H_a(z; w) \cdot \langle V(w, \bar{w}) \rangle_{E_a, E_b, E_c} = \langle \text{Tr} \mu_a^2[Z](z) V(w, \bar{w}) \rangle$$

- Insertion of V requires parabolic structure at z of specific weight

$$V(z, \bar{z}) = \int dv d\bar{v} e^{va_\alpha b_\beta c_\gamma Z^{\alpha\beta\gamma}(z) - \text{c.c.}}$$

- Addition kernel with unipotent regular singularities is a correlation function of V 's

$$K(a, b, c) = \mathcal{Z}[a, b, c] \int \prod_i |dv_i|^2 e^{\sum_{i,j} v_i v_j \mathcal{G}(z_i, z_j; a_i, b_i, c_i; a_j, b_j, c_j) - \text{c.c.}}$$

Regular punctures of general weight

A Mellin transform

- Deformed vertex operator

$$V_{\lambda, \bar{\lambda}}(z, \bar{z}) = \int dv d\bar{v} v^\lambda \bar{v}^{-\bar{\lambda}} e^{vZ^{+++}(z) - \bar{v}\bar{Z}^{+++}(\bar{z})}$$

- Correlation function is Mellin transform of Gaussian:

$$K(a, b, c) = \mathcal{Z}[a, b, c] \int \prod_i |dv_i|^2 v_i^{\lambda_i} \bar{v}_i^{-\bar{\lambda}_i} e^{\sum_{i,j} v_i v_j \mathcal{G}(z_i, z_j; a_i, b_i, c_i; a_j, b_j, c_j) - \text{c.c.}}$$

Hecke operators and spectral flow modules

A role for Weyl modules

$$\mathcal{H}_a(z, \bar{z}) \cdot \mathcal{Z}[E_a, E_b, E_c] = \langle S_a(z, \bar{z}) \rangle_{E_a, E_b, E_c}$$

$$S_a(z, \bar{z}) = S_b(z, \bar{z}) = S_c(z, \bar{z})$$

$$\partial_z^2 S_a(z, \bar{z}) + H_a(z) S_a(z, \bar{z}) = 0$$

- Minimal Hecke modification makes $Z(z)$ anti-periodic.
- Hecke operator inserts a special Ramond module for symplectic bosons

$$Z^{\alpha\beta\gamma}(z) S_a(0,0)[v] \sim \frac{S_a(0,0)[c^{\alpha\beta\gamma} v]}{z^{\frac{1}{2}}} + \dots$$
$$[c^{\alpha\beta\gamma}, c^{\alpha'\beta'\gamma'}] = \epsilon^{\alpha\alpha'} \epsilon^{\beta\beta'} \epsilon^{\gamma\gamma'}$$

A wonderful Weyl module

Quantizing the nice locus

- Non-chiral description:

$$S_a = S_a \left[\int |dt|^2 e^{t \det_{2 \times 2} u - c.c.} \right] \quad c^{-\cdots} = u \cdot \quad c^{+\cdots} = \partial_u$$

- Secretly symmetric

$$\mu_a[c] \circ \int |dt|^2 e^{t \det_{2 \times 2} u - c.c.} = 0 \quad \mu_b[c] \circ \int |dt|^2 e^{t \det_{2 \times 2} u - c.c.} = 0$$

$$\mu_c[c] \circ \int |dt|^2 e^{t \det_{2 \times 2} u - c.c.} = 0$$

Arakawa's VOA

Chiral algebras of class S, Sicilian boundary VOAs

- Chiral algebras $V_k[\mathfrak{g}]$ labelled by ADE Lie algebra and integer
 - k copies of critical Kac-Moody subalgebras, glued along center
- Conformal blocks of $V_k[\mathfrak{g}]$ as twisted D-module on $\mathrm{Bun} \times \mathrm{Bun} \dots \times \mathrm{Bun}$
 - GL dual to structure sheaf of diagonal
 - Obviously intertwines Hitchin Hamiltonians
- Non-chiral CFTs?

Operations on Arakawa's VOA

Gluing or removing

- Whittaker reduction of one copy $V_k[\mathfrak{g}] \rightarrow V_{k-1}[\mathfrak{g}]$
- Diagonal quantum DS reduction $V_k[\mathfrak{g}] \times V_{k'}[\mathfrak{g}] \rightarrow V_{k+k'-2}[\mathfrak{g}]$
 - Non-chiral version of quantum DS reduction: 2d gauge theory

$$Z_{k+k'-2,\mathfrak{g}}(a_1, \dots, a_{k-1}, b_2, \dots, b_{k'}) = \int_{\text{Bun}} d\mu_c Z_{k,\mathfrak{g}}(a_1, \dots, a_{k-1}, c) Z_{k',\mathfrak{g}}(c, b_2, \dots, b_{k'})$$

