Chiral algebra

~~Gauge theory~~ tools for the Analytic Geometric Langlands program

Davide Gaiotto

Structure of the talk

- An intertwining kernel for Gaudin Hamiltonians
- The SL(2) addition kernel K(a,b,c)
- A Vertex Algebra interpretation of K(a,b,c)
- Hitchin Hamiltonians intertwining properties
- Hecke operators intertwining properties
- Arakawa's Vertex Algebras and addition (multiplication) kernels

Appetizer: a Gaudin intertwiner

A peculiar formula

- Gaudin Hamiltonians for half-densities on $(\mathbb{C}P^1)^n$

$$H_i^{(a)} = \sum_{j\neq i} \frac{e_i f_j + 2h_i h_j + f_i e_j}{z_i - z_j} \qquad f_i = \partial_{a_i} \qquad h_i = a_i \partial_{a_i} + \frac{1}{2} \qquad e_i = -a_i^2 \partial_{a_i} - a_i$$

- Intertwining condition:

$$H_i^{(a)} K(a,b,c) = H_i^{(b)} K(a,b,c) = H_i^{(c)} K(a,b,c)$$
$$\bar{H}_i^{(a)} K(a,b,c) = \bar{H}_i^{(b)} K(a,b,c) = \bar{H}_i^{(c)} K(a,b,c)$$

- Algebraic solution:

$$K(a,b,c) = \frac{1}{|\det_{n\times n} A|}$$

$$A^i_j = \frac{(a_i - a_j)(b_i - b_j)(c_i - c_j)}{z_i - z_j} \qquad i \neq j$$
$$A^i_i = 0$$

The SL(2) addition kernel

Gaudin -> Hitchin

- Genus 0, parabolic points z_i, Bun_0
    - Lines $(1, a_i)$ at parabolic points z_i, modulo SL(2)
    - Gauge-fix three points: a_1, a_2, a_3, etc.

$$\sqrt{d\mu_a} = |a_1 - a_2||a_1 - a_3||a_2 - a_3||da_4 da_5 \cdots da_{n+3}|$$

$$K(a,b,c) = \frac{1}{|\det A|}\sqrt{d\mu_a}\sqrt{d\mu_b}\sqrt{d\mu_c}$$

- K(a,b,c) intertwines Hitchin's Hamiltonians on $Bun_0 \times Bun_0 \times Bun_0$
- Claim: also intertwines aGL Hecke operators!

The Lame' addition kernel

Genus 0, four points

- Fix $z_1 = 0, z_2 = 1, z_3 = \infty, z_4 = z$
- Gauge-fix $a_1 = 0, a_2 = 1, a_3 = \infty, a_4 = a$ etcetera.
- Hitchin Hamiltonians -> Lame' operator $\partial_a a(a-1)(a-z)\partial_a + a$

$$K(a,b,c) = \frac{1}{|\det A|}|da||db||dc|$$

$$\det A \sim 1 + \frac{a^2b^2c^2}{z^2} + \frac{(1-a)^2(1-b)^2(1-c)^2}{(1-z)^2} - 2\frac{abc}{z} - 2\frac{(1-a)(1-b)(1-c)}{1-z} - 2\frac{abc}{z}\frac{(1-a)(1-b)(1-c)}{1-z}$$

- Why addition: det A=0 is support of addition along smooth fibers of Hitchin system, with zero at a=z

$$x^2 = a(a-1)(a-z)u \qquad \omega = \frac{dadu}{x} = \frac{dadx}{a(a-1)(a-z)}$$

The SL(2) addition kernel

Genus 2 and higher, no parabolic points

$$K(a,b,c) = \frac{1}{\left|\det \bar{\partial}_{E_a \otimes E_b \otimes E_c \otimes K^{\frac{1}{2}}}\right|}$$

- E: rank 2 associated bundle
- Singular at Theta divisor where $E_a \otimes E_b \otimes E_c \otimes K^{\frac{1}{2}}$ has sections
- Parabolic points: build A from Green's function

$$K(a,b,c) = \frac{1}{\left|\det \bar{\partial}_{E_a \otimes E_b \otimes E_c \otimes K^{\frac{1}{2}}}\right|} \frac{1}{|\det A|}$$

Preliminary facts

The magic of 2 x 2 x 2=8

- Hamiltonian action of SL(2) x SL(2) x SL(2) on $\mathbb{C}^8$
- Quadratic moment maps μ_a, μ_b, μ_c
- Unique invariant quartic polynomial: $\mathrm{Tr}\,\mu_a^2 = \mathrm{Tr}\,\mu_b^2 = \mathrm{Tr}\,\mu_c^2$
- Notation: coordinates $s^{\pm\pm\pm}$
- Nice Lagrangian submanifold: $s^{\pm\pm\pm} = a^\pm b^\pm c^\pm$
    - Vanishing moment maps
    - Useful quantization $\delta^{(2)}\left(\det_{\beta\gamma} s^{+\beta\gamma}\right) \simeq \delta^{(2)}\left(\det_{\alpha\gamma} s^{\alpha+\gamma}\right) \simeq \delta^{(2)}\left(\det_{\alpha\beta} s^{\alpha\beta+}\right)$

Trifundamental Symplectic Bosons

Aka 4 beta-gamma systems

- Symplectic boson VOA attached to symplectic vector space $\mathbb{C}^8$

$$Z^{\alpha\beta\gamma}(z)Z^{\alpha'\beta'\gamma'}(w) \sim \frac{\epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}\epsilon^{\gamma\gamma'}}{z-w}$$

- $\widehat{\mathfrak{sl}}(2)$ Kac-Moody subalgebras at critical level

$$J_a = \mu_a[Z],\ J_b = \mu_b[Z],\ J_c = \mu_c[Z]$$

- Centers of Kac-Moody subalgebras coincide!

$$\mathrm{Tr}\,\mu_a^2[Z] = \mathrm{Tr}\,\mu_b^2[Z] = \mathrm{Tr}\,\mu_c^2[Z]$$

The non-chiral symplectic boson

A free theory

$$\mathcal{Z}(E_a, E_b, E_c) = \int DZ\, D\bar{Z} e^{\int_C (Z,\bar{\partial}Z)-(\bar{Z},\partial\bar{Z})} = \frac{1}{\left|\det \bar{\partial}_{E_a\otimes E_b\otimes E_c\otimes K^{\frac{1}{2}}}\right|}$$

$$\left\langle Z^{\alpha\beta\gamma}(z) Z^{\alpha'\beta'\gamma'}(w)\right\rangle_{E_a,E_b,E_c} = \int DZ\, D\bar{Z} e^{\int_C (Z,\bar{\partial}Z)-(\bar{Z},\partial\bar{Z})} Z^{\alpha\beta\gamma}(z) Z^{\alpha'\beta'\gamma'}(w) = \frac{\mathcal{G}^{\alpha,\beta,\gamma;\alpha',\beta',\gamma'}(z,w)}{\left|\det \bar{\partial}_{E_a\otimes E_b\otimes E_c\otimes K^{\frac{1}{2}}}\right|}$$

- Z is a section of $E_a \otimes E_b \otimes E_c \otimes K^{\frac{1}{2}}$
- (,) is symplectic pairing
- Partition function is an half-density on Bun x Bun x Bun

Sugawara vectors and Hitchin Hamiltonians

A central point

$$H_a(z) \cdot \mathcal{Z}[E_a, E_b, E_c] = \left\langle \mathrm{Tr}\mu_a^2[Z] \right\rangle$$

$$H_a(z) \cdot \mathcal{Z}[E_a, E_b, E_c] = H_b(z) \cdot \mathcal{Z}[E_a, E_b, E_c] = H_c(z) \cdot \mathcal{Z}[E_a, E_b, E_c]$$

- K(a,b,c) = Z(a,b,c) intertwines Hitchin’s Hamiltonians!
- Easy to generalize to real bundles: chiral algebra on surface with involution

Regular punctures

A nice singularity

$$V(z,\bar{z}) = \int dv d\bar{v}\, e^{vZ^{+++}(z) - \bar{v}\bar{Z}^{+++}(\bar{z})}$$

$$H_a(z;w)\cdot \langle V(w,\bar{w})\rangle_{E_a,E_b,E_c} = \left\langle \mathrm{Tr}\mu_a^2[Z](z)\, V(w,\bar{w})\right\rangle$$

- Insertion of V requires parabolic structure at z
- Insertion of V induces simple poles in $\mathrm{Tr}\mu_a^2[Z](z)$
- Addition kernel with unipotent regular singularities is a correlation function of V's

$$K(a,b,c) \equiv \langle V_{a_1,b_1,c_1}(z_1)\cdots V_{a_n,b_n,c_n}(z_n)\rangle = \int \prod_i |dv_i|^2 e^{v\cdot \mathcal{G}(z_i,z_j)\cdot v - \mathrm{c.c.}}$$