- Still associative? Tested for $V_4[\mathfrak{sl}_2]$ on 4pt sphere
- Wavefunction decomposition? $Z_{k,\mathfrak{g}}(a_1, \dots, a_k) = \sum_{\rho \sim \bar{\rho}} C_{\rho, \bar{\rho}}^{k-2} \prod_i \psi_{\rho, \bar{\rho}}(a_i)$

Other generalizations

Beyond addition

- Many predictions of kernels dual to specific sheaves
 - Analogues to Lafforgue's talk
- Example: $\mathrm{Sp}(2n) \times \mathrm{SO}(2n+2)$ symplectic bosons dual to $\mathrm{SO}(2n+1)$ in $\mathrm{SO}(2n+2)$
- Challenge: for many other examples levels are not critical.
- Example: $\mathrm{Sp}(2n) \times \mathrm{SO}(2n)$ symplectic bosons dual to $\mathrm{SO}(2n)$ in $\mathrm{SO}(2n+1)$
 - What to do? D-modules can be twisted, but what about wavefunctions?
 - Twist may be needed for lack of section?

Appetizer: a Gaudin intertwiner

A peculiar formula

- Gaudin Hamiltonians for half-densities on $(\mathbb{C}P^1)^n$

$$H_i^{(a)} = \sum_{j \neq i} \frac{e_i f_j + 2h_i h_j + f_i e_j}{z_i - z_j} \quad f_i = \partial_{a_i} \quad h_i = a_i \partial_{a_i} + \frac{1}{2} \quad e_i = -a_i^2 \partial_{a_i} - a_i$$

- Intertwining condition:

$$\begin{aligned} H_i^{(a)} K(a, b, c) &= H_i^{(b)} K(a, b, c) = H_i^{(c)} K(a, b, c) \\ \bar{H}_i^{(a)} K(a, b, c) &= \bar{H}_i^{(b)} K(a, b, c) = \bar{H}_i^{(c)} K(a, b, c) \end{aligned}$$

- Algebraic solution:

$$K(a, b, c) = \frac{1}{|\det_{n \times n} A|}$$

$$\begin{aligned} A_j^i &= \frac{(a_i - a_j)(b_i - b_j)(c_i - c_j)}{z_i - z_j} & i \neq j \\ A_i^i &= 0 \end{aligned}$$

The $\text{SL}(2)$ addition kernel

Gaudin \rightarrow Hitchin

- Genus 0, parabolic points z_i, Bun_0
 - Lines $(1, a_i)$ at parabolic points z_i , modulo $\text{SL}(2)$
 - Gauge-fix three points: a_1, a_2, a_3 , etc.

$$\sqrt{d\mu_a} = |a_1 - a_2| |a_1 - a_3| |a_2 - a_3| |da_4 da_5 \cdots da_{n+3}|$$

$$K(a, b, c) = \frac{1}{|\det A|} \sqrt{d\mu_a} \sqrt{d\mu_b} \sqrt{d\mu_c}$$

- $K(a, b, c)$ intertwines Hitchin's Hamiltonians on $\text{Bun}_0 \times \text{Bun}_0 \times \text{Bun}_0$
- Claim: also intertwines aGL Hecke operators!

The Lame' addition kernel

Genus 0, four points

- Fix $z_1 = 0, z_2 = 1, z_3 = \infty, z_4 = z$
- Gauge-fix $a_1 = 0, a_2 = 1, a_3 = \infty, a_4 = a$ etcetera.
- Hitchin Hamiltonians -> Lame' operator $\partial_a a(a-1)(a-z)\partial_a + a$

$$K(a, b, c) = \frac{1}{|\det A|} |da||db||dc|$$

$$\det A \sim 1 + \frac{a^2 b^2 c^2}{z^2} + \frac{(1-a)^2(1-b)^2(1-c)^2}{(1-z)^2} - 2\frac{abc}{z} - 2\frac{(1-a)(1-b)(1-c)}{1-z} - 2\frac{abc}{z} \frac{(1-a)(1-b)(1-c)}{1-z}$$

- Why addition: $\det A=0$ is support of addition along smooth fibers of Hitchin system, with zero at $a=z$

$$x^2 = a(a-1)(a-z)u \quad \omega = \frac{dadu}{x} = \frac{dadx}{a(a-1)(a-z)}$$

The $\text{SL}(2)$ addition kernel

Genus 2 and higher, no parabolic points

$$K(a, b, c) = \frac{1}{\left| \det \bar{\partial}_{E_a \otimes E_b \otimes E_c \otimes K^{\frac{1}{2}}} \right|}$$

- E : rank 2 associated bundle
- Singular at Theta divisor where $E_a \otimes E_b \otimes E_c \otimes K^{\frac{1}{2}}$ has sections
- Parabolic points: build A from Green's function

$$K(a, b, c) = \frac{1}{\left| \det \bar{\partial}_{E_a \otimes E_b \otimes E_c \otimes K^{\frac{1}{2}}} \right|} \frac{1}{|\det A|}$$