Regular punctures of general weight

A Mellin transform

- Deformed vertex operator

$$V_{\lambda,\bar{\lambda}}(z,\bar{z}) = \int dv d\bar{v}\, v^{\lambda} \bar{v}^{-\bar{\lambda}}\, e^{vZ^{+++}(z) - \bar{v}\bar{Z}^{+++}(\bar{z})}$$

- Correlation function is Mellin transform of Gaussian:

$$K(a,b,c) = \int \prod_i |dv_i|^2\, v_i^{\lambda_i} \bar{v}_i^{-\bar{\lambda}_i}\, e^{v\cdot\mathcal{G}(z_i,z_j)\cdot v - \text{c.c.}}$$

Hecke operators and spectral flow modules

A role for Weyl modules

$$\mathcal{H}_a(z,\bar{z}) \cdot \mathcal{Z}[E_a, E_b, E_c] = \langle S_a(z,\bar{z}) \rangle_{E_a, E_b, E_c}$$

$$S_a(z,\bar{z}) = S_b(z,\bar{z}) = S_c(z,\bar{z})$$

$$\partial_z^2 S_a(z,\bar{z}) + H_a(z) S_a(z,\bar{z}) = 0$$

- Minimal Hecke modification makes Z(z) anti-periodic.
- Hecke operator inserts a special Ramond module for symplectic bosons

$$Z^{\alpha\beta\gamma}(z) S_a(0,0)[v] \sim \frac{S_a(0,0)[c^{\alpha\beta\gamma} v]}{z^{\frac{1}{2}}} + \cdots$$

$$[c^{\alpha\beta\gamma}, c^{\alpha'\beta'\gamma'}] = \epsilon^{\alpha\alpha'} \epsilon^{\beta\beta'} \epsilon^{\gamma\gamma'}$$

A wonderful Weyl module

Quantizing the nice locus

- Non-chiral description:

$$S_a = S_a\left[\int |dt|^2 e^{t\,\mathrm{det}_{2\times 2}\, u - \mathrm{c.c}}\right]$$

$$c^{-\cdots} = u\,\cdot \qquad c^{+\cdots} = \partial_u$$

- Secretly symmetric

$$\mu_a[c]\circ \int |dt|^2 e^{t\,\mathrm{det}_{2\times 2}\, u - \mathrm{c.c}} = 0$$

$$\mu_b[c]\circ \int |dt|^2 e^{t\,\mathrm{det}_{2\times 2}\, u - \mathrm{c.c}} = 0$$

$$\mu_c[c]\circ \int |dt|^2 e^{t\,\mathrm{det}_{2\times 2}\, u - \mathrm{c.c}} = 0$$

Arakawa's VOA

Chiral algebras of class S, Sicilian boundary VOAs

- Chiral algebras $V_k[\mathfrak{g}]$ labelled by ADE Lie algebra and integer
  - k copies of critical Kac-Moody subalgebras, glued along center
- Conformal blocks of $V_k[\mathfrak{g}]$ as twisted D-module on Bun x Bun … x Bun
  - GL dual to structure sheaf of diagonal
  - Obviously intertwines Hitchin Hamiltonians
- Non-chiral CFTs?

Operations on Arakawa's VOA

Gluing or removing

- Whittaker reduction of one copy $V_k[\mathfrak{g}] \to V_{k-1}[\mathfrak{g}]$
- Diagonal quantum DS reduction $V_k[\mathfrak{g}] \times V_{k'}[\mathfrak{g}] \to V_{k+k'-2}[\mathfrak{g}]$
  - Non-chiral version of quantum DS reduction: 2d gauge theory

$$Z_{k+k'-2,\mathfrak{g}}(a_1, \cdots, a_{k-1}, b_2, \cdots, b_{k'}) = \int_{\mathrm{Bun}} d\mu_c Z_{k,\mathfrak{g}}(a_1, \cdots, a_{k-1}, c) Z_{k',\mathfrak{g}}(c, b_2, \cdots, b_{k'})$$

  - Still associative? Tested for $V_4[\mathfrak{sl}_2]$ on 4pt sphere
- Wavefunction decomposition? $Z_{k,\mathfrak{g}}(a_1, \cdots, a_k) = \sum_{\rho \sim \bar{\rho}} C^{k-2}_{\rho,\bar{\rho}} \prod_i \psi_{\rho,\bar{\rho}}(a_i)